

# The Temperature of a System as a Function of the Multiplicity and its Rate of Change

Rodolfo A. Frino

Electronics Engineer  
Degree from the National University of Mar del Plata - Argentina  
rodolfo\_frino@yahoo.com.ar  
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## Abstract

*In this paper I derive the formula for the temperature of a thermodynamic system as a function of the multiplicity (number of microstates) and its the rate of change with respect to the absorbed or lost energy. Then the formula is used to obtain the “temperature-microstates-energy” relation for a black hole assuming that the number of microstates is proportional to the energy of the emitted photon through either tunnelling effect or Hawking radiation.*

**Keywords:** *macroscopic state, multiplicity, microscopic states, Boltzmann constant, Boltzmann entropy formula, absolute temperature, entropy, Hawking radiation.*

## 1. Nomenclature

I shall use the following nomenclature for the constants and variables used in this paper

$\mu$  = multiplicity: number of microstates corresponding to a given macroscopic state  
 $W$  = another symbol for multiplicity ( $\mu = W$ )  
 $N$  = total possible number of microstates of the system  
 $P$  = probability of a given macroscopic state of the system  
 $k_B$  = Boltzmann constant  
 $S$  = system's entropy change  
 $Q$  = net absorbed or lost heat (heat is a form of energy) (heat change)  
 $E$  = net absorbed or lost energy (energy change)  
 $T$  = absolute temperature of the system  
 $dQ$  = infinitesimal net absorbed or lost energy  
 $dE$  = infinitesimal net absorbed or lost energy  
 $dS$  = infinitesimal entropy change  
 $A$  = proportionality constant  
 $T_{BH}$  = absolute temperature of a black hole  
 $f_\nu$  = frequency of the Hawking radiation (or frequency of a virtual photon)

## 2. Thermodynamic Equations

Let us consider the basic equations we shall use in Section 3. These equations are

### Equation 1 The Second Principle of Thermodynamics (Clausius' entropy definition)

The second principle of thermodynamics states:

*It is impossible to carry out a transformation whose only final result is to transfer heat from a body at a given temperature, to a body at a higher temperature (Clausius' postulate).*

The German scientist Clausius defined the infinitesimal change in the entropy of a system as follows [1]

Irreversible transformations

$$dS \geq \frac{dQ}{T} \quad (2.1 \text{ a})$$

This means that the infinitesimal change in the entropy during an irreversible transformation is equal to or greater than the quotient between the infinitesimal change in the heat absorbed (or emitted) by the system and the system temperature. In an irreversible transformation the entropy of the system plus the entropy of the environment increases. For example, the entropy of the system can increase while the entropy of the environment can remain constant. This is the case, for example, of a free adiabatic expansion of a gas contained in one half of a container while in the other half there is vacuum. When we open a valve so that the gas can go from one half to the empty one, the gas expands adiabatically (we assume that the wall of the container do not allow the transfer of heat to or from the environment) so that the entropy of the system increases while the entropy of the environment does not change.

Reversible transformations

$$dS = \frac{dQ}{T} \quad (2.1 \text{ b})$$

This means that the infinitesimal change in the entropy during a reversible transformation can be calculated by dividing the infinitesimal change in the heat absorbed (or emitted) by the system by the system temperature. For reversible transformations, if the system absorbs an amount of heat  $dQ$  from the environment, the entropy of the environment decreases by  $dQ/T$ . On the other hand the entropy of the system increases by  $dQ/T$ , so that the net change of entropy of the system and the environment is zero.

The second principle of thermodynamics can be expressed quantitatively in terms of the change in entropy  $S = \Delta S$  as follows

Irreversible transformations

$$S = \Delta S = S_f - S_i \geq \int_i^f \frac{dQ}{T} \quad (2.1 \text{ c})$$

In an irreversible transformation

$$S_f > S_i \quad (2.1 \text{ d})$$

Reversible transformations

$$S = \Delta S = S_f - S_i = \int_i^f \frac{dQ}{T} \quad (2.1 \text{ e})$$

In a reversible transformation

$$S_f = S_i \quad (2.1 \text{ f})$$

Thus we have linked the second principle of thermodynamics with the entropy of a system. In Section 3 we only use equation (2.1 b). Equations (2.1 a) and equations (2.1 c) through (2.1 f) are given for completeness purposes.

## Equation 2 The Probability equation

Let us define the probability  $P$  that a system is in a given macrostate by the following ratio

$$P = \frac{\mu}{N} \quad (2.2 \text{ a})$$

where  $\mu$  is the number of microstates that create a given macroscopic state (A macroscopic state is also known as a thermodynamic state). In other words, the number of microstates,  $\mu$ , for a given macroscopic state, is the number of ways in which the macroscopic state can be produced or created. The number of microstates is also denoted by  $W$  which is known as *multiplicity*.  $N$  is the total possible number of microstates (In the case of a gas,  $N$  is the total number of combinations of molecular velocities).

Because equation (2.2 a) can be written as

$$\mu = W = NP \quad (2.2 \text{ b})$$

we can say that the multiplicity of a system is proportional to the probability.

## Equation 3 The Boltzmann entropy formula

The Boltzmann formula, which has been considered the second most famous formula of physics, is

$$S = k_B \ln W \quad (3.1 \text{ a})$$

Substituting  $W$  in equation (3.1 a) by equation (2.2 b) we get

$$S = k_B \ln(NP) \quad (3.1 \text{ b})$$

It is worthy to remark that the entropy is a macroscopic quantity while the number of microstates is a microscopic quantity.

### 3. Derivation of the Temperature of a System as a Function of the Multiplicity and its the Rate of Change with Respect to the Absorbed or Lost Energy

To make things easier let us abbreviate the name of the formula we want to derive. Thus I shall call it: *the Temperature-Microstates-Energy* relation. Let us start the derivation by differentiating equation (3.1 b) with respect to the absolute temperature,  $T$ . This yields

$$\frac{dS}{dT} = k_B \frac{d}{dT} [\ln(NP)] \quad (3.1)$$

$$\frac{dS}{dT} = k_B \left( \frac{1}{NP} \right) \frac{d}{dT} (NP) \quad (3.2)$$

$$\frac{dS}{dT} = k_B \left( \frac{1}{NP} \right) N \frac{dP}{dT} \quad (3.3)$$

$$\frac{dS}{dT} = \frac{k_B}{P} \frac{dP}{dT} \quad (3.4)$$

$$dS = \frac{k_B}{P} dP \quad (3.5)$$

We shall assume that the system suffers a reversible transformation. We can eliminate  $dS$  from equation (3.5) by substituting it with the second side of equation (2.1 b). This gives

$$\frac{dQ}{T} = \frac{k_B}{P} dP \quad (3.6)$$

Equation (3.6) can be re-written as

$$\frac{1}{T} = \frac{k_B}{P} \frac{dP}{dQ} \quad (3.7)$$

or

$$T = \frac{P}{k_B} \frac{dQ}{dP} \quad (3.8)$$

Because heat is a form of energy we can write

$$dQ = dE \quad (3.9)$$

Consequently we can re-write equation (3.8) as follows

$$T = \frac{P}{k_B} \frac{dE}{dP} \quad (3.10)$$

Now let us express the definition of probability given by equation (2.2 a) in differential form. This gives

$$dP = \frac{1}{N} d\mu \quad (3.11)$$

Now in equation (3.10) we eliminate  $P$  and  $dP$  with equations (2.2 a) and (3.11), respectively. These substitutions produce the following result

$$T = \frac{\mu}{N k_B} \frac{N dE}{d\mu} \quad (3.12)$$

and after simplification we get

$$T = \frac{1}{k_B} \mu \frac{dE}{d\mu} \quad (3.13)$$

A more convenient way of writing the above formula is as follows

#### Temperature-Microstates-Energy relation

$$T = \frac{1}{k_B} \frac{\mu}{\frac{d\mu}{dE}} \quad (3.14)$$

This is the *Temperature-Microstates-Energy* relation for reversible systems. Where  $\frac{d\mu}{dE}$  is the rate of change of the microstates of the system (multiplicity) with respect to the absorbed or lost energy. To obtain the final temperature-energy change expression for a particular system, we have to either find a mathematical relation between the number of microstates,  $\mu$ , and the absorbed or lost energy,  $E$ , or postulate one. In the next section I shall illustrate this point by applying equation (3.14) to the physics of black holes.

## 4. The Temperature-Energy Change Relation for Black Holes

Now want to find the *Temperature-Energy Change* relation for black holes. Thus we shall introduce the following postulate:

### *Postulate*

*The number of microstates,  $\mu$ , of a black holes is proportional to the energy,  $E_\nu$ , of the emitted photon through either tunnelling effect or Hawking radiation.*

Mathematically this means that

$$\mu = A E \quad (4.1)$$

where  $A [J^{-1}]$  is a proportionality constant. Let us differentiate equation (4.1) with respect to the energy. This gives

$$\frac{d\mu}{dE} = A \quad (4.2)$$

Substituting  $\mu$  and  $\frac{d\mu}{dE}$  in equation (3.14) with the second side of equations (4.1) and (4.2), respectively we obtain

$$T = \frac{E}{k_B} \quad (4.3)$$

Finally, because we applied equation (3.14) to black holes,  $T$  becomes  $T_{BH}$ . The energy lost by the system (black hole),  $E$ , is the energy of the escaping photon,  $E_v$  (the virtual photon that escapes to empty space through the Hawking evaporation mechanism and becomes real). Thus we can write:

$$T_{BH} = \frac{E_v}{k_B} \quad (4.4)$$

and solving for  $E_v$

$$E_v = E_{\text{escaping } \gamma} = k_B T_{BH} \quad (4.5)$$

Thus the energy of the escaping photon,  $E_v = E_{\text{escaping } \gamma}$ , is proportional to the absolute temperature,  $T_{BH}$ , of the black hole [2].

## 5. Conclusions

We can draw the following conclusions

- (1) Assuming a linear relationship between the number of microstates of a black hole and the energy of the emitted photon (the virtual photon that escapes to empty space and becomes real), we found that this energy is proportional to the temperature of the black hole. Mathematically:

$$E_v = E_{\text{escaping } \gamma} = k_B T_{BH} \quad (5.1)$$

Therefore the temperature of the black hole can be determined by measuring the frequency of the emitted photon through the formula

$$T_{BH} = \frac{h}{k_B} f_v \quad (5.2)$$

It is worthy to remark that the present astronomical techniques do not allow us to detect the Hawking radiation.

- (2) This model indicates that the Hawking radiation is monochromatic.
- (3) Formula (3.14) allow us, also, to postulate non-linear relationships between the number of microstates of a black hole and the energy of the emitted photon. To determine the validity of these postulates, the predictions of the theory based on them, should be compared with future measurement of black hole temperatures.

## REFERENCES

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