Comments on Overdetermination of Maxwell's Equations

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Abstract—Maxwell's equations seem over-determined, which have 6 unknowns and 8 equations. It is generally believed that Maxwell's divergence equations are redundant and may be ignored when they are satisfied at some time t_0. In the paper, the definition of differential linear dependence is used to explain the over-determined problem. Maxwell's equations are well-determined in this definition, and two divergence equations are not redundant which have equal status with curl ones.

1. INTRODUCTION

In linear algebraic system, the over-determined equations, whose independent equations are more than unknowns, have no solution. This is well known. In linear differential equations, most equations are well-determined. But there are some exceptions, for example Maxwell's equations which have 6 unknowns and 8 equations[1]. It should be noted that Maxwell's equations' solutions exist and are unique.

Maxwell's equations are:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{4}$$

Two divergence equations (Eq.(3,4)) are usually omitted and only two curl ones are solved in electromagnetics[1, 2]. In fact taking the divergence of Eq.(1,2) and using the continuous condition that sources satisfy $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$, gives

$$\frac{\partial}{\partial t}\left(\varepsilon_0 \nabla \cdot \mathbf{E} - \rho\right) = 0, \quad \frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \tag{5}$$

so that if Eq.(3,4) are satisfied at some time t_0 , it is inferred they are automatically fulfilled for any $t > t_0$. Therefore two divergence equations can be seen the initial conditions of two curl ones, and two left curl ones are well-determined. This explanation was first introduced by J. A. Stratton[1].

However, It has been shown that this conclusion is not correct for boundary-initial value problems[2]. Jiang think two divergences are not redundant, and both must be solved. I do agree that two divergences must be solved in any fields and this explanation is not appropriate. In the following I will explain the over-determined problem in another way.

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2. DEFINITION OF DIFFERENTIAL LINEAR DEPENDENCE

There are linear partial differential equations:

$$\begin{cases} \sum_{ij} a_{ij}^{(1)} \frac{\partial y_j}{\partial x_i} + f_1 = 0 \\ \sum_{ij} a_{ij}^{(2)} \frac{\partial y_j}{\partial x_i} + f_2 = 0 \\ \qquad \vdots \\ \sum_{ij} a_{ij}^{(n)} \frac{\partial y_j}{\partial x_i} + f_n = 0 \end{cases} \tag{6}$$

where x_i are independent variables; y_j are dependent variables; $a_{ij}^{(k)}$ are coefficients; and f_k are non-homogeneous items. And I make $Z_k = \sum_{ij} a_{ij}^{(k)} \frac{\partial y_j}{\partial x_i} + f_k$.

Two linear dependence definitions are as following.

Definition I: In algebra, when there are coefficients (c_k), not all zero, such that $\sum_{k=1}^{n} c_k Z_k \equiv 0$; the Eq.(6) is linear dependent.

This definition can be referred in any algebraic textbook. Maxwell's equations are over-determined in definition I. Now I generalize the definition of linear dependence in differential equations.

Definition II (differential linear dependence): When there are coefficients (d_{kl}), not all zero, such that $\sum_k c_k Z_k + \sum_{kl} d_{kl} \frac{\partial Z_k}{\partial x_l} \equiv 0$, the Eq.(6) are thought as differential linear dependent.

If $d_{kl} \equiv 0$, definition II becomes definition I. The difference between definition I and II is: I take one (or more) differentiation of Z_k. I think definition II is more appropriate than definition I in fields of differential equations.

3. ELECTROMAGNETICS

Before Maxwell's equations are discussed, we talk the div-curl over-determined system whose solution exists and is unique[2].

$$\nabla \times \mathbf{E} = \mathbf{S} \tag{7}$$

$$\nabla \cdot \mathbf{E} = \rho' \tag{8}$$

where $\rho', \mathbf{S}$ are known sources; the function $\mathbf{S}$ must satisfy the compatibility condition: $\nabla \cdot \mathbf{S} \equiv 0$. This is well known. Eq.(7) and (8) are basic equations, and both must be solved. Equation $\nabla \cdot \mathbf{S} \equiv 0$ is compatibility condition, which is not needed to be solved. The div-curl system is over-determined in definition I, but it is well-determined in definition II because of $\nabla \cdot \nabla \times \mathbf{E} \equiv 0$.

From above point of view, in fact Stratton's explanation is equivalent to that Eq.(5) are regarded as the compatibility conditions of Maxwell's curl equations. That is to say, Eq.(4) is the compatibility condition of Eq.(1), and Eq.(3) is the compatibility condition of Eq.(2). However, the compatibility conditions (Eq.(3,4)) are different from the compatibility conditions $\nabla \cdot \mathbf{S} \equiv 0$ of Eq.(7). The function $\mathbf{S}$ is a known one, therefore it is not needed be sovled. But compatibility conditions Eq.(3,4) include unknowns $(\mathbf{B}, \mathbf{E})$. If they are not solved, unphysical solutions perhaps appear[2].

3.1. Electrostatic fields

If $\mathbf{S} \equiv 0, \rho' = \frac{\rho}{\varepsilon_0}$, Eq.(7,8) become basic equations of electrostatic fields

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \tag{9}$$

$$\mathbf{E} = -\nabla\phi, \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \quad \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \tag{10}$$

The compatibility condition of Eq.(9) is trivial (because of $\mathbf{S} \equiv 0$). Therefore we cannot obtain non-trivial conclusions like Eq.(5) in electrostatic fields, and divergence equation here is basic equation, not compatibility condition. We can perhaps ignore divergence equations in Maxwell's equations because of Stratton's explanation. However, *this explanation cannot be applied on electrostatic fields*.

As we all know, Eq.(9) is equivalent to Eq.(10). However, Eq.(9) is over-determined, while Eq.(10) is well-determined. This is an embarrassing situation. Why does this difference exist in both equivalent equations? As we all know $\nabla \cdot \nabla \times \mathbf{E} \equiv 0$, Eq.(9) is differential linear dependent in definition II. There are only two linear independent equations in a curl vector one. Therefore, Eq.(9) has only three linear independent equations in definition II, which are equal to unknowns. If definition I is used in differential equations' fields, we have to face the embarrassing situation: Eq.(9) is equivalent to Eq.(10), but Eq.(9) is over-determined, while Eq.(10) is well-determined. If definition II is used, both equations are well-determined.

Now, we can see that Stratton's explanation is not suitable for electrostatic fields (and div-curl system Eq.(7,8)). If definition of linear dependence is changed from I to II, the overdetermination of electrostatic fields (and div-curl system Eq.(7,8)) can be explained. So I think definition II is more suitable than definition I.

3.2. Maxwell's equations

Now I will discuss Maxwell's equations. There are two curl equations in Maxwell's equations. As we all know

$$\nabla \cdot \nabla \times \mathbf{E} \equiv 0,\ \nabla \cdot \nabla \times \mathbf{B} \equiv 0 \tag{11}$$

Therefore, there are two differential linear dependent equations, and the number of independent equations is six, which are equal to unknowns. Now, Maxwell's equations are well-determined in definition II. All eight equations have equal status, and all of them are basic equations, and no one is the initial condition of another. All eight equations should be solved in electromagnetics, while not omitting divergence equations. If only two curl equations are solved, which are under-determined in definition II, solutions perhaps are not unique.

From above, Stratton's explanation is equivalent to that two divergence equations are regarded as compatibility conditions. Two divergence equations have two properties: basic equations and compatibility conditions. As the property of basic equations, two divergence equations must be solved. As the property of compatibility conditions, two divergence equations can be ignored. And the property of basic equations is more important than the property of compatibility conditions. Therefore two divergence equations cannot be omitted. I think Stratton's explanation is not incorrect, but inappropriate.

4. OTHER SYSTEMS

4.1. Elasticity equilibrium equations

Apart from Maxwell's equations, there are some other over-determined differential equations in definition I. In the following, we will discuss elasticity equations[3]. The equation in strain form is:

$$\nabla \cdot [2\mu\mathbf{\Gamma} + \lambda\mathbf{I}\, trace\,(\mathbf{\Gamma})] = 0 \tag{12}$$

$$\nabla \times \mathbf{\Gamma} \times \nabla = 0 \quad (\Leftrightarrow \varepsilon_{pij}\varepsilon_{qks}\partial_i\partial_s\Gamma_{jk} = 0) \tag{13}$$

where $\mathbf{\Gamma}$ is the second order symmetric strain tensor; μ, λ are lame coefficients; $\mathbf{I}$ is the unit second order tensor; ε_{pij} is a Levi-Civita symbol. Eq.(12) is equilibrium equations, and Eq.(13) is compatibility equations. Eq.(12,13) has 6 knowns and 9 equations, whose solutions exist and are unique. As we all know

$$\nabla \cdot (\nabla \times \mathbf{\Gamma} \times \nabla) \equiv 0, \quad (\nabla \times \mathbf{\Gamma} \times \nabla) \cdot \nabla \equiv 0 \tag{14}$$

Because $\mathbf{\Gamma}$ is symmetric, the two above identities only have 3 different component equations (3 differential linear dependent equations). Therefore the independent equations in elasticity equations are six (9-3=6), which are equal to unknowns. Eq.(12,13) is well-determined in definition II.

4.2. Einstein Field Equation

Einstein field equation has 10 equations. The 4 Bianchi identities reduce the independent equations to six. Generally, 4 coordinate conditions are added to fix the freedoms. In definition II, the 14 equations (10 equations +4 coordinate conditions) are well-determined, and the equations' solutions exist and are unique.

5. SUGGESTION FOR COMPUTATIONAL ELECTROMAGNETICS

Maxwell's equations are well-determined in definition II, but the discrete scheme of Maxwell's equations is over-determined which has 6 unknowns and 8 algebraic equations. The least square method is usually adopted to solve it. Solving over-determined algebraic equations is not the best choice. We wish to solve well-determined algebraic equations. I will give a suggestion which can realize it. Transformations (Eq.(17)) are substituted into Eq.(1-4). And Coulomb gauge condition ($\nabla \cdot \mathbf{A} = 0$) is adopted. We can obtain vector and scalar potential Maxwell's equations.

$$\nabla^2 \phi = -\rho/\varepsilon_0 \tag{15}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu_0 \varepsilon_0 \frac{\partial \nabla \phi}{\partial t} = -\mu_0 \mathbf{J} \tag{16}$$

Vector and scalar potential transformations are

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{17}$$

If we can create divergence-free schemes in which Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) is automatically satisfied, the discrete schemes of Eqs.(15,16) are well-determined algebraic equations. These schemes do not have shortcomings of least square methods.

6. DISCUSSION

I have generalized the definition of linear dependence in differential equations; and discussed Maxwell's equations and elasticity equations by this definition. Both equations are over-determined in definition I, but both are well-determined in definition II.

If we use definition I, there are some conclusions against common sense. Paradox (a): As we all know, generally over-determined systems' solutions do *not* exist. But the solutions of over-determined Maxwell's equations do exist. In order to change Maxwell's equations to well-determined ones, people ignore two divergence equations. The solutions of left two curl equations (under-determined in definition II) are not unique, and spurious solutions perhaps appear in electromagnetics[2]. Paradox (b): Eq.(9) is equivalent to Eq.(10). But Eq.(9) is over-determined, while Eq.(10) is well-determined. *Stratton's explanation cannot interpret the overdetermination of electrostatic fields (Eq.(9)) and div-curl system (Eq.(7,8)).*

If we use definition II, all paradoxes disappear. All eight well-determined Maxwell's equations are solved to ensure solutions unique; and both Eq.(7,8) and Eq.(9,10) are well-determined.

There are some conjectures:

$\boxed{1}$: If Eq.(6), whose solution exists and is unique, is over-determined in definition I, it must be well-determined in definition II.

$\boxed{2}$: If Eq.(6), whose solution exists, is under-determined in definition II and it is well-determined in definition I, the solution must be non-unique.

$\boxed{3}$: If Eq.(6) is over-determined in definition II, the solution does not exist.

The conjectures seem obvious, but the proof is not easy. If all the conjectures are correct, definition I should be changed to definition II.

REFERENCES

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3. M.H. Sadd, Elasticity, Theory, Applications, and Numerics, Oxford: Elsevier Butterworth-Heinemann, (2005) chapter 5