Power series convergence sums

Chelton D. Evans and William K. Pattinson

Abstract

Calculating the radius and interval of convergence with power series at infinity. By using non-reversible arithmetic, either by factoring, comparison or application of the logarithmic magnitude relation, convergence or divergence may be determined. We interpret uniform convergence with a convergence sum.

1 Introduction

While convergence testing for power series is straight forward, we mirror the tests with convergence sums [2]. The theory is general as it calculates in a different way the radius of convergence, intervals and theorems.

In power series convergence sums we find application of non-reversible multiplication, Theorem 1.3.

'Convergence sums' theory extensively uses power series at infinity. By threading a continuous curve through a monotonic sequence and interchanging between the continuous form and the sequence. This one idea leads to the integral test in both directions [2, Theorem 8.1]. We also use power series at infinity to describe a derivative of a sequence [3].

The power series representation at infinity is interesting because historically the power series has played a role in applications, continuity, uniform convergence, limit interchanges, partial differentiation, solution validity, and many other matters.

For example, we can represent a trigonometric function at infinity. As power series are analytic, the property is applicable over the infinite domain too.

With power series, a generalization of the geometric series is extensively used for function representation and approximation. Fourier series, partial differential equations and other applied topics also appear in number theory of partitions with generating functions.

It happens that simplifying a sum at infinity, by reasoning of magnitude of $\sum a_n x^n|_{n=\infty}$, is a different experience, and is another way of determining convergence. The reasoning is often algebraic, arguing with magnitudes and factoring.

A power series is a geometric series; we know that $1+x+x^2+\dots$ is convergent when $|x|<1$. The convergence can also be derived at infinity by comparing against a convergent p-series.

Intuitively a fraction less than one multiplied by another fraction less than one infinitely many times, is infinitely small.

Theorem 1.1. *If* $|x| < 1$ *then* $\sum x^n|_{n=\infty} = 0$ *converges, and has radius of convergence* $r = 1$.

Proof. Comparing against the convergent p-series. $\sum x^n \; z \; \sum \frac{1}{n^\alpha}|_{n=\infty}$ converges when $\alpha > 1$. $x = 0$ is a solution. Solving for x, $x^n \; z \; \frac{1}{n^\alpha}|_{n=\infty}$, $x^n n^\alpha \; z \; 1|_{n=\infty}$, $\ln(x^n n^\alpha) \; (\ln z) \; \ln 1|_{n=\infty}$, $n \ln x + \alpha \ln n \; (\ln z) \; 0|_{n=\infty}$, $n \ln x \; (\ln z) \; 0|_{n=\infty}$, $\ln x^n \; (\ln z) \; 0|_{n=\infty}$, $x^n \; z \; e^0|_{n=\infty}$, $x^n \; z \; 1|_{n=\infty}$, $z = <$ then $|x| < 1$. (Solving for $z = \leq$ leads to $x = 1$ which in the sum diverges hence this case is excluded). $\square$

Proposition 1.1. $\sum x^n|_{n=\infty}$ *diverges when* $|x| > 1$.

Proof. For convergence, $\sum a_n|_{n=\infty}$ requires $a_n|_{n=\infty} = 0$. When $|x| > 1$ then $x^n|_{n=\infty} \neq 0$. $\square$

Definition 1.1. *The radius of convergence is absolute convergence of* $\sum a_n x^n|_{n=\infty}$*, solving Theorem 1.2,* $|a_n^{\frac{1}{n}} r||_{n=\infty} < 1$ *about the origin.* x *is absolutely convergent about the origin within* $(-r, r)$.

Definition 1.2. *The interval of convergence includes the radius of convergence, and the end points which need to be tested separately.*

A power series convergence test, Theorem 1.2 transforms the series at infinity to evaluate the radius of convergence, a distance about which the sum converges. Unimportant terms in the sums product, which are not required to determine convergence or divergence, become transients. Applying non-reversible arithmetic, these variables and constants can be removed.

Since a power series about a point can be translated to the origin, the calculation of the radius of convergence and the interval of convergence may be applied to infinite series of the form $\sum_{k=1}^{\infty} a_k (x-c)^k$.

While solving absolute convergence finds the general interval, the end points of the interval need to be tested separately for the interval of convergence [4, Properties of functions represented by power series, p.431].

Example 1.1. *Determine the radius of convergence of* $\sum n(\frac{x}{2})^n|_{n=\infty} = 0$.

Transform the power series by bringing the n term into the product and simplifying.

$$\begin{aligned}
&\sum n(\frac{x}{2})^n|_{n=\infty} \\
&= \sum (n^{\frac{1}{n}}\frac{x}{2})^n|_{n=\infty} && (n^{\frac{1}{n}}|_{n=\infty} = 1) \\
&= \sum (\frac{x}{2})^n|_{n=\infty} = 0 && \text{(for convergence)} \\
&|\frac{x}{2}| < 1, \quad |x| < 2 && \text{(Theorem 1.1)} \\
&\textit{radius of convergence } r = 2
\end{aligned}$$

Theorem 1.2. *Transform the sum $\sum a_n x^n = \sum (b_n x)^n|_{n=\infty}$. For convergence $\sum (b_n x)^n|_{n=\infty} = 0$, solving for $|b_n x| < 1$, the radius of convergence $r = \frac{1}{|b_n|}|_{n=\infty}$ If r exists $x = (-r, r)$ converges. For the interval of convergence the end points need to be tested.*

Proof. Let $b_n = a_n^{\frac{1}{n}}$, $\sum a_n x^n|_{n=\infty} = \sum (a_n^{\frac{1}{n}} x)^n|_{n=\infty} = 0$ by Theorem 1.1 when $|a_n^{\frac{1}{n}} x||_{n=\infty} < 1$

The end points $b_n x|_{n=\infty} = 1$ evaluated separately using the Criteria E3 [2]. □

A primary technique in simplifying products is to apply the inverse log and exponential functions, then non-reversible arithmetic.

$$ab = e^{\ln(ab)} = e^{\ln a + \ln b} = e^{\ln a} \text{ when } \ln a \succ \ln b$$

Example 1.2. $\sum n(\frac{x}{2})^n|_{n=\infty} = \sum e^{\ln n + n \ln \frac{x}{2}}|_{n=\infty} = \sum e^{n \ln \frac{x}{2}}|_{n=\infty} = \sum (\frac{x}{2})^n|_{n=\infty}$, *$|\frac{x}{2}| < 1$, $|x| < 2$, radius of convergence $r = 2$.*

By application of 'logarithmic magnitude' we can directly simplify the product. Given positive functions and relation $f \; z \; g$, when $\ln f \prec \ln g$ then by definition we say $f \prec\!\prec g$. We then apply a non-reversible product theorem, if $a \prec\!\prec b$ then $ab = b$ [1, Part 5]. For power series, this allows the simplification of product terms.

Theorem 1.3. *For positive a and b, if $a \succ\!\succ b|_{n=\infty}$ then $\sum ab = \sum a|_{n=\infty}$*

Proof. $(\sum ab = \sum e^{\ln(ab)} = \sum e^{\ln a + \ln b} = \sum e^{\ln a} = \sum a)|_{n=\infty}$ since $a \succ\!\succ b$ means $\ln a \succ \ln b$. □

Example 1.3. *As $\left(\frac{x}{2}\right)^n \succ\!\succ n|_{n=\infty}$, $\sum n(\frac{x}{2})^n|_{n=\infty} = \sum (\frac{x}{2})^n|_{n=\infty}$.*

To establish the logarithmic magnitude relationship, solve the comparison. $n \; z \; (\frac{x}{2})^n|_{n=\infty}$, $\ln n \; (\ln z) \; n \ln \frac{x}{2}|_{n=\infty}$, $(\ln z) = \prec$ then by definition $z = \prec\!\prec$.

Smaller infinities in the product/division may be simplified, thereby making the sum easier to solve for convergence. It is not always easy to identify the dominant term. From [1, Part 5], products can be converted to sums by taking the logarithm, and solving the relation.

Example 1.4. *Show* $\sum cn^p x^n = \sum x^n|_{n=\infty}$. $\sum cn^p x^n|_{n=\infty} = \sum e^{\ln(cn^p x^n)}|_{n=\infty} = \sum e^{\ln c + p\ln n + n\ln x}|_{n=\infty}$ $= \sum e^{n\ln x}|_{n=\infty} = \sum x^n|_{n=\infty}$, *because* $n\ln x \succ p\ln n + \ln c|_{n=\infty}$.

Proposition 1.2. *When p and c are constant then* $\sum ca_n n^p x^n = \sum a_n x^n|_{n=\infty}$

Proof. By similar argument to Example 1.4. $\sum ca_n n^p x^n|_{n=\infty} = \sum e^{\ln(ca_n n^p x^n)}|_{n=\infty}$. $\ln(ca_n n^p x^n)|_{n=\infty}$ $= \ln c + \ln a_n + p\ln n + n\ln x|_{n=\infty} = \ln a_n + n\ln x|_{n=\infty} = \ln a_n x^n|_{n=\infty}$, as $n\ln x \succ \ln c|_{n=\infty}$ and $n\ln x \succ p\ln n|_{n=\infty}$. Reversing the exponential and logarithmic operations, $\sum e^{\ln(ca_n n^p x^n)}|_{n=\infty}$ $= \sum e^{\ln a_n x^n}|_{n=\infty} = \sum a_n x^n|_{n=\infty}$ □

Example 1.5. *Find the radius and interval of convergence for* $\sum_{n=1}^{\infty} \frac{(x-5)^n}{(n+2)3^n}$.

$$\begin{aligned}
\sum \frac{(x-5)^n}{(n+2)3^n}|_{n=\infty} & & \text{(Need only consider the point at infinity)}\\
= \sum \frac{(x-5)^n}{3^n}|_{n=\infty} & & (\text{as } 3^n \succ\succ n+2|_{n=\infty})\\
= \sum (\frac{x-5}{3})^n|_{n=\infty} = 0 & & \text{(for convergence)}\\
|\frac{x-5}{3}| < 1 & & \\
|x-5| < 3 & & (\text{radius } r = 3)\\
2 < x < 8 & &
\end{aligned}$$

Test the intervals end points. Case $x = 2$, $\sum(\frac{2-5}{3})^n|_{n=\infty} = \sum(-1)^n|_{n=\infty} = \infty$ *diverges. Case* $x = 8$, $\sum(\frac{8-5}{3})^n|_{n=\infty} = \sum 1|_{n=\infty} = \infty$ *diverges. Radius of convergence is* 3*, interval of convergence is* $x = (2, 8)$.

Basic arithmetic is used in solving these problems, $a^{bc} = (a^b)^c = (a^c)^b$. Where the raised powers are interchanged. The best way is to evaluate the triple from the base upwards. E.g. $2^{15} = 2^{3\cdot 5} = (2^3)^5 = (2^5)^3 = 32768$. If we write without correct bracketing, the order can be ambiguous, evaluating from the top down, $2^{3^5} = 2^{(3^5)} = 2^{243}$, $2^{5^3} = 2^{(5^3)} = 2^{125}$.

Example 1.6. *[4, 11.7.10]. Determine the radius of convergence for* $\sum_{n=1}^{\infty} 3^{n^{\frac{1}{2}}} \frac{z^n}{n}$. $\sum 3^{n^{\frac{1}{2}}} \frac{z^n}{n}|_{n=\infty}$ $= \sum (3^{\frac{1}{2}})^n \frac{z^n}{n}|_{n=\infty} = \sum (3^{\frac{1}{2}} z)^n \frac{1}{n}|_{n=\infty} = \sum (3^{\frac{1}{2}} z)^n|_{n=\infty}$ *when* $|3^{\frac{1}{2}} z| < 1|_{n=\infty}$, $|z| < \frac{1}{3^{\frac{1}{2}}}$, $r = \frac{1}{3^{\frac{1}{2}}}$

Example 1.7. *[4, 11.7.12]. Determine the radius of convergence for* $\sum_{n=1}^{\infty}(1+\frac{1}{n})^{n^2} z^n$. $\sum(1+\frac{1}{n})^{n^2} z^n|_{n=\infty} = \sum((\frac{n+1}{n})^n z)^n|_{n=\infty} = \sum(ez)^n|_{n=\infty} = 0$ *when* $|ez| < 1$, $r = \frac{1}{e}$

By Stirling's formula we know $(n!)^{\frac{1}{n}}|_{n=\infty} = e^{-1}n$. This can be used in determining radius of convergence with factorial expressions.

Example 1.8. *[5, Example 5, p.795]. Determine the radius of convergence.* $\sum \frac{(2n)!}{(n!)^2} y^n|_{n=\infty}$ $= \sum (\frac{((2n)!)^{\frac{1}{n}}}{(n!)^{\frac{2}{n}}} y)^n|_{n=\infty} = \sum (\frac{(((2n)!)^{\frac{1}{2n}})^2}{((n!)^{\frac{1}{n}})^2} y)^n|_{n=\infty} = \sum (\frac{(2n)^2}{n^2} y)^n|_{n=\infty} = \sum (4y)^n|_{n=\infty} = 0$ *when* $|4y| < 1$, $|y| < \frac{1}{4}$, $r = \frac{1}{4}$

Example 1.9. *[6, 3.3.7.c, p.98], given* $\sum a_n x^n|_{n=\infty}$ *has radius of convergence* R, $R \prec \infty$. *Solve the radius of convergence* r *for* $\sum \frac{n^n}{n!} a_n x^n|_{n=\infty}$.

Solving for R. $\sum a_n x^n|_{n=\infty} = \sum (a_n^{\frac{1}{n}} x)^n|_{n=\infty} = 0$ *when* $|a_n^{\frac{1}{n}} x| < 1|_{n=\infty}$, $|x| < \frac{1}{|a_n^{\frac{1}{n}}|}|_{n=\infty}$, $R = \frac{1}{|a_n^{\frac{1}{n}}|}|_{n=\infty}$. $\sum \frac{n^n}{n!} a_n x^n|_{n=\infty} = \sum (\frac{n}{(n!)^{\frac{1}{n}}} a_n^{\frac{1}{n}} x)^n|_{n=\infty} = \sum (e a_n^{\frac{1}{n}} x)^n|_{n=\infty} = 0$ *when* $|e a_n^{\frac{1}{n}} x| < 1|_{n=\infty}$, $e|x| < R$, $|x| < \frac{R}{e}$ *radius of convergence* $r = \frac{R}{e}$

Example 1.10. *[6, 3.3.7.d, p.98] given* $\sum a_n x^n|_{n=\infty}$ *has radius of convergence* R, *as above* $R = \frac{1}{|a_n^{\frac{1}{n}}|}|_{n=\infty}$, $R \prec \infty$. *Solve the radius of convergence* r *for* $\sum a_n^2 x^n|_{n=\infty}$.

$\sum a_n^2 x^n|_{n=\infty} = \sum (a_n^{\frac{2}{n}} x)^n|_{n=\infty}$, $|a_n^{\frac{2}{n}} x| < 1|_{n=\infty}$, $|x| < |a_n^{-\frac{2}{n}}||_{n=\infty}$, $|x| < R^2$ *then radius of convergence* $r = R^2$.

Considering power series with the Alternating Convergence Theorem(ACT), we can determine convergence with functions that can be represented with these power series, for example log and trigonometric functions.

Example 1.11. *Show* $\ln(1 + x) = \sum_{k=0}^{n} (-1)^k \frac{x^{k+1}}{k+1}|_{n=\infty}$ *converges when radius of convergence* $r = 1$. $\sum (-1)^n \frac{x^{n+1}}{n+1}|_{n=\infty}$ *converges by the ACT (See Theorem [3, Theorem 3.1]) if* $\frac{x^{n+1}}{n+1}|_{n=\infty} = 0$. *Solve* $\frac{x^{n+1}}{n+1}|_{n=\infty} = 0$. *When* $|x| < 1$, $\frac{x^{n+1}}{n+1}|_{n=\infty} = x^{n+1}|_{n=\infty} = 0$ *as* $x^{n+1} \succ\succ n + 1|_{n=\infty}$, *radius of convergence* $r = 1$. *More simply without* $\succ\succ$, $\frac{x^{n+1}}{n+1}|_{n=\infty} = x^{n+1} \cdot \frac{1}{n+1} = 0 \cdot 0 = 0$.

Example 1.12. *Determining the radius of convergence of* $\operatorname{atan} x$ *follows the same reasoning as Example 1.11, in determining convergence consider* $\operatorname{atan} x$ *at infinity,* $\operatorname{atan} x = \sum (-1)^n \frac{x^{2n+1}}{2n+1}|_{n=\infty}$.

For negative x, *factoring out the negative sign leaves the positive case, hence need only consider* $x = (0, \infty)$.

When $x \neq 1$ *we observe that* x^{2n+1} *'log dominates'* $2n + 1$. x^{2n+1} z $2n + 1|_{n=\infty}$, $(2n + 1) \ln x$ $(\ln z)$ $\ln(2n + 1)|_{n=\infty}$, $(2n + 1) \ln x \succ \ln(2n + 1)|_{n=\infty}$, *then* $x^{2n+1} \succ\succ (2n + 1)$. *[* $\ln z = \succ$, $z = e^{\succ} = \succ\succ$ *] Then* $\sum (-1)^n \frac{x^{2n+1}}{2n+1}|_{n=\infty} = \sum (-1)^n x^{2n+1}|_{n=\infty}$.

Case $x > 1$, $\sum (-1)^n x^{2n+1}|_{n=\infty} = \infty$ *diverges. Case* $x = (0, 1)$, $\sum (-1)^n x^{2n+1}|_{n=\infty} = 0$ *converges. Hence the radius of convergence* $r = 1$.

For the interval of convergence, test the end points. Case $x = 1$ *converges by the ACT.*

$\sum \frac{(-1)^n 1^{2n+1}}{2n+1}|_{n=\infty} = \sum \frac{(-1)^n}{2n+1}|_{n=\infty} = 0$ *converges by ACT as* $\frac{1}{2n+1}|_{n=\infty} = 0$. *Case* $x = -1$, *Case* $x = 1$ *converges by the ACT.* $\sum \frac{(-1)^{n+1}}{2n+1}|_{n=\infty} = 0$ *Interval of convergence is* $[-1, 1]$.

Example 1.13. *Determine radius of convergence for* $\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}|_{n=\infty}$. *Determine* $\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}|_{n=\infty}$. *Assume* x *is positive as sign can be factored out. Solve* $\frac{x^{2n+1}}{(2n+1)!}|_{n=\infty} = 0$, $x^{2n+1} \quad z \quad (2n+1)!|_{n=\infty}$, $(2n+1) \ln x \quad (\ln z) \quad \sum_{k=1}^{2n+1} \ln k|_{n=\infty}$, $(2n+1) \ln x \quad (\ln z) \quad \int^{2n+1} \ln n \, dn|_{n=\infty}$, *since* $\int \ln n \, dn = n \ln n|_{n=\infty}$, $(2n+1) \ln x \quad (\ln z) \quad (2n+1) \ln(2n+1)|_{n=\infty}$. $(2n+1) \ln x \prec (2n+1) \ln(2n+1)|_{n=\infty}$, $\ln z = \prec$, $z = e^{\prec} = \prec$, $\frac{x^{2n+1}}{(2n+1)!}|_{n=\infty} = 0$, *by ACT the series is convergent for all* x. *Similarly the same result for* $\cos x$.

In considering properties of power series, we again find parallel theorems with the standard theorems.

Theorem 1.4. *[4, Theorem 11.9, pp.432–433]* $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, *the differentiated series* $\sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$ *also has radius of convergence* r.

The termwise differentiation and integration theorems given in [5, Theorems 3 and 4, pp.643–644], that the power series differentiated and integrated have the same radius of convergence, follows from a finite number power of n being simplified at infinity, demonstrated by $\sum c n^p x^n = \sum x^n|_{n=\infty}$.

Our assumption is that if a convergence sum is an infinireal, it can be integrated and differentiated, by treating each term separately.

Theorem 1.5. *Termwise differentiation and integration of the power series have the same radius of convergence.*

$$\sum a_n x^n|_{n=\infty} = \frac{\partial}{\partial x} \sum a_n x^n|_{n=\infty} = \int \sum a_n x^n \, \partial x|_{n=\infty}$$

Proof. $\frac{\partial}{\partial x} \sum a_n x^n|_{n=\infty} = \sum a_n n x^{n-1}|_{n=\infty} = \sum a_n n x^n|_{n=\infty} = \sum e^{\ln(a_n n x^n)}|_{n=\infty} = \sum e^{\ln a_n + \ln n + n \ln x}|_{n=\infty} = \sum e^{\ln a_n + n \ln x}|_{n=\infty} = \sum a_n x^n|_{n=\infty}$. Similarly $\int \sum a_n x^n \, \partial x|_{n=\infty} = \sum a_n \frac{1}{n+1} x^{n+1}|_{n=\infty} = \sum a_n \frac{1}{n+1} x^n|_{n=\infty} = \sum e^{\ln(a_n \frac{1}{n+1} x^n)}|_{n=\infty} = \sum e^{\ln a_n - \ln(n+1) + n \ln x}|_{n=\infty} = \sum e^{\ln a_n + n \ln x}|_{n=\infty} = \sum a_n x^n|_{n=\infty}$ □

Example 1.14. *Find the radius of convergence of the sum,* $\sum \binom{n}{2} x^n|_{n=\infty} = \sum \frac{n!}{(n-2)!2!} x^n|_{n=\infty} = \sum n(n-1) x^n|_{n=\infty}$ *[5, Example 1, p.799].*

$\sum n(n-1) x^n|_{n=\infty}. = \sum x^n|_{n=\infty}$, *as* $x^n \succ\!\succ n(n-1)$, *or by bringing the* n *terms into the power,* $\sum n(n-1) x^n|_{n=\infty} = \sum (n^{\frac{1}{n}} (n-1)^{\frac{1}{n}} x)^n|_{n=\infty} = \sum x^n|_{n=\infty}$, *radius of convergence* $r = 1$.

By application of Theorem 1.5, partially integrating, $\sum n(n-1) x^n|_{n=\infty} = \int \sum n(n-1) x^n \partial x|_{n=\infty} = \sum n(n-1) \frac{1}{n+1} x^{n+1}|_{n=\infty} = \int \sum (n-1) x^{n+1} \partial x|_{n=\infty} = \sum (n-1) x^{n+2} \frac{1}{n+2}|_{n=\infty} = \sum x^{n+2}|_{n=\infty}$, *radius of convergence* $r = 1$.

For general testing, the ratio test is simpler to implement.

Example 1.15. *Determine the radius of convergence of* $\sum \frac{1\cdot 3\cdot 5 \ldots \cdot (2n-1)}{2\cdot 4\cdot 6 \ldots \cdot (2n)} \frac{x^n}{n}|_{n=\infty}$.

$\sum \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{x^n}{n}|_{n=\infty} = \sum(\prod \frac{2n-1}{2n}|_{n=\infty}) \cdot \frac{x^n}{n}|_{n=\infty} = \sum(\prod \frac{2n}{2n}|_{n=\infty}) \cdot \frac{x^n}{n}|_{n=\infty} = \sum \frac{x^n}{n}|_{n=\infty}$ $= \sum x^n|_{n=\infty} = 0$ *when* $|x| < 1$ *then* $r = 1$

With the ratio test: Let $a_n = \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{x^n}{n}$, $|\frac{a_{n+1}}{a_n}| < 1|_{n=\infty}$, $|\prod_{k=1}^{n+1} \frac{2k-1}{2k} \cdot \frac{x^{n+1}}{n+1} \prod_{k=1}^{n} \frac{2k}{2k-1} \cdot \frac{n}{x^n}| < 1|_{n=\infty}$, $|\frac{2n+1}{2n+2}x| < 1|_{n=\infty}$, $|x| < 1$, $r = 1$

We consider continuity at infinity. By considering The convergence sums, if they differ near a point and at a point, then the sum is discontinuous at a point.

Example 1.16. *[5, Example 2, p.815]. Show* $\sum \frac{x^2}{(1+x^2)^n}|_{n=\infty}$ *is a discontinuous sum.*

Case $x = 0$, $\sum \frac{0^2}{(1+0^2)^n}|_{n=\infty} = \sum \frac{0}{1^n}|_{n=\infty} = \sum 0|_{n=\infty}$.

Case $x \neq 0$, $\sum \frac{x^2}{(1+x^2)^n}|_{n=\infty} = \sum(\frac{x^{\frac{2}{n}}}{(1+x^2)})^n|_{n=\infty} = \sum(\frac{1}{(1+x^2)})^n|_{n=\infty} = \sum \alpha^n|_{n=\infty}$, $\alpha = \frac{1}{1+x^2} \neq 0$.

Comparing the convergence sums, $\sum 0 \;z\; \sum \alpha^n|_{n=\infty}$, $0 \;z\; \alpha^n|_{n=\infty}$, $0 \neq \alpha^n|_{n=\infty}$ *as* 0 *is not an infinitesimal and* $\alpha^n \in \Phi$ *is. Alternatively,* $\sum 0 \;z\; \sum \alpha^n|_{n=\infty}$, $0 \;z\; \int \alpha^n \, dn|_{n=\infty}$, $0 \neq \alpha^n \ln \alpha|_{n=\infty}$.

Both sums converge, as when realized their convergence sum is zero. Since the convergence sum is not continuous about $x = 0$, *the convergence sum is not uniform continuous about* $x = 0$. *Hence, while the sum is convergent, the sum is not uniformly convergent.*

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RMIT University, GPO Box 2467V, Melbourne, Victoria 3001, Australia
chelton.evans@rmit.edu.au