An Interesting Perspective to the P versus NP Problem

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Abstract. We discuss the P versus NP problem from the perspective of addition operation about polynomial functions. Two contradictory propositions for the addition operation are presented. With the proposition that the sum of k ($k \leq n$) polynomial functions on n always yields a polynomial function, we prove that $P = NP$, considering the maximum clique problem. And with the proposition that the sum of k polynomial functions may yield an exponential function, we prove that $P \neq NP$ by constructing an abstract decision problem. Furthermore, we conclude that $P = NP$ and $P \neq NP$ if and only if the above propositions hold, respectively.

Keywords: *Complexity; P versus NP Problem; NP-Complete Problem; Turing Machine; Addition Operation; Binomial Theorem*

1 Introduction

As one of the most important problems in mathematics and computer science, the P versus NP problem is to determine whether every language accepted by some nondeterministic algorithm in polynomial time is also accepted by some (deterministic) algorithm in polynomial time [1]. Since first mentioned in a 1956 letter written by Kurt Gödel to John von Neumann [2] and precisely stated in 1971 by Stephen Cook [3], the problem has been considered by many papers [4]. However, it is still open [5]. For detailed introduction of the P versus NP problem, please refer to the excellent survey articles by eminent authors (see [4]-[10]).

In this paper, we will discuss the P versus NP problem based on the addition operation of polynomial functions. It is often thought that the sum of k ($k \leq n$) polynomial functions on n always yields a polynomial function. However, we can also get a contradiction if we accept this proposition. So we present another proposition that the sum of k polynomial functions may yield an exponential function. With the propositions, we find that $P = NP$ and $P \neq NP$ can both be proved, respectively.

The rest of this paper is organized as follows. Section 2 presents two simple and interesting properties about the binomial theorem. Section 3 presents an interesting contradiction and two contrary propositions for the addition operation. Section 4 discusses the P versus NP problem according to the propositions. Finally, Section 5 concludes this paper.

2 Story of Binomial Theorem

Firstly, let's remember the binomial theorem which is also called Yang Hui triangle in China [11].

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \cdots + C_n^{n-1} a b^{n-1} + C_n^n b^n \qquad (1)$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ for $k = 0, 1, \cdots, n$ and $C_n^{k+1} = C_{n-1}^k + C_{n-1}^{k+1}$ for $k = 0, 1, \cdots, n-1$.

We present two simple and interesting properties about Eq. (1) in the following.

Lemma 1. $2^n = C_n^0 + C_n^1 + \cdots + C_n^{n-1} + C_n^n$.

Proof. Replacing $a = b = 1$ in Eq. (1), the lemma follows. □

Lemma 2. *Let k be an arbitrary integer number such that $0 \leq k < n-1$. We have that $C_n^{k+1} = C_{n-1}^k + C_{n-2}^k + \cdots + C_{k+1}^k + C_k^k$.*

Proof. Noting that $C_n^{k+1} = C_{n-1}^k + C_{n-1}^{k+1} = C_{n-1}^k + (C_{n-2}^k + C_{n-2}^{k+1}) = \cdots = C_{n-1}^k + C_{n-2}^k + \cdots + (C_{k+1}^k + C_{k+1}^{k+1})$ and $C_{k+1}^{k+1} = C_k^k$, the lemma follows. □

3 An Interesting Contradiction

Now, let's come to an interesting contradiction. Let $h_1(n), h_2(n), \cdots, h_k(n)$ be arbitrary k polynomial functions on n, and $H(n) = h_1(n) + h_1(n) + \cdots + h_k(n)$, where $k \leq n+1$.

Is $H(n)$ polynomial or exponential? Noting that $H(n) \leq k \max\limits_{1 \leq j \leq k}\{h_j(n)\} \leq n \max\limits_{1 \leq j \leq k}\{h_j(n)\}$, you may say $H(n)$ is polynomial since $\max\limits_{1 \leq j \leq k}\{h_j(n)\}$ is polynomial. However, an contradiction can be obtained if we accept the following three propositions, which all sound rational.

Proposition 1. *Polynomial functions and exponential functions are different.*

Proposition 2. *Mathematical induction is rational.*

Proposition 3. *$H(n)$ is always polynomial.*

Assume that $h_i(n) = C_n^i$ for $i = 0, 1, \cdots, n$, where $C_n^i = \frac{n!}{i!(n-i)!}$. From Lemma 1, we know that $H(n) = h_0(n) + h_1(n) + \cdots + h_n(n) = C_n^0 + C_n^1 + \cdots + C_n^n = 2^n$, implying that $H(n)$ is exponential. However, applying mathematical induction, we can also get that $H(n)$ is polynomial as following.

In fact, noting Proposition 3, it is sufficient to prove that C_n^0, C_n^1, $\cdots$, C_n^n are all polynomial. Firstly, it is obvious that C_n^0 is polynomial. Now we suppose that C_n^k is polynomial for some k $(0 \leq k \leq n)$ and try to prove that C_n^{k+1} is also polynomial. Noting that $C_k^k < C_{k+1}^k < C_{k+2}^k < \cdots < C_{n-1}^k < C_n^k$ and C_n^k is polynomial, we have that $C_k^k, C_{k+1}^k, \cdots, C_{n-1}^k$ are all polynomial. Remembering

Lemma 2 and Proposition 3, we have that C_n^{k+1} is polynomial. So, we also get that $H(n)$ is polynomial, contradicting to Proposition 1.

Why does the contradiction happen? Firstly, it is easy for us to accept Proposition 1. Secondly, we have to accept the mathematical induction. So we conclude that Proposition 3 may be not right, and have the following contrary proposition.

Proposition 4. *There exist k $(k \leq n)$ polynomial functions on n, i.e., $h_1(n)$, $h_2(n)$, $\cdots$, $h_k(n)$, such that $H(n)$ is exponential, where $H(n) = h_1(n) + h_2(n) + \cdots + h_k(n)$.*

4 Discussion for the P versus NP Problem

Two subsections are considered according to the Proposition 3 and Proposition 4, respectively.

4.1 Proposition 3 holds

Considering the maximum clique problem, which is NP-complete [12], we will prove $P = NP$ in the following.

Theorem 1. *$P = NP$ if Proposition 3 holds.*

Proof. Given a graph G with n vertices, we can find the maximum clique of G by Enumerative Algorithm. Noting that there are at most C_n^i cliques with i vertices for $i = 0, 1, \cdots, n$, and the worst case run time of verifying a clique is polynomial, it is sufficient to prove that $C_n^0 + C_n^1 + \cdots + C_n^{n-1} + C_n^n$ is polynomial.

From the analysis in the above section, we can get that $C_n^0 + C_n^1 + \cdots + C_n^{n-1} + C_n^n$ is polynomial accepting Proposition 2 and Proposition 3.

The theorem follows. □

4.2 Proposition 4 holds

We will prove that $P \neq NP$ in the following. It is often thought that proving $P \neq NP$ involves proving a superpolynomial lower bound on the run time of any algorithm for some NP-complete problem such as SAT [6]. However, instead of any NP-complete problem, we will construct an abstract problem Π, such that $\Pi \in NP$ and $\Pi \notin P$.

From Proposition 4, we know that there exist k $(k \leq n)$ polynomial functions on n, i.e., $h_1^*(n), h_2^*(n), \cdots$, and $h_k^*(n)$, such that $H^*(n)$ is exponential, where $H^*(n) = h_1^*(n) + h_2^*(n) + \cdots + h_k^*(n)$.

Remember that the return of a decision problem is just a "yes" or "no". Let π_i $(i = 1, 2, \cdots, k)$ denote an abstract decision problem with input I_i, where the length of I_i is n and the worst case run time for π_i is $h_i^*(n)$. Moreover, we let Π be a decision problem which is to ask if there exists a "yes" in the returns of $\pi_1, \pi_2, \cdots$, and π_k. Note that the input of Π are $I_1, I_2, \cdots$ and I_k with a total length of $nk < n^2$.

Theorem 2. *$P \neq NP$ if Proposition 4 holds.*

Proof. It is sufficient to prove that $\Pi \in NP$ and $\Pi \notin P$.

Note that $I_1, I_2, \cdots$ and I_k are unrelated. So, for any deterministic Turing Machine, the worst case of solving Π is to check the k returns of $\pi_1, \pi_2, \cdots$ and π_k. And the total run time is $h_1^*(n) + h_2^*(n) + \cdots + h_k^*(n)$. Noting that $H^*(n) = h_1^*(n) + h_2^*(n) + \cdots + h_k^*(n)$ and $H^*(n)$ is exponential, we get that $\Pi \notin P$.

For a non-deterministic Turing Machine, it is sufficient to check the return of one of the k decision problems with polynomial run time. So it is that $\Pi \in NP$.

The theorem follows. □

4.3 Further Discussion

Moreover, we have the following conclusions.

Corollary 1. *$P = NP$ if and only if Proposition 3 holds.*

Proof. From Theorem 1, we know that $P = NP$ if Proposition 3 holds. We only have to prove that Proposition 3 holds if $P = NP$. In fact, if otherwise Proposition 3 does not hold, then Proposition 4 holds, implying $P \neq NP$ remembering Theorem 2. The corollary follows. □

Corollary 2. *$P \neq NP$ if and only if Proposition 4 holds.*

Proof. The proof is similar as for Corollary 1. □

5 Conclusion

In this paper, we discuss the P versus NP problem from the perspective of addition operation about polynomial functions. And we conclude that $P = NP$ and $P \neq NP$ if and only if Propositions 3 and 4 hold, respectively.

However, it is still hard for us to understand the two contrary Propositions 3 and 4. Buddha Sakyamuni said that inequality of heart yields annoyance. May be that the polynomial and exponential functions are not absolutely different but naturally interrelated.

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