New formula of the mobius function

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Theorem 1

\[
\sum_{1 \leq n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1
\]

Proof)

\(x = 1 \Rightarrow 1 = 1\)
\(x = 2 \Rightarrow 2 - 1 = 1\)
\(x = 3 \Rightarrow 3 - 1 - 1 = 1\)

\(x = 29\) and \(x = 30\) cases
\(1 - 1 - 1 + 1 + 1 + 1 = 0\) increase.

\[
\sum_{1 \leq n \leq 30} \mu(n) \left\lfloor \frac{30}{n} \right\rfloor = 1
\]

General cases are similarly.

Theorem 1

\[
\sum_{1 \leq n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1
\]

Another proof)

\[
\sum_{n|m} \mu(n) = 1 (m = 1)
\]

\[
\sum_{n|m} \mu(n) = 0 (m \neq 1)
\]
Clearly, theorem 1 is got.

More generally,

Theorem 2
For real number $x$. (For example $x = 1.5$)

$$\sum_{1 \leq n \leq x} \mu(n)\left\lfloor \frac{x}{n} \right\rfloor = 1$$

Remark:

$$\sum_{1 \leq n \leq x} \mu(n) = O\left( \sum_{1 \leq n \leq x} \mu(n)\left\lfloor \frac{x}{n} \right\rfloor \right) = O(1)$$

is not true.