Comments on Overdetermination of Maxwell's Equations

LIU Changli (刘长礼)*
Institute of Applied Physics and Computational Mathematics, Beijing China, 100094

Maxwell's equations seem overdetermined, which have six unknowns and eight equations. It is generally believed that Maxwell's divergence equations are redundant, and both equations are thought as initial conditions of curl ones. Because of this explanation, two divergence equations usually are not solved in computational electromagnetics. A circular logical fallacy of this explanation is found, and two divergence equations, which are not redundant, but fundamental, cannot be ignored in computational electromagnetics.

Maxwell's equations in vacuum seem overdetermined, which have six unknowns ($\mathbf{B}$, $\mathbf{E}$) and eight equations[2].

$$\nabla\cdot\mathbf{B}=\lambda\left(t-t_0\right)^2 \tag{1}$$

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0} \tag{2}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{4}$$

where, $\mathbf{B}$ is the magnetic induction; $\mathbf{E}$ is the electric field; μ_0,ε_0 are the electromagnetic constants; $\mathbf{J},\rho$ are the current and charge densities. $\mathbf{J},\rho$ usually are known sources. *Unless otherwise stated, constant* $\lambda\equiv 0$.

Two divergence equations (Eqs.(1,2)) are usually omitted and only two curl ones are solved in computational electromagnetics[2]. In fact taking the divergence of Eqs.(3,4) and using the continuity equation that sources satisfy $\nabla\cdot\mathbf{J}+\frac{\partial}{\partial t}\rho=0$, gives

$$\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{E}-\frac{\rho}{\varepsilon_0}\right)=0,\quad \frac{\partial\nabla\cdot\mathbf{B}}{\partial t}=0 \tag{5}$$

so that if Eqs.(1,2) are satisfied at some time t_0, it is inferred they are automatically fulfilled for any $t>t_0$. Therefore two divergence equations can be seen as the initial conditions of two curl ones. Here, Eq.(5) is regarded as the corollary of two curl equations (Eqs.(3,4)). J. A. Stratton[2] firstly introduced this explanation in 1941.

However, Jiang[1] think two divergences are not redundant, and both must be solved. In fact, Stratton's explanation has a circular logical fallacy . In the following, firstly we talk the circular logical error, secondly we discuss overdetermination of Maxwell's equations.

Firstly we talk the single div-curl overdetermined system,

$$\nabla\times\mathbf{u}=\mathbf{S} \tag{6}$$

$$\nabla\cdot\mathbf{u}=\rho \tag{7}$$

where, $\mathbf{u}$ is the unknown; $\rho,\mathbf{S}$ are sources. Taking the divergence of Eq.(6), the compatibility condition $\nabla\cdot\mathbf{S}=0$ is botained. If $\nabla\cdot\mathbf{S}\neq 0$, solutions of Eq.(6) do not exist. Therefore, with proper boundary conditions, the equation ($\nabla\cdot\mathbf{S}=0$) is equivalent to the existence of the solutions for Eq.(6), while not the corollary of Eq.(6). It must be noted that the equal sign in $\nabla\cdot\mathbf{S}=0$ does not automatically hold. For example, if $\mathbf{S}=\lambda\mathbf{r}$, then $\nabla\cdot\mathbf{S}=\nabla\cdot\lambda\mathbf{r}=3\lambda\neq 0$; while solutions of Eq.(6) do not exist in this situation. If $\nabla\cdot\mathbf{S}=0$ is thought as a corollary of Eq.(6), a circular logical fallacy must be in it.

Maxwell's equations are double div-curl systems. From above point of view, Stratton's explanation is equivalent to that Eq.(5) is regarded as the compatibility conditions of Maxwell's curl equations. *Stratton's explanation is based on the existence of solutions for Maxwell's curl equations.* If $\lambda\neq 0$ in Eq.(1), we get $\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{B}\right)=2\lambda\left(t-t_0\right)$, which has conflicts with Eq.(5). The reason of this conflicts is that solutions of Eqs.(1-4) do not exist for any $t>t_0$. Here, Stratton's explanation is not correct.

Stratton's explanation[2] prior demands that solutions of Maxwell's curl equations do exist with proper boundary & initial conditions (which is equivalent to suppose equal signs in Eq.(5) hold), and then Eq.(5) is deduced again from Eqs.(3,4). In another words, a hypothesis is used in advance, which is equivalent to suppose Eq.(5) hold, then Eq.(5) is obtained as a corollary. This is a circular logic. Like single div-curl system, equal signs in Eq.(5) do not automatically hold (Making Eqs.(1,2) hold for all times is the only way to guarantee equal signs in Eq.(5) work). The equal signs in Eq.(5) are thought automatically hold in Stratton's explanation, and Eq.(5) is thought as the corollary of curl equations (Eqs.(3,4)). However, the logic is circular, and Eq.(5) is not the corollary of curl ones but the precondition. Ignoring two divergence equations in computational electromagnetics losts the theoretical fundamental. (Dear readers, if you clearly know what conditions are needed to ensure solutions of Maxwell's curl equations exist, then you can understand where is wrong about Stratton's explanation.)

Now, I talk overdetermination of Maxwell's equations. In general relativity[3], Einstein field equations have ten equations. The four Bianchi identities reduce the independent equations from ten to six[3]. Generally, four harmonic coordinates are added to fix the freedoms[3]. Similarly, both $\nabla\cdot\nabla\times\mathbf{E}\equiv 0$, $\nabla\cdot\nabla\times\mathbf{B}\equiv 0$ identities reduce the independent Maxwell's curl equations from six to four. Including two divergence equations, there are six independent equations in Maxwell's equations.

In fact, a generalized definition can be employed to describe it. There are first-order linear partial differential equations as following

$$\begin{cases} \sum_{ij} a_{ij}^{(1)} \frac{\partial y_j}{\partial x_i} + f_1 = 0 \\ \vdots \\ \sum_{ij} a_{ij}^{(n)} \frac{\partial y_j}{\partial x_i} + f_n = 0 \end{cases} \tag{8}$$

where y_j are unknowns; $a_{ij}^{(k)}$ are coefficients; and f_k are non-homogeneous items. Let $Z_k = \sum_{ij} a_{ij}^{(k)} \frac{\partial}{\partial x_i} y_j + f_k$.

Two linear dependence definitions are as following.

Definition I: In algebra, when there are coefficients (c_k), not all zero, such that $\sum_{k=1} c_k Z_k = 0$; the Eqs.(8) are linear dependent.

This definition can be referred in any algebraic textbook. Maxwell's equations are over-determined in the definition I.

Definition II (differential linear dependence): When there are coefficients (c_k, d_{kl}), not all zero, such that $\sum_k c_k Z_k + \sum_{kl} d_{kl} \cdot \frac{\partial}{\partial x_l} Z_k = 0$, the Eqs.(8) are thought as linear dependent. If $d_{kl} \equiv 0$, this definition degenerates into the definition I.

In the definition II, the div-curl system (Eqs.(6,7)), Maxwell's equations, Einstein field equations (ten equations plus four harmonic coordinates) and elasticity equilibrium equations in strain (or stress) formulation are well-determined.

In summary, Stratton's explanation demands Eq.(5) hold in advance (which is equivalent to that solutions of Maxwell's equations do exist with proper boundary & initial conditions), and then Eq.(5) is deduced as the corollary of Eqs.(3,4). The circular logical relationship is wrong, and guaranteeing Eqs.(1,2) hold for all times is the only way to make equal signs in Eq.(5) work in electromagnetics. Neglecting two Gauss's laws in computational electromagnetics is not correct. Both identities ($\nabla \cdot \nabla \times \mathbf{E} \equiv 0$, $\nabla \cdot \nabla \times \mathbf{B} \equiv 0$) are the true origin of overdetermination of Maxwell's equations.

There are some conjectures about the definition II.

$\boxed{1}$ If Eqs.(8), whose solutions exist and are unique, are over-determined in the definition I, then they must be well-determined in the definition II.

$\boxed{2}$ If Eqs.(8), whose solutions exist, are under-determined in the definition II and are well-determined in the definition I, then the solutions must be non-unique.

$\boxed{3}$ If Eqs.(8) are over-determined in the definition II, then the solutions do not exist.

The conjectures seem obvious, but the proof is not easy. If all the conjectures are correct, the definition I should be changed to the definition II.

* E-Mail: LIUCL78@pku.org.cn

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[2] Stratton, J.A., Electromagnetic Theory, New York: McGraw-Hill Book Company, 5-6 (1941).

[3] Wald, R.M., General Relavtivity, Chicago: University of Chicago Press, 259-260 (1984).