Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory
Part 6 Sequences and calculus in $*G$

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Abstract

With the partition of positive integers and positive infinite integers, it follows naturally that sequences are also similarly partitioned, as sequences are indexed on integers. General convergence of a sequence at infinity is investigated. Monotonic sequence testing by comparison. Promotion of a ratio of infinite integers to non-rational numbers is conjectured. Primitive calculus definitions with infinitary calculus, epsilon-delta proof involving arguments of magnitude are considered.

1 Introduction

The discovery of infinite integers leads to the obvious existence of the infinitesimals [6, Part 1], as dividing 1 by an infinite integer is not a real number. However, it also does so much more. For theorems, the separation of the finite and infinite is possible, rather than having a single theorem which addresses both cases.

As sequences are indexed by integers, we similarly find that sequences can be partitioned. The following investigates some of the mechanics of sequences, particularly at infinity. For example it could be that over time sequences supersede sets. Sequences can be viewed as more primitive structures.

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2 Sequences and functions

We have extended the sequence notation to include intervals, as we deem that 'the order' is the most important property.

Definition 2.1. *Join or concatenate two sequences.* $(a) + (b) = (a, b)$ *A sequence can be deconstructed.* $(a, b) = (a) + (b)$ *The operator* $+$ *is not commutative,* $(b) + (a) = (b, a)$*, and in general* $(b, a) \neq (a, b)$*.*

Definition 2.2. *Compare sequences component wise on relation* z*.* $(a_1, a_2, \ldots)\ z\ (b_1, b_2, \ldots)$ *then* $(a_1\ z\ b_1, a_2\ z\ b_2, \ldots)$

Example 2.1. *While we often use functions to state a comparison of sequences, we may use sequence notation. Here, infinite positive integers are implied.* $(n) \prec (n^2)|_{n=\infty}$

Definition 2.3. *If a set has a 'less than' relation, the sequence of the set is ordered. If* X *is the set, let* (X) *be the sequence of* X *with the order relation.*

Example 2.2. $(+\Phi)$ *is the ordered sequence of positive infinitesimals,* $(\mathbb{N}_\infty)$ *is the ordered sequence of infinite positive integers,* $(+\Phi^{-1})$ *is the ordered sequence of positive infinite numbers,* $(\mathbb{N})$ *is the ordered sequence of natural numbers,* $(\mathbb{N}_<)$ *ordered sequence of finite natural numbers.*

We can iterate over infinity in the following way. Consider the infinite sequence $1, 2, 3, \ldots$ We can express this with a variable n. i.e. $1, 2, 3, \ldots, n, n+1, n+2, n+3, \ldots|_{n=\infty}$.

Definition 2.4. *Any integer sequence can be composed of both finite and infinite integers.*

$$(1, 2, 3, \ldots, k)|_{k<\infty} + (\ldots, n-1, n, n+1, n+2, n+3, \ldots)|_{n=\infty}$$

$$(\mathbb{N}_<) + (\mathbb{N}_\infty)$$

With the establishment of the existence of the infinite integers, since a sequence is indexed by integers we can partition the sequence into finite and infinite parts. Further all sequences with integer indices, implicitly or explicitly are of this form. A finite sequence is deconstructed with no infinite part.

Definition 2.5. *Define a sequence at infinity* $(a_n)|_{n=\infty}$ *to iterate over the whole infinite interval,*

$$(\ldots, a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}, \ldots)|_{n=\infty}$$

or to count from a point onwards, generally in a positive direction.

$$(a_n, a_{n+1}, a_{n+2}, \ldots)|_{n=\infty}$$

The concept of a sequence at infinity is particularly important, as we now can separate and partition finite and infinite numbers.

Definition 2.6. *A sequence can be deconstructed into both finite and infinite parts.*

$$(a_1, a_2, \ldots) = (a_1, a_2, \ldots, a_k)|_{k<\infty} + (\ldots, a_n, a_{n+1}, \ldots)|_{n=\infty}$$

$$(a_1, a_2, \ldots) = (a_k)|_{1 \leq k < \infty} + (a_{\mathbb{N}_\infty})$$

$$(a_1, a_2, \ldots) = (a_{\mathbb{N}_<}) + (a_{\mathbb{N}_\infty})$$

What is striking is that at infinity there is no minimum or maximum elements. If $n = \infty$ is an infinity, so is $n-1$, $n-2$, ... Similarly for the continuous variable. If $x = \infty$, so is $x-1$, $x-2$, ... While infinity has no lower or upper bound, we may find it useful to define the 'ideal' min and max elements, as these can describe an interval.

Definition 2.7. *Ideal minimum and maximum numbers*
Let $\min(\mathbb{N}_\infty)$ *be an ideal minimum of the lowest positive infinite integer.*
Let $\max(\mathbb{N}_\infty)$ *be an ideal maximum of the highest positive infinite integer.*
Let $\min(+\Phi^{-1})$ *be an ideal minimum of the lowest positive infinite number.*
Let $\max(+\Phi^{-1})$ *be an ideal maximum of the highest positive infinite number.*

$$(\ldots, n-1, n, n+1, \ldots)|_{n=\infty} = (\min(\mathbb{N}_\infty), \ldots, \max(\mathbb{N}_\infty)) = (\mathbb{N}_\infty)$$
$$(\ldots + [x-1, x) + [x, x+1) + [x+1, x+2) + \ldots)|_{x=\infty} = (\min(+\Phi^{-1}), \max(+\Phi^{-1})) = (+\Phi^{-1})$$

Considering $1, 2, 3, 4, \ldots$, we believe that the infinity was thought of as an open set, $(1, 2, \ldots)$, but with infinite integers, this may be better expressed with an interval notation $[1, 2, \ldots, \infty]$.

Definition 2.8. *An interval can be deconstructed into real and infinite real parts*

$$(\alpha, \infty] = (x)|_{\alpha < x < \infty} + (+\Phi^{-1})$$

That there is no infinite integer lower bound often does not matter. Once we arrive at infinity, we may iterate from a chosen point onwards.

Example 2.3. *In describing the function* $\frac{1}{n}$ *as a sequence, we often say* $(\frac{1}{2}, \frac{1}{3}, \ldots)$ *which includes both finite and infinitesimal numbers. By considering the infinite sequence,* $(\frac{1}{n})|_{n=\infty}$ *we now are describing the infinitesimals only.*

We would like to iterate over infinity for various reasons. On occasion it is necessary to iterate not over all the infinities, but between two infinities. For example, like with NSA(Non-Standard Analysis), iterate between two infinities ω and 2ω. In our notation, we can start counting at infinity, till we reach the next infinity. Construct an auxiliary sequence for this purpose.

$$(a_{n+1}, a_{n+2}, \ldots a_{2n}) = (b_1, b_2, \ldots b_n) \text{ where } b_k = a_{n+k}|_{n=\infty}$$

From another perspective, we can use the same notation to iterate over all infinity. Iterating over infinity at infinity, so that the finite part is removed, $(a_{n+k})|_{k=n=\infty}$ iterates over all the infinite elements.

While there is no lower infinite bound for the infinite integers, this is not really a problem as we need not consider all infinite elements, but elements from a certain point onwards, hence Definition 2.5.

We require sequences at infinity when building other structures at infinity. The ordering property of sequences is separate to sets, which by their definition are unordered.

Sequences can be transformed and or rearranged, from one sequence to another, with an infinity of elements, in such a way to guarantee a property based on the order. Our subsequent papers [9], [10] both require sequences in the ideas and proofs.

Sequences are not restricted to discrete variables. As we can consider a function as a continuous sequence of points, we extend the sequence notation to the continuous variable. We would then consider the index which is also a continuous variable, the domain.

Example 2.4. *Partition the interval,* $[\alpha, \infty] = (x)|_{x=[\alpha,\infty]} = (x)|_{[\alpha, x<\infty)} + (x)|_{+\Phi^{-1}}$ *or* $[\alpha, +\mathbb{R}) + (+\Phi^{-1})$.

Definition 2.9. *We say a function is "monotonically increasing" if* $f(x+\delta) \geq f(x)$*, "monotonically decreasing" if* $f(x+\delta) \leq f(x)$*,* $\delta \in \Phi$.

Definition 2.10. *We say a sequence is "monotonically increasing" if* $a_{n+1} \geq a_n$*, "monotonically decreasing" if* $a_{n+1} \leq a_n$.

Definition 2.11. *We say a sequence or function has "monotonicity" if the sequence or function is monotonic: monotonically increasing or monotonically decreasing.*

Determine if a function is monotonic by comparing successive terms and solving for the relation. For a continuous function we can often take the derivative. However, for sequences this may not be possible.

Conjecture 2.1. *We can determine the monoticity of sequence* $a_n|_{n=\infty}$ *by solving for relation* z *in* $*G$, $a_{n+1} \; z \; a_n|_{n=\infty}$*, or if it exists its continuous version* $a(n+1) \; z \; a(n)$.

Example 2.5. *Determine if the sequence* $(a_n)|_{n=\infty}$ *is monotonic.* $a_n = \frac{1}{n^2}$*, compare sequential terms,* $a_{n+1} \; z \; a_n|_{n=\infty}$, $\frac{1}{(n+1)^2} \; z \; \frac{1}{n^2}|_{n=\infty}$, $n^2 \; z \; (n+1)^2|_{n=\infty}$, $n^2 \; z \; n^2+2n+1|_{n=\infty}$, $0 \; z \; 2n+1|_{n=\infty}$, $z = <$, $a_{n+1} < a_n$ *and the sequence is monotonically decreasing.*

Example 2.6. *Test if the sequence* $(a_n)|_{n=\infty}$ *is monotonic,* $a_n = \frac{1}{n^{\frac{1}{2}}+(-1)^n}|_{n=\infty}$*. Let* $j = (-1)^n$, $a_n \;\; z \;\; a_{n+1}|_{n=\infty}$, $\frac{1}{n^{\frac{1}{2}}+j} \;\; z \;\; \frac{1}{(n+1)^{\frac{1}{2}}-j}|_{n=\infty}$, $(n+1)^{\frac{1}{2}} - j \;\; z \;\; n^{\frac{1}{2}} + j|_{n=\infty}$, $(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}} \;\; z \;\; 2j|_{n=\infty}$*, using the binomial theorem* $(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}} = n^{-\frac{1}{2}}|_{n=\infty} = 0$, $0 \;\; z \;\; 2j|_{n=\infty}$, $0 \;\; z \;\; (-1)^n|_{n=\infty}$, $z = <, >, \ldots$ *then* $(a_n)|_{n=\infty}$ *is not monotonic.*

Example 2.7. *An example of when not to apply infinitary argument simplification* $a+b=a$.

Test if the sequence $a_n = \frac{1}{n^{\frac{1}{2}}+(-1)^n}|_{n=\infty}$ *is monotonic.* $n^{\frac{1}{2}} \succ (-1)^n|_{n=\infty}$*, if we say* $a_n = \frac{1}{n^{\frac{1}{2}}+(-1)^n}|_{n=\infty} = \frac{1}{n^{\frac{1}{2}}}|_{n=\infty}$*, the sequence* $(\frac{1}{n^{\frac{1}{2}}})|_{n=\infty}$ *is easily shown to be monotonic. However example 2.6 shows the sequence is not monotonic. Even though the magnitude is infinitely small compared with the other function, the property of monoticity by adding* $(-1)^n$ *was changed.*

3 Convergence

We now move on to a more theoretical use of at-a-point definition. The Cauchy convergence, Cauchy sequence and limit, can be defined at infinity instead of both a finite and infinite perspective definition. Given that a number system exists at infinity and zero, this is more than justified, and since the definitions may be more primitive, may subsume the standard definitions.

The problem with both the limit existence and the Cauchy sequence convergence is that they both define convergence to the point to include the point of convergence in the same space. While this is incredibly useful it is a subset of a more general convergence.

For example Hille [5, p.17, Theorem 1.3.1] already assumes complex numbers, $z_k \in \mathbb{C}$ and defines Cauchy convergence

$$|z_m - z_n| < \epsilon \text{ for } m, n > M(\epsilon).$$

Then provides the following corollary from the definition, expressed as a limit. [5, p.71 (4.1.12)]

$$\lim_{m,n\to\infty} ||z_m - z_n|| = 0$$

However turning this around, the corollary is the more primitive operation, that being their difference is zero. Make this the definition, defining a sequence as converging at infinity (Definition 3.3) and derive the Cauchy sequence (Definition 3.4).

Definition 3.1. *Convergence is the negation of divergence.*

Definition 3.2. *A sequence with singularities diverges.*

Theorem 3.1. *A sequence without singularities before infinity can only diverge at infinity.*

Proof. Every finite sequence converges because the number of terms is finite and the terms are not singularities. □

Corollary 3.1. *For a sequence without finite singularities, convergence or divergence is determined at infinity.*

Proof. Since convergence is defined as the negation of divergence (Definition 3.1), and divergence can only happen at infinity (Theorem 3.1), then both convergence and divergence can only be completely determined at infinity. Note: this does not contradict finite sums converging, as at infinity their sum is 0. □

Definition 3.3. *A sequence (a_n) converges at infinity if given $\{m, n\} \in \mathbb{N}_\infty$:*

$$a_m - a_n|_{\forall m,n=\infty} \simeq 0$$

Definition 3.4. *A Cauchy sequence converges if the sequence $(x_n)|_{n=\infty}$ converges (Definition 3.3) and the finite and infinite numbers are the same type of number. $n < \infty$ then $x_n \in W$ and $x_m|_{m=\infty} \in W$*

In Definition 3.3 of sequence convergence, the number types can be different as Φ is composed of the infinireals. Cauchy convergence Definition 3.4 derives from defining convergence Definition 3.3.

Similarly the limit definition changes. Define evaluation at a point, then define the limit as the evaluation at the point and in the same space. Definition 4.5 the limit derives from Definition 4.4 evaluation at a point.

The idea of a sequence not being convergent because it is not 'complete' is a narrow view.

Consider the computation of two integer sequences a_n an b_n where their ratio for finite values is always rational, but what they are approximating is not. The Cauchy sequence convergence does not explain the differing number types, only convergence.

The limit fails to be defined when a ratio between these two sequences is considered. This is a simple operation. The best answer that can explain the calculation is that at infinity the ratio is promoted, a rational approximation at infinity can be promoted to a transcendental number.

Conjecture 3.1. *There exists ratios of infinite integers of the form $\frac{\mathbb{N}_\infty}{\mathbb{N}_\infty}$ which can be transfered to real numbers.*

Since all irrationals including transcendental numbers are calculated by integer sequences, such a restriction on the ratio of two integer sequences not converging is absurd. Such sequences do converge at infinity.

If $\frac{a_n}{b_n}|_{n=\infty}$ converges at a point not in the limit.

If $\{a_n, b_n\} \in \mathbb{N}$ are integers, $\frac{a_n}{b_n} \in \mathbb{Q}$, but $\frac{a_n}{b_n} \to \mathbb{Q}'$ then $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist. However $\frac{a_n}{b_n}|_{n=\infty}$ does not have this restriction. $\frac{a_n}{b_n}|_{n=\infty} \in \frac{\mathbb{J}_\infty}{\mathbb{J}_\infty} \in \mathbb{Q}_\infty$, but $\mathbb{Q}_\infty$ for any non-rational number approximation is promoted to $\mathbb{Q}'$, if the approximation exists in $\mathbb{R}$.

These rational approximations are common. All calculations of numbers in $\mathbb{R}$ are reduced to integer calculations. However such calculations need to be explained in a higher dimension number, at least with $\mathbb{R}_\infty$ because infinite integers are involved.

Example 3.1. *Construct an integer sequence to approximate $\sqrt{3}$. Hence, we consider $\sqrt{3}$ as the ratio of two infinite integers at infinity. $(x-1)^2 = 3$ has a solution $x = 1 + \sqrt{3}$. $x^2 - 2x + 1 = 3$, $x = 2 + \frac{2}{x}$. Develop an iterative scheme. $x_{n+1} = 2 + \frac{2}{x_n}$. Assume an integer solution, $x_n = \frac{a_n}{b_n}$. $\frac{a_{n+1}}{b_{n+1}} = 2 + \frac{2b_n}{a_n}$, $\frac{a_{n+1}}{b_{n+1}} = \frac{2a_n + 2b_n}{a_n}$. Let $b_{n+1} = a_n$ then $a_{n+1} = 2a_n + 2a_{n-1}$. For two initial values, $a_0 = 1$, $a_1 = 1$, $a_2 = 2a_1 + 2a_0 = 2 \cdot 1 + 2 \cdot 1 = 4$, the sequence* *generated is* $(1, 1, 4, 10, 28, 76, 208, 568, 1552, \ldots)$. $\sqrt{3} = \frac{a_n}{b_n} - 1 = \frac{a_n + 2a_{n-1}}{a_n}|_{n=\infty}$.

The way around this difficulty by saying it is not important through definition is problematic. Simply promoting the two numbers being divided to the same number system (as demonstrated by Hille [5, p.17, Theorem 1.3.1]), so that by definition and only by definition they are the same number type; and therefore the ratio converges in the same space, is incomplete.

The alternative Definition 3.3 define the same concept more generally. Admittedly the problem of 'promotion' is not explained, but is acknowledged.

While this may be a controversial finding, it suggests either that notions of convergence have not been entirely settled, or that they are incomplete. Especially for the most basic operations.

In summary, there are two problems with the Cauchy sequence. It is derived from a more general convergence, and is better explained in a space at infinity, with infinite integers.

4 Limits and continuity

The standard epsilon definition of a limit Definition 4.1 can be improved by explicitly defining $\{\epsilon, \delta\} \in \Phi^+$ as, by the conditions, these numbers become infinitesimals. Therefore this is an implicit infinitesimal definition.

Definition 4.1. *The symbol* $\lim_{x \to p} f(x) = A$ *means that for every* $\epsilon > 0$*, there is a* $\delta > 0$ *such that*

$$|f(x) - A| < \epsilon \text{ whenever } 0 < |x - p| < \delta.$$

If we consider a limit definition [1, p.129] given by Apostol, we can generalize the definition in $*G$ to include infinitesimals, thereby making the definition explicit. A statement with infinitesimals, let $\epsilon \in +\Phi$: $|f(x) - A| < \epsilon$ can be equivalently expressed: $f(x) - A \in \Phi \cup \{0\}$.

Definition 4.2. *The symbol* $\lim_{x \to p} f(x) = A$ *in* $*G$ *means that when* $x - p \in \Phi$ *then* $f(x) - A \in \Phi \cup \{0\}$.

Example 4.1. $\lim_{n \to \infty} \frac{n^3 + \frac{1}{n}}{4n^3} = \lim_{n \to \infty} \frac{3n^2 - n^{-2}}{12n^2} = \lim_{n \to \infty} \frac{6n + 2n^{-3}}{24n} = \lim_{n \to \infty} \frac{6 - 6n^{-4}}{24} = \frac{1}{4} - \frac{1}{4n}|_{n=\infty}$

The limit $\lim_{n \to \infty} \frac{a_n}{b_n}$ in $\mathbb{R}$ implicitly applies a transfer $*G \mapsto \mathbb{R}$ [8, Part 4]. A limit in $*G$ (Definition 4.2) by default does not do a transfer, but this is easily done.

Proposition 4.1. *If a limit exists in* $\mathbb{R}$ *then* $*G \mapsto \mathbb{R}$*: in* $*G$, $\mathrm{st}(\lim_{x\to p} f(x)) = \lim_{x\to p} f(x))$ *in* $\mathbb{R}$.

Proof. $\mathbb{R}$ is a subset of $*G$. Then a transfer must exist, since limits are actually calculated in $*G$. During the transfer, infinitesimals are mapped to zero, $\Phi \mapsto 0$. □

Limits and continuity are tied together in $\mathbb{R}$, however we will see that this is often not the case in $*G$. For example, we can have a discontinuous staircase function in $\mathbb{R}$ which is continuous in $*G$.

However, we can similarly define continuity in $*G$ with limits. Since this is before the transfer principle $*G \mapsto \mathbb{R}$ is applied then there is no paradox. We believe that the definition of continuity via limits has the advantage of an 'at-a-point' perspective.

Consider the 'principle of variation', which for a continuous variable is the 'law of adequality' [13, p.5]: $d(f(x)) = f(x+\delta) - f(x)$ leads to the derivative, as a ratio of infinitesimals. $df(x) = f(x+dx) - f(x)$, $\frac{df(x)}{dx} = \frac{f(x+dx)-f(x)}{dx}|_{dx=0}$

However, the principle of variation is also applicable to discrete change, where $dn = (n+1) - n = 1$ is a change in integers, which we interpret to derive a derivative of a sequence [10].

Continuity can be defined either by the principle of variation or the limit. Continuity can be expressed as a variation; taking two points infinitesimally close, and their difference is an infinitesimal.

Definition 4.3. *A function* $f : *G \to *G$ *is continuous at* x*;* $f(x), x, y \in *G$*;* $\delta x, \delta y \in \Phi$*;*

$$y = f(x)$$

$$y + \delta y = f(x + \delta x)$$

Definition 4.4. *A function* $f : *G \to *G$ *is continuous at* $x = a$ *and has been evaluated to* L*:* $f(x)|_{x=a} = L$*,*

$$\text{If } \forall x : x \simeq a \text{ then } f(x) \simeq L.$$

$x \simeq a$ is equivalent to $x + \delta = a$ when $\delta \in \Phi$ or $x = a$ [7, Part 2 Definition 3.7].

Lemma 4.1. *Definition 4.3 implies Definition 4.4.*

Proof. Consider Definition 4.3. $y + \delta y = f(x+\delta x)|_{x=a}$, $y \simeq f(x+\delta x)|_{x=a}$, let $L = f(x)|_{x=a}$, $L \simeq f(x+\delta x)|_{x=a}$, $L \simeq f(x)|_{x\simeq a}$ □

A definition of a limit, less general than a definition of evaluation at a point is given.

Definition 4.5. *A function $f : *G \to *G$ is a limit at $x = a$ if $f(x)$ is continuous at $x = a$ and is the same number type W.*

$$\textit{If } f(x)|_{x=a} = L \textit{ and } \{f(x),\, L\} \in W \textit{ then } f(x)|_{x=a} \textit{ is a limit.}$$

The symbol $\lim\limits_{x\to a} f(x) = L$ in $*G$ means that when $x \simeq a$ then $f(x) \simeq L$.

Considering the larger picture between the two-tiered number systems. What is being claimed is that providing a finite and infinite perspective definition does not describe well what is happening, particularly when "at infinity" simplifies the explanation. These ideas of defining at infinity extend into many other definitions.

As there appear to be different kinds of arithmetics and convergence as governed by what happens at infinity, defining convergence in general at infinity makes more sense.

Example 4.2. *Define $x > 1$: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Then we can have an infinite sum with positive terms that has a negative solution. Let $\frac{1}{1-x} = -w$, solve for x, $x = \frac{1+w}{w}$. Case $w = 2$, $x = \frac{3}{2}$, $\sum_{k=0}^{\infty}(\frac{3}{2})^k = -2$.*

Not only has a positive sum of terms become negative, we needed an infinity of terms to interpret the sum, for the sum to have meaning.

This example is relevant because our applications following this series [9, Convergence sums] define convergence at infinity, as does Robinson's non-standard analysis (here infinity defined as not finite is interpreted as successive orders of numerical infinities ω).

5 Epsilon-delta proof

We again look at the limit in the guise of the epsilon-delta proof. [4] comments on the generalization in multidimensional space with the norm and open balls.

For complex proofs, NSA has been successful as an alternative to Epsilon-Delta management, and other proofs solving in the higher number system and transferring the results back into the reals. We would expect our calculus to also be useful in proving propositions and theorems, with similar purpose to NSA but in another way.

An epsilon-delta definition proof, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$, in a minimalistic sense is not a finite inequality, but an inequality at infinity with infinitesimals, as we can derive the finite inequality. By the transfer principle, project the statement from $*G$ into $\mathbb{R}$. E.g. $\delta_1 \in \Phi$; $\delta_2 \in \mathbb{R}^+$; $(*G, |x - x_0|\delta_1) \mapsto (\mathbb{R}, |x - x_0|\delta_2)$

An abstraction, by removing the inequality relation, expressing the relation as a variable which is infinitesimal, hence a more direct reasoning.

Definition 5.1. *The Epsilon-Delta Proof with* Φ

$$\textit{If } x - x_0 = \Phi \textit{ then } f(x) - f(x_0) = \Phi, \, n = \infty.$$

Example 5.1. *[3, Epsilon-Delta Proof]* $f(x) = ax + b$*;* $a, b \in \mathbb{R}$*;* $a \neq 0$*. Show* $f(x)$ *is continuous.*

$$\begin{aligned} x - x_0 &= \Phi \\ f(x) - f(x_0) &= (ax + b) - (ax_0 + b) \\ &= a(x - x_0) = \Phi \qquad \text{(as } a\Phi = \Phi\text{)} \end{aligned}$$

An Epsilon-Delta definition and proof with a similar structure [4] could be given where the real numbers are replaced by $*G$. For example it may not be enough that the numbers are infinitesimals (Definition 5.1), but we may require the infinitesimals to be continually approaching 0. (See Proposition 6.2)

$$\delta_n \to 0 \text{ replaced with } \delta_n \succ \delta_{n+1}$$

Definition 5.2. *The Epsilon-Delta Proof in* $*G$

$$\delta_n = x - x_0 \in \Phi; \;\; \epsilon_n = f(x) - f(x_0) \in \Phi; \;\; n = \infty$$

$$\textit{If } \delta_n \succ \delta_{n+1} \textit{ then } \epsilon_n \succ \epsilon_{n+1}$$

6 A two-tiered calculus

We can work in $*G$ and project back or transfer to $\mathbb{R}$ or $\overline{\mathbb{R}}$, or $*G$. The overall reason for doing this was a separation of the finite and infinite domains, thereby separating and isolating the problem.

With this separation, we will seek further mathematics. The order of one variable reaching infinity before another is common as a partial differential equation. If the variables are described by sequences, a partial derivative can be described as one variable reaching infinity before another.

This raises the possibility of combinations of variables reaching infinity in different orders. Uniform convergence, absolute sum convergence and other concepts, which for example highlight when the problem is independent of the order, could be investigated with orderings.

Turning the problem around, if the ordering does not matter, we only need to find the solution of one ordering to determine the whole solution. (for example sum rearrangements at infinity [11])

We introduce the following sequence definitions as a consequence of a two-tiered calculus. It is possible for a sequence to plateau in $*G$ and project back to a convergent sequence in $\mathbb{R}$. Similarly a divergent sequence could plateau in $*G$ and diverge in $\overline{\mathbb{R}}$. We need to be able to describe arbitrarily converging and diverging sequences to guarantee certain properties and avoid the plateau. Hence, additional requirements are needed to manage the sequences.

Definition 6.1. *We say $x_n \to 0$ then $x_n \in \Phi$ and is decreasing in magnitude: $|x_{n+1}| \leq |x_n|$.*

Definition 6.2. *We say $x_n \to \infty$ then $x_n \in \Phi^{-1}$ and is increasing in magnitude: $|x_{n+1}| \geq |x_n|$.*

Since a variable may be expressed as a point, the sequences described can be extended to the continuous variable. An adaptable notation, given that we may need different sequences for particular problems and theory.

For example, $x \to \infty$, (x) indefinitely increases and is positive monotonic, $f(x)|_{x=\infty} = \ldots$

The other type of sequences in general use are a partition. For example, for all $x > x_0$.

Definition 6.3. *In context, a variable $x \to 0$ can be described at zero by $x \in \Phi$ or Definition 6.1 or Definition 6.6 or other as $|_{x=0}$.*

Definition 6.4. *In context, a variable $x \to \infty$ can described at infinity by $x \in \Phi^{-1}$ or Definition 6.2 or Definition 6.7 or other as $|_{x=\infty}$.*

Definition 6.5. *A 'subsequence' is a sequence formed from a given sequence by deleting elements without changing the relative position of the elements.*

Definition 6.6. *We say a sequence is 'indefinitely decreasing' in magnitude. $x_n \to 0$; n, $n_2 \in \mathbb{N}_\infty$; there exists $n_2 : n_2 > n$ and $x_{n_2} \prec x_n$.*

Definition 6.7. *We say a sequence is 'indefinitely increasing' in magnitude. $x_n \to \infty$; n, $n_2 \in \mathbb{N}_\infty$; there exists $n_2 : n_2 > n$ and $x_{n_2} \succ x_n$.*

Proposition 6.1. *If $x_n \to 0$ is indefinitely decreasing there exists a subsequence (ν_n) : $\nu_{n+1} \prec \nu_n|_{n=\infty}$ and $\nu_n \to 0$*

Proof. Since x_n is decreasing in magnitude, we can always choose a subsequent much-less-than term. □

We use infinity arguments with order in the proof of Proposition 6.2. Normally we would send $h \to 0$ before $\delta \to 0$. However, if the solution is independent of the infinity, consider $\delta \to 0$ before $h \to 0$. Then we reason that the derivative must be an infinitesimal.

Proposition 6.2. *If $\delta_n \to 0$ is indefinitely decreasing and strictly positive monotonic decreasing then*

$$D\delta_n|_{n=\infty} \in -\Phi$$

Proof. $h \in +\Phi$; Strictly monotonic decreasing δ_n then $\delta_{n+1} < \delta_n$, $\delta_{n+1} - \delta_n < 0$, $\frac{\delta_{n+1}-\delta_n}{h} < 0$, $D\delta_n = \frac{\delta_{n+1}-\delta_n}{h}|_{h=0}$ is negative.

Consider the infinite state where $\delta_n \to 0$ before $h \to 0$. Since δ_n can be made arbitrarily small, then $\delta_{n+1} - \delta_n \prec h$, $\delta_{n+1} - \delta_n \in \Phi$, $\frac{\delta_{n+1}-\delta_n}{h}|_{h=0} \in \Phi$, $D\delta_n \in \Phi$. □

Lemma 6.1. *If $f(x)$ and $g(x)$ are positive monotonic functions, with relation $z : z \in \{<, \leq, >, \geq\}$, $f(x)\ z\ g(x)$ for all x in a given domain, such a relation can be reformed to a positive inequality: $\phi > 0$ or $\phi \geq 0$*

Proof. A less than relation can always be expressed as a greater than relation by swapping the arguments sides. If $f < g$ then $g > f$. If $f \leq g$ then $g \geq f$. If $f(x) > g(x)$ then $f(x) - g(x) > 0$. If $f(x) \geq g(x)$ then $f(x) - g(x) \geq 0$. □

Proposition 6.3. *$z \in \{>, \geq\}$; If $f(x)\ z\ 0$ and $f(x)$ is monotonically increasing then $Df(x)\ z\ 0$ where $f(x)$ is not constant.*

Proof. $h \in \Phi^+$, $f(x+h)\ z\ f(x)$, $f(x+h) - f(x)\ z\ 0$, $\frac{f(x+h)-f(x)}{h}\ z\ 0$, $\frac{f(x+h)-f(x)}{h}|_{h=0}\ z\ 0$, $Df(x)\ z\ 0$ □

Proposition 6.4. *$z \in \{>, \geq\}$; If $f(x)$ and $g(x)$ are positive monotonic functions: $f(x)\ z\ g(x)$ over a positive domain then $Df(x)\ z\ Dg(x)$, where $f(x) - g(x)$ is monotonically increasing.*

Proof. Reorganise the relation to be positive, Lemma 6.1. Apply Proposition 6.3. □

Proposition 6.5. *If $f \succ 0$ and f is a positive monotonic increasing function then ignoring integration constants and integrating in a positive interval, $\int f \succ 0$.*

Proof. Since integrating an infinitesimal or infinity does not result in 0, then integral must have a much-greater-than relationship with 0. □

Theorem 6.1. *Let f and g be positive monotonic functions: $f \succ g$. If integrated over a positive interval ignoring integration constants then $\int f \succ \int g$.*

Proof. $f \succ g$, $f - g \succ 0$, apply Proposition 6.5. □

This infinitesimal and infinitary analysis is more suited to a functional approach and does not explicitly use sets, compared with NSA. Hence the complexity of use would likely make this calculus more accessible. We have found, empirically, different solutions to problems and in many cases simpler reasoning than with standard calculus, such as found in [2]. That is, we have constructed a new calculus of sum convergence [9].

The following propositions define continuity and calculus in $*G$.

Proposition 6.6. *A function $f : *G \mapsto *G$ is uniformly continuous if $f(x) \simeq f(y)$ when $x, y \in *G$ and $x \simeq y$.*

Proposition 6.7. *A function $f : *G \mapsto *G$ is differentiable at $x \in *G$ if there exists $b \in *G$:*

$$\frac{f(x) - f(a)}{x - a} \simeq b \text{ when } x \simeq a$$

Many of the classical results can be proved in $*G$. Assuming the Taylor series in $*G$ with arbitrary truncation, that is a well-behaved function, prove Newton's method.

Theorem 6.2. *When $f(x_{n+1}) \prec f(x_n)|_{n=\infty}$ and $x_{n+1} \simeq x_n|_{n=\infty}$ then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}|_{n=\infty}$*

Proof. $h, f(x_n) \in \Phi;\ x_n,\ f^{(w)}(x_n) \in *G$

$$\begin{aligned}
f(x_n + h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!}f'^2(x_n) + \ldots|_{n=\infty}|_{h=0} & \qquad \text{(Assume continuity)} \\
f(x_n + h) = f(x_n) + hf'(x_n)|_{n=\infty}|_{h=0} & \qquad \text{(Choose } h = x_{n+1} - x_n\text{)} \\
f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)f'(x_n)|_{n=\infty} & \qquad \text{(Non-reversible arithmetic)} \\
(f(x_n) - f(x_{n+1}) = f(x_n)|_{n=\infty} & \text{ as } f(x_n) \succ f(x_{n+1})|_{n=\infty}) \\
0 = f(x_n) + (x_{n+1} - x_n)f'(x_n)|_{n=\infty} & \qquad \text{(Solve for } x_{n+1}\text{)} \\
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}|_{n=\infty} & \qquad \text{(Transfer principle } *G \mapsto \mathbb{R}\text{)}
\end{aligned}$$

□

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