The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory. III

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today

Siehe, sie wissen, was ich gesagt habe. (Johannes, Kapitel 18.)

Abstract:
Some mathematical aspects of holonomy in the Nambu-Goldstone theorem are discussed. A unified theorem of the normal, generalized, and anomalous Nambu-Goldstone theorems are presented.

It is well-known fact that the dispersion relation of spin waves in a ferromagnet and an antiferromagnet of $SU(2)$ Heisenberg models with coupling constant J are given by $E(k) = 2J(1-\cos k)$ (ferromagnet) and $E(k) = 2J\sin k$ (antiferromagnet). Even though the number of broken generators are two in the both cases, the number of observed Nambu-Goldstone (NG) bosons is one for ferromagnet and two for antiferromagnet. Such an anomalous behavior, such as a ferromagnet, in the NG theorem is frequently found in a system with explicitly broken Lorentz symmetry [9,10]. It was revealed in Part I of this paper [13] that a bosonic kaon condensation model with a finite chemical potential gives a mode-mode coupling between two broken generators (Q^1, Q^2) of $SU(2)$, by the following Lagrangian of the Nambu-Goldstone (NG) sector of the spontaneous symmetry breaking:

$$\mathcal{L} = -\Phi_0^\dagger\Big[g^{-1}(\partial_\nu\partial^\nu - 2i\mu\partial_0 - \mu^2)g\Big]\Phi_0, \tag{1}$$

where, μ is a chemical potential, $g \in SU(2)$, and Φ_0 is the vacuum state. In Refs. [11,16], it was pointed out that the term proportional to μ gives a Berry phase. The corresponding part of the Berry phase in our Lagrangian is expressed in the following form after expanding the following parametrization of a group element $g = \exp(i \sum_A \chi^A Q^A)$:

$$2i\mu\Phi_0^\dagger g^{-1}\partial_0 g\Phi_0 = -2\mu \sum_{A>B} (\chi^A\partial_0\chi^B - \chi^B\partial_0\chi^A)\Phi_0^\dagger[Q^B, Q^A]\Phi_0 + \cdots . \quad (2)$$

Here, (χ^A, χ^B) are the NG bosons associated with the broken generators (Q^A, Q^B) ($A, B = 1, 2$ in the case of $SU(2)$ kaon condensation). The VEVs of the commutators $[Q^A, Q^B]$ describe mode-mode couplings between pairs of NG bosons of (Q^A, Q^B) realized over a ground state of the system, and they give a (quasi-)Heisenberg algebra in general [9,13]. Since the part of the Lagrangian also causes a non-vanishing contribution of the Berry phase in a theory, we argue the following theorem in this paper:

Theorem: *A mode-mode coupling algebra of quantum fluctuations associated with a spontaneous symmetry breaking of the anomalous Nambu-Goldstone theorem, which is explicitly realized on the ground state of the system via a 1-form, is represented globally by a holonomy of a symplectic manifold.*

Namely, when a Maurer-Cartan form $g^{-1}dg$ has a non-vanishing vacuum expectation value, it gives a (quasi-)Heisenberg algebra which implies a Heisenberg-type uncertainty relation between a pair of NG bosons would be found in the vacuum state of the system. Simultaneously, the Maurer-Cartan form can give a non-vanishing contribution to the integral of Berry phase (holonomy),

$$\gamma(C) \sim \oint_C g^{-1}dg. \quad (3)$$

Here, we should mention that we do not have to take a VEV of $g^{-1}dg$ for our definition of the Berry phase.

Needless to say, an NG mode is given by a generator of an algebra of a broken symmetry. A mode-mode coupling of the anomalous NG theorem explicitly appears from a non-vanishing off-diagonal contribution of generators of the algebra after taking a VEV, and it defines a symplectic vector space quite generally in a diagonal breaking scheme (all generators except the Cartan subalgebra are broken), since a set of broken generators (the number of broken generators in a diagonal breaking is always even) gives symplectic structures pairwisely, due to the odd nature of any Lie bracket $[X, Y] = -[Y, X]$. A geometric realization of such an algebra over a symplectic space is given as a holonomy (Berry phase) via a connection 1-form (now, a Maurer-Cartan form). Here, we will show the observation given here is quite general.

As we have mentioned above, it is a trivial fact that a symplectic structure is always found in a Lie algebra due to the definition of a Lie bracket. Thus, in that sense, a symplectic structure always exists in any tangent space of group manifolds of NNG (normal NG), GNG (generalized NG), and ANG (anomalous NG) cases [12,13]. What a special aspect of such a symplectic structure in ANG is its *explicit* appearance via the Heisenberg algebra which will be found from a VEV of a Lagrangian. A symplectic manifold always has a symplectic connection, and which can be utilized to define a holonomy, namely a Berry phase. Hence, such a holonomy can explicitly appear in a case of the ANG theorem, while it cannot be found in any VEV of NNG and GNG cases since they have no example of a generation of a Heisenberg-type algebra coming from a VEV of a Lie algebra of a symmetry. It must be emphasized that the group action of $Sp(1)$ ($\simeq Spin(3)$) on the space of a pair of broken generators is *not* a symmetry of the system in general.

A theory on symplectic structures assoicated with Lie groups is well-established in literature [1,2,3,4,5,6,15]. A symplectic homogeneous space [1,5,6] is frequently found in a diagonal breaking scheme of NG theorem. The definition of a symplectic homogeneous space is given as follows. Let G be a semisimple connected Lie group, and let H be a connected closed subgroup of G, let Ω be a G-invariant symplectic form. A symplectic homogeneous space is given by the triple (G, H, Ω). If we choose Z from an element of a Cartan subalgebra (or, a linear combination of Cartan subalgebras) of $\mathrm{Lie}(G)$, then such a G-invariant symplectic form Ω can be given by the Killing form $\omega_Z(G) = -K(Z, [X, Y])$ ($X, Y, Z \in \mathrm{Lie}(G)$) for a fixed Z. Then, a symplectic homogeneous space is obtained as a coset $G/O_G(Z)$, where $O_G(Z) = \{g \in G | \mathrm{Ad}(g)Z = Z\}$ is an adjoint orbit. Thus, we can say a diagonal breaking of a Lie group in NG theorem (for example, $SU(2) \to U(1)$, $SU(3) \to U(1) \times U(1)$, ...) generally defines an associated symplectic homogeneous space: In a diagonal breaking, Z belongs to the space of symmetric generators. Here, one should notice the fact carefully that a diagonal breaking does not fix a *representation* of Z. Due to the orthogonality of basis/generators of a Lie algebra, the similarity/relation of the Killing form $\omega_Z(G) = -K(Z,[X,Y])$ and the part of our Lagrangian $-2\mu\sum_{A>B}(\chi^A\partial_0\chi^B - \chi^B\partial_0\chi^A)\Phi_0^\dagger[Q^B,Q^A]\Phi_0$ (namely, a Maurer-Cartan 1-form $g^{-1}dg$) is obvious. Since a diagonal breaking chooses and fixes the Cartan subalgebra, the Maurer-Cartan form has a correspondence (injective, surjective, or bijective) with the Killing form. As we have mentioned above, this type of symplectic homogeneous space does not distinguish between any type of NG theorem (NNG, GNG, ANG). In the NNG theorem, the effective potential V_{eff} or the low-energy effective Lagrangian L_{eff} of a theory has no explicit dependence on the bosonic coordinates of broken generators, while a dependence will be found in V_{eff} or L_{eff} of the GNG or ANG cases [12,13]. Thus, we can say a symplectic structure of symplectic homogeneous space is implicit in an NNG case, while it becomes apparent (explicitly realized) in an ANG case. While, a direction (of the Cartan subgroup) orthogonal to a symplectic (sub)space of G appears explicitly in a case of GNG theorem [12]. Hence we have arrived at the ultimate understanding of the notion of symmetry breakings, summarized as the following "unified" theorem.

Theorem: *In a diagonal breaking, the GNG theorem is an extension of the NNG theorem toward the direction of Cartan subgroup of a Lie group G, while the ANG theorem gives the symplectic subspace orthogonal to the space of the Cartan subgroup explicitly via a Lagrangian. This fact indicates us that the GNG and ANG are only generalizations of the NNG theorem, according to the decomposition of a vector space to the vertical and horizontal subspaces.*

A Berry phase, namely a holonomy, is a Lie group arises from a parallel transport of a vector bundle (i.e., a connection 1-form). The definition of Berry phase in quantum mechanics is

$$\gamma_n(\mathcal{C}) = -\Im\oint_{\mathcal{C}}\langle n(X(t))|\partial_X|n(X(t))\rangle dX - i\ln\langle n(X(0))|n(X(T))\rangle. \quad (4)$$

Here, X denotes the parameter space for defining the holonomy, and the logarithmic term contributes in the case when $|n\rangle$ is not single-valued. A Berry curvature of $U(1)$-holonomy can be interpreted as a magnetic field defined over the parameter space.

Let us examine a geometric aspect of the mode-mode coupling term of our Lagrangian $2i\mu\Phi_0^\dagger g^{-1}\partial_0 g\Phi_0$ as a Maurer-Cartan 1-form $g^{-1}dg \in \mathrm{Lie}(G)$. Let M be a manifold, let $\pi : P \to M$ be a principal G-bundle, and let

$\omega \in \Omega^1(P) \otimes \mathrm{Lie}(G)$ be a Lie(G)-valued 1-form. The Maurer-Cartan form belongs to ω, and a holonomy is defined by [15],

$$\gamma(C) \;=\; \oint_C \omega. \tag{5}$$

The curvature 2-form is also defined as $\Omega = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P) \otimes \mathrm{Lie}(G)$. Needless to say, $\Omega = 0$ is the condition of locally flat ω, and the Maurer-Cartan form satisfies this condition. A holonomy group Hol is an automorphism of a tangent space of a manifold M: $Hol_x(M) \subset \mathrm{Aut}(T_xM)$. Of particular importance (and appropriate for the definition) here is a case of M as a group manifold. The Borel-Lichnerowicz theorem (1952) states that any holonomy group of a Riemannian manifold is a Lie group. In the diagonal breaking of NG theorem (NNG, GNG, ANG), this type of holonomy is given by a symplectic connection 1-form, and the corresponding curvature is a symplectic curvature 2-form. Some interesting theorems for us are summarized as follows:

- Any symplectic manifold has a symplectic connection [4].
- Any space in which a *-product of deformation quantization is naturally defined always has a symplectic connection [4].
- Any Poisson manifold (a symplectic manifold is a special case of a Poisson manifold) has a unique *-product [8].

A diagonal breaking of the NG theorem (NNG, GNG, ANG) is a case where all generators except the Cartan subalgebra are broken. Especially in a case of ANG theorem, a symplectic space is found explicitly/apparently in each pair of broken generators coupled through a physical space of our Lagrangian, and thus the total space of broken generators are pairwisely decoupled. Since the number of pairs is estimated by $n_{NG} = \frac{1}{2}(\mathrm{dim}G - \mathrm{rank}G)$ (recall that rank$G$ gives the dimension of the Cartan subalgebra/subgroup of G), the symplectic structure in that case is expressed by

$$\overbrace{Sp(1) \times Sp(1)}^{n_{NG}} \;\simeq\; \bigotimes_{l=1}^{n_{NG}} (\mathbf{R}^2/\mathbf{Z}^2)_l. \tag{6}$$

Here, $\mathbf{R}^2/\mathbf{Z}^2$ is called as a symplectic torus. Namely, the space of broken generators (the NG sector) are decomposed into a product of two-dimensional symplectic spaces (a product of sympletic tori), $(M = \mathbf{R}^{2n_{NG}}, g) = (\mathbf{R}^2, g_1) \otimes \cdots \otimes (\mathbf{R}^2, g_{n_{NG}})$ ($g, g_1, \cdots, g_{n_{NG}}$ are metrics). Again, we mention that such a structure of a product of symplectic tori will be found in any case of NNG, GNG, and ANG. The characteristic aspect of the ANG theorem is that it gives the structure apparently via a VEV (VEVs of Lie brackets) of the Lagrangian of a theory. For each symplectic torus, a symplectic connection ∇ can be defined, and a holonomy will be defined by using it. Then, from the de Rham splitting theorem [15], the holonomy group of total space is decomposed as $Hol_p(M) = Hol(1) \times \cdots \times Hol(n_{NG})$. This is the Berry phase generically observed in a diagonal breaking scheme of ANG theorem:

Theorem: *The total holonomy group (Berry phase) of a diagonal breaking scheme of the ANG theorem is given by $Hol(1) \times \cdots \times Hol(n_{NG})$, where each holonomy group of the direct product is $Sp(1)$.*

In summary, our logic presented here is given by the following scheme:

A diagonal breaking in NG theorem $\to$ a symplectic homogeneous space $\to$ a Heisenberg algebra via a finite VEV of a Lagrangian in a situation of ANG $\to$ symplectic connection 1-form $\to$ holonomy (Berry phase) in a symplectic manifold $\to$ curvature 2-form $\to$ Chern-Weil homomorphism $\to$ characteristic class.

In Part I, it was shown that a (quasi-)Heisenberg algebra is generally obtained in a diagonal breaking of a Lie group G. Such a situation can be found not only in cases of $SU(N)$ or $SO(N)$ but also in the three-dimensional conformal group of $SL(2,\mathbf{R})$, $[L,T] = T, [L,H] = -H, [H,T] = \frac{1}{2}L$ (L; dilatation, T; special conformal transformation, H; translation). After taking a VEV such as $\langle T \rangle = \langle H \rangle = 0$, and $\langle L \rangle \neq 0$, then a Heisenberg algebra is obtained. A symplectic structure is found in the two-dimensional space (H,T), and $Sp(1)$ acts on this space of broken generators. Thus, one can consider an $Sp(1)$ holonomy (Berry phase) of this system.

As we have discussed in the Part I of this paper, in the case of $SU(2)$ kaon condensation model, the Nambu-Goldstone bosonic coordinates (χ_1, χ_2) associated with the broken generators form a cylinderical coordinates $(r,\phi) = (\sqrt{\chi_1^2 + \chi_2^2}, \mathrm{Arg}(\chi_1 + i\chi_2))$. The one-loop effective potential V_{eff} of the kaon condensation model shows a periodicity of trigonometric function such that $V_{eff} \sim \cos r$, while it is completely flat along with the angular variable ϕ. Thus, the symmetry realized in the vacuum of the physical system is $U(1)$

rotation along ϕ, while these variables (r, ϕ) is expressed in the symplectic space of (χ_1, χ_2) which is subjected by a group action of $Sp(1)$. The periodicity and flatness reflect the Heisenberg uncertainty of the three-dimensional Heisenberg algebra Lie(H_3) coming from the VEVs of Lie$(SU(2))$. We should mention that the group action of $Sp(1)$ acts on the group manifold (or, the vector space) of (χ_1, χ_2), and not on, for example, the effective potential V_{eff}. It should be noticed that (r, ϕ) are interpreted as action-angle variables of a harmonic oscillator (see Part II of this paper) which is expressed by the canonical conjugate pair (χ_1, χ_2) (since $\chi_1^2 + \chi_2^2 = P^2 + X^2$ where P and X are momentum and position, respectively). Therefore, a dynamics of phase transition of the kaon condensation approaches to the "limit cycle" $r = r_0$, a set of stationary points. Since (χ_1, χ_2) form a canonical conjugate pair of three-dimensional Heisenberg algebra, the vacuum $(\chi_1^2 + \chi_2^2 - \hbar)|\Psi\rangle = 0$ ($\hbar \propto \langle Q^3 \rangle$) corresponds to the minimal uncertainty condition, namely a coherent state of a harmonic oscillator. In this case, a mapping, defined over V_{eff}, which describes a relaxation process of the dynamics of phase transition can be considered. If such a mapping is hyperbolic and has a positive Lyapunov exponent, then a chaos associated with the relaxation dynamics of phase transition might take place. Such a map of a positive Lyapunov exponent cannot be given by a $U(1)$ group action, while a group action of $Sp(1)$ which might be given as a toral automorphism of a symplectic torus on the space (χ_1, χ_2), might causes a hyperbolic dynamics [2,7].

References

[1] A. Yu. Alekseev and A. Z. Malkin, *Symplectic Structures Associated to Lie-Poisson Groups*, Commun. Math. Phys. Vol.162, 147-173 (1994).

[2] V. Arnol'd, *Mathematical Methods of Classical Mechanics* (GTM 60, Springer, 1978).

[3] A. Ballesteros, A. Blasco, and F. Musso, *Classification of Real Three-Dimensional Poisson-Lie Groups*, arXiv:1202.2077.

[4] P. Bieliavsky, M. Cahen, S. Gutt, J. Rawnsley, and L. Schwachhöfer, *Symplectic Connections*, arXiv:math.SG/0511194.

[5] N. Boumuki, *On Symplectic Homogeneous Spaces and Adjoint Orbits*, RIMS proceedings (2005).

[6] B.-Y. Chu, *Symplectic Homogeneous Spaces*, Trans. Am. Math. Soc. Vol.197, 145-159 (1974).

[7] J.-P. Eckmann and D. Ruelle, *Ergodic Theory of Chaos and Strange Attracters*, Rev. Mod. Phys. Vol.57, 617-656 (1985).

[8] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, arXiv:q-alg/9709040.

[9] Y. Nambu, *Spontaneous Breaking of Lie and Current Algebras*, J. Stat. Phys. Vol.115, 7-17 (2004).

[10] H. B. Nielsen and S. Chadha, *On How To Count Goldstone Bosons*, Nucl. Phys. B105, 445 (1976).

[11] S. M. Radošević, M. R. Pantić, M. V. Pavkov-Hrvojević, and D. V. Kapor, *Magnon Energy Renormalization and Low-Temperature Thermodynamics of O(3) Heisenberg Ferromagnets*, arXiv:1306.5584.

[12] T. Ohsaku, *Dynamical Mass Generations and Collective Excitations in the (Supersymmetric-)Nambu−Jona-Lasinio Model and a Gauge Theory with Left-Right-Asymmetric Majorana Mass Terms* , arXiv:0802.1286.

[13] T. Ohsaku, *The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory*, arXiv:1312.0916, (71 pages). (The Part I of this paper.)

[14] T. Ohsaku, *The Anomalous Nambu-Goldstone Theorem in Relativistic/Nonrelativistic Quantum Field Theory. II*, (24 pages). (Submitted to arXiv.org, 25 Feb. 2014, and removed at the end of August 2014.)

[15] L. J. Schwachhoöfer, *Holonomy Groups and Algebras* (C. Bär et al. (eds.), *Global Differential Geometry*, Springer, 2012).

[16] H. Watanabe and H. Murayama, *The Effective Lagrangian for Nonrelativistic Systems*, arXiv:1402.7066.