This document explores the links between the proposed “grand unified theory (GTU)” based on a SU(5) model and the theory involving extended Lie products (ELP). A special focus is given to some EM fields obtained via a Cartan-like procedure rebuilding the general theory of relativity (GTR) with the variations of the basis vectors up to the second order (the so-called GTR2 – see link in the bibliography below). The link between the GTR and the GTR2 is better understood and explained. My hope is a reconstruction of the results actually obtained by the standard model via the intervening of the ELPs for the GTR2-induced electromagnetic fields. A big part of the way in that direction has been made but the goal is not yet totally reached; thus: yet in evolution.

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1. Extended Lie’s products, Lie algebras and their representations

1.1. Remark 01: presumption of a link with the grand unified theory

The C-Lie algebra \( \mathfrak{sl}(5, \mathbb{C}) \) = \{ \mathbf{x} \in \mathfrak{gl}(5, \mathbb{C}) \mid \text{Trace} (\mathbf{x}) = 0 \} \ [02; p. 6, examples 1.11-2] and the \( \mathbb{R} \)-Lie algebra \( \mathfrak{su}(5) \) \ [02; p. 6, examples 1.11-4] appear spontaneously within the theory of the (E) question (TEQ) developed in a four dimensional context. The TEQ is nothing but the one of decomposed extended Lie’s brackets (ELP). It is known since a long time (1973; \[i\]) that \( \mathfrak{su}(5) \) may be an ideal context for the description of a grand unified theory (GUT). Basics data about the Lie algebras can be read at the beginning of the document \[02\]. Despite of the fact that the decay of the proton has never been observed, shedding a shadow on that proposed version of the GUT, I shall confront the TEQ with it.

1.2. Lie algebra structure

The concept of extended Lie’s product, as its name evocate, is directly inspired by the one of Lie product. With that concept, a vector space \( E_4(\mathbb{C}) \) wins a Lie algebra structure \[02; p. 5, definition 1.1\]. The \( k \) field introduced in \[02\] is identified here with the set of all complex numbers: \( \mathbb{C} \). The C-Lie algebras \( (E_4(\mathbb{C}), [A] \wedge) \) -where \( ^{(5)}[A] \) represents the reduced cube \( ^{(4)} \mathbf{\nabla} A \) and where \( [A] \wedge \) denotes the extended Lie product at hand built on a traceless anti-Hermitian matrix \( ^{(5)}[A] \in M_5(\mathbb{C}) \) \ [02; p. 6, examples 1.11-4]- are fundamentally anti-commutative.

1.3. Remark 02: guessing the existence of an adjoint representation

The ability to represent the Lie algebras is a central thematic. Some fundamental data are given in \[02; § 2, pp. 7-8\]. Applying these ideas to the TEQ, in peculiar \[02; p. 7, definition 2.1\], looking a representation of \( (E_4(\mathbb{C}), [A] \wedge) \) in \( \mathfrak{gl}(E_4(\mathbb{C})) \) is a legitimate duty. As an evident fact the trivial split of a given ELP yields a natural representation in \( M_4(\mathbb{C}) \). This seemingly gives at least one concrete answer to that quest of representation. The trivial split, denoted \( [A] \Phi(\mathbf{^{(4)} projectile}) \), is a tool allowing the definition of a function \( [A] \Phi \) which may in fact be seen as defining a kind of adjoint representation \[02; p. 7, example 2.3\]; in extenso:

\[
\begin{align*}
\downarrow^{(d) \mathbf{projectile}} \\
\downarrow_{[A] \Phi} \\
|^{(4) \mathbf{target}} \rightarrow [A] \Phi(\mathbf{^{(4)} projectile}) \rightarrow [A] \Phi(\mathbf{^{(4)} projectile}) \rightarrow |^{(4) \mathbf{target}} > = |[A] \wedge(\mathbf{^{(4)} projectile}, \mathbf{^{(4)} target}) >
\end{align*}
\]

And:

\( [A] \Phi \equiv \text{ad} \) for “adjoint representation of”

1.4. Remark 03: exploring the foundations of extended Lie’s products

This fact may be seen as a result of a sort of duality in the definition of ELPs. Indeed, at least one reduced cube, that is in fact a traceless anti-Hermitian matrix \( ^{(5)}[A] \in M_5(\mathbb{C}) \) is needed for the definition of at least one ELP. Consequently: each and any ELP defines itself and one adjoint representation \( [A] \Phi \equiv \text{ad} \in \mathfrak{J}(E_4(\mathbb{C}), M_4(\mathbb{C})) \), the set of all functions with a source in \( E_4(\mathbb{C}) \) and an image in \( M_4(\mathbb{C}) \). The image of an adjoint representation is in \( M_4(\mathbb{C}) \); that means: it could eventually happen that a part of that image coincides with the matrix defining the adjoint representation. At this stage, it’s a little bit complicated to say more on that topic but following that way of thinking it may be useful to recall the Perian decompositions \( [a] \). Applying it to any element \( ^{(5)}[A] \) will
irremediably yield a triplet \((^4[A], (^4_1a, ^4_2a), a)\) of \(M_4(C) \times E_4(C)^2 \times C\).

And then: “Why not suppose the existence of some special conditions for which:

\[
\begin{align*}
_{^4p} & \downarrow \\
_{^4t} & \to ([^4[A] \to a = <([^4_1a, ([^4[A] - [^4a] \Phi ([^4p])). |[^4_1a] > + |[^4_2a]) >^\alpha?)
\end{align*}
\]

These situations, if they exist, relate the definition of a given ELP with a “degree of realization of its trivial split” after the action of some projectile, \((^4p)\), on a fixed target, \((^4t)\). If one of the two vectors \((^4_1a)\) or \((^4_2a)\) is proportional either to the projectile or to the target, this measure of the difference between a trivial and a non-trivial split is just a polynomial of degree two depending on the projectile or respectively of the target. This is a funny but extremely complicated idea, difficult to formalize and, because of that, I leave it for later.

1.5. Remark 04: existence of an adjoint representation (starting the proof)

Anyway, not only non-trivial splits may be envisaged but the function \([^4a] \Phi\) will really be an adjoint representation if it defines Lie algebra morphism.

\[
_{[A]} \Phi(a)._{[B]} \Phi(b) - _{[A]} \Phi(b)._{[A]} \Phi(a) = \ldots
\]

Concretely, each term (the position is labeled with the pair \((\alpha, \beta)\)) is:

\[
\forall (\alpha, \beta) \text{ given: } m_{\alpha\beta} = \lambda_{\alpha\beta} \cdot _{[A]} \Phi(a)._{[B]} \Phi(b) - _{[B]} \Phi(b)._{[A]} \Phi(a)
\]

That sum can be decomposed:

\[
m_{\alpha\beta} = \sum_{\lambda < \mu} \lambda_{\alpha\beta} \cdot \lambda_{\alpha\beta} \cdot _{[A]} \Phi(a)._{[B]} \Phi(b) = \ldots
\]

That formalism is a clear proof that a commutator involving two trivial decompositions/splits is a deformed exterior product. On the other hand, for a trivially decomposed extended exterior product:
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\[ \Phi(\lambda) \wedge (a, b) = [\ldots A_{\chi\beta}^\alpha \cdot [\ldots A_{\alpha\chi}^\beta \wedge (a, b)]^\lambda \ldots] = [\ldots A_{\chi\beta}^\alpha \cdot \sum_\lambda \sum_\mu A_{\lambda\mu}^\chi, (a^\lambda, b^\mu) \ldots] \]

The formalism is an invitation. Let me compare:

\[ \sum_\lambda \sum_\mu (A_{\chi\alpha}^\lambda \cdot A_{\beta\alpha}^\mu - A_{\chi\alpha}^\mu \cdot A_{\beta\alpha}^\lambda) \cdot (b^\mu, a^\mu - a^\alpha, b^\alpha) \text{ and } \sum_\lambda \sum_\mu A_{\chi\beta}^\alpha \cdot A_{\lambda\mu}^\chi, (a^\chi, b^\mu - a^\alpha, b^\lambda). \]

Equality is obtained for any pair \((a, b)\) as soon as it is possible to write:

\[ \forall \lambda, \mu > \chi: A_{\chi\alpha}^\lambda \cdot A_{\beta\alpha}^\mu - A_{\chi\alpha}^\mu \cdot A_{\beta\alpha}^\lambda = A_{\chi\beta}^\alpha \cdot A_{\alpha\chi}^\lambda \]

Is it really possible? If yes: “What does this imply?” “Is it related to the Lie’s algebra structure?” Let me explore. Up to now there is only one known property for the cube \(\nabla A\) at hand, namely its anti-symmetry: \(A_{\lambda\mu}^\chi + A_{\mu\lambda}^\chi = 0\); this seems to be from no real utility since the previous relation concerns cases where \(\lambda < \mu\). The pairs \((\alpha, \beta)\) label a position in a square \((4-4)\) matrix. The unique certitude is actually the existence of a set of six identities connecting the 24 components of \(\nabla A\):

\[
\begin{align*}
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
\end{align*}
\]

In order to get an idea about what it represents, let me adopt a “bottom to top” approach and just add a supplementary condition on these components:

\[ A_{\chi\beta}^\alpha + A_{\alpha\chi}^\beta = 0 \]

There is zero difficulty to prove the compatibility between both constraints and the fact that they reduce the number of the components of \(\nabla A\) to only four. What is the impact on the six relations? At the level of the principles, every component with a repeated index vanishes. Let me consider the first relation of coherence:

\[
\begin{align*}
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
A_{\alpha\chi}^\beta \cdot A_{\beta\alpha}^\chi &= A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta, A_{\alpha\chi}^\beta = A_{\alpha\chi}^\beta,
\end{align*}
\]

The details depend then on the position in the matrix.
But whatever that position will be, it will be easy to state the exactitude of the first relation of coherence. The same maneuver can be repeated with the five others. What can be deduced from this? As explained previously, a double reduced cube \( ^{(4)} \nabla \mathbf{A} \) is just some vector of \( E_4(\mathbb{R}) \); let me denote it with \( ^{(4)} \mathbf{c} \). It has just been proved that:

\[
[c \Phi ^{(4)} \mathbf{a}, c \Phi ^{(4)} \mathbf{b}] = c \Phi (c \wedge ^{(4)} \mathbf{a}, ^{(4)} \mathbf{b})
\]

1.6. Lemma

The function \( c \Phi \) where \( ^{(4)} \mathbf{c} \) denotes the double reduced cube \( ^{(4)} \nabla \mathbf{A} \) allows the construction of a morphism between \( \{E_4(\mathbb{R}), \wedge \} \) and \( \{M_4(\mathbb{R}), [..., ...]\} \). This is interesting and this is furthermore a pleasant result but it can be suspected that it doesn’t really learn more than that a kind of Lie derivative can be built with \( \wedge \). It would be quite more interesting and general to get the proof for the existence of morphisms between \( \{E_4(\mathbb{R}), [\mathbf{a}] \wedge \} \) and \( \{M_4(\mathbb{R}), [..., ...]\} \). What can also be guessed is the fact that strategies must be discovered to manage the equations of coherence in a clever manner avoiding industrial calculations.

1.7. Remark 05: existence of an adjoint representation (the proof)

Let me remark that instead of doing unpleasant calculations, perhaps is it clever to look for another property characterizing the extended Lie’s products and to compare it then with these relations of coherence. And of course:

\[
\triangle_n \mathbf{a}[x, \triangle_n \mathbf{a}[y, z]] = [x^\alpha, [y, z]_\wedge^\beta, [y, z]_\wedge^\gamma]. [e_{\beta\gamma}, e_\lambda]_\wedge
\]

\[
= \{x^\alpha, A_{\beta\gamma}, \mathbf{v}, z^{\mu}, x^\beta, A_{\alpha\gamma}, \mathbf{v}, z^{\mu}, A_{\alpha\beta}, \mathbf{v}, e_\lambda\}
\]
Following the same logic but in permuting:
\[
\Delta \triangleleft A[\Delta \triangleleft A[x, y], z] = - \{A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\}. e^\chi
\]

And:
\[
\Delta \triangleleft A[y, \Delta \triangleleft A[x, z]] = \{A_{\alpha \beta} \chi, (y^\alpha. A_{\lambda \mu} \beta - y^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\}. e^\chi
\]

This allows:
\[
\Delta \triangleleft A[x, \Delta \triangleleft A[y, \Delta \triangleleft A[z, x]]] + \Delta \triangleleft A[z, \Delta \triangleleft A[x, y]] = \{A_{\alpha \beta} \chi, (x^\alpha. A_{\lambda \mu} \beta - x^\beta. A_{\lambda \mu} \alpha). y^\chi, z^\mu\} - \{A_{\alpha \beta} \chi, (y^\alpha. A_{\lambda \mu} \beta - y^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\} + \{A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu\}
\]

\[
\equiv \{A_{\alpha \beta} \chi, (x^\alpha. A_{\lambda \mu} \beta - x^\beta. A_{\lambda \mu} \alpha). y^\chi, z^\mu\} - \{A_{\alpha \beta} \chi, (y^\alpha. A_{\lambda \mu} \beta - y^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\} + \{A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu\}
\]

\[
\equiv \{A_{\alpha \beta} \chi, (x^\alpha. A_{\lambda \mu} \beta - x^\beta. A_{\lambda \mu} \alpha). y^\chi, z^\mu\} - \{A_{\alpha \beta} \chi, (y^\alpha. A_{\lambda \mu} \beta - y^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\} + \{A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu\}
\]

\[
\equiv \{A_{\alpha \beta} \chi, (x^\alpha. A_{\lambda \mu} \beta - x^\beta. A_{\lambda \mu} \alpha). y^\chi, z^\mu\} - \{A_{\alpha \beta} \chi, (y^\alpha. A_{\lambda \mu} \beta - y^\beta. A_{\lambda \mu} \alpha). x^\chi, z^\mu\} + \{A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu\}
\]

\[
\equiv (A_{\alpha \beta} \chi, (x^\alpha. A_{\lambda \mu} \beta - x^\beta. A_{\lambda \mu} \alpha). y^\chi, z^\mu) - (A_{\beta \alpha} \chi, (x^\beta. A_{\lambda \mu} \alpha - x^\alpha. A_{\lambda \mu} \beta). y^\chi, z^\mu) + (A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu)
\]

All this has been obtained in re-ordering the indices. It may also be rewritten \((\alpha \rightarrow \beta, \beta \rightarrow \chi, \chi \rightarrow \alpha)\):

\[
(A_{\beta \alpha} \chi, (x^\beta. A_{\lambda \mu} \alpha - x^\alpha. A_{\lambda \mu} \beta). y^\chi, z^\mu) + (A_{\alpha \beta} \chi, (z^\alpha. A_{\lambda \mu} \beta - z^\beta. A_{\lambda \mu} \alpha). x^\chi, y^\mu)
\]

Taking care of the natural anti-symmetry imposed to the cubes defining the extended Lie’s products, this is:

\[
2. (- A_{\beta \alpha} \chi, A_{\lambda \mu} \beta - A_{\lambda \mu} \alpha). y^\chi, z^\mu = \{04; \S 1.5, p.15, (1.4)\}. x^\alpha, y^\chi, z^\mu
\]

A direct and visual comparison with the relations of coherence achieves to convince the reader that:

1.8. Theorem 01
The necessary and sufficient condition insuring that \(E_d(C), [a] \wedge \) is equipped with a \(C\)-Lie structure algebra is the same condition insuring that \([a] \Phi \) defines morphisms between \(E_d(C), [a] \wedge \) and \(M_d(C), [...,...]); namely:

\[
[a] \Phi (\Phi^{(a)} a, [a] \Phi (\Phi^{(a)} b)) = [a] \Phi ([a] \wedge (\Phi^{(a)} a, \Phi^{(a)} b))
\]

This generalizes the previous lemma to any anti-symmetric cube \(\Phi \Delta A\) not necessarily reduced once more time via \(A_{\beta \alpha} \chi + A_{\alpha \beta} \chi = 0\). The theory of the (E) question (TEQ) is thus now equipped with an
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adjoint representation (see: [02; example 2.3, pp. 7-8]). Note that the C-Lie algebra $\mathfrak{sl}(5, \mathbb{C})$ introduced at the very beginning of that document is relatively intensively studied in [02; §4, pp. 12-15] and needs at this stage for the purpose of the TEQ not further analyze. I shall only remark that the structure is known while the representations are not. This fact alone is perhaps a sufficient motivation encouraging me to develop the TEQ further since it could yield at least one family of representations of the C-Lie algebra $\mathfrak{sl}(5, \mathbb{C})$.

1.9. Corollary
The lemma remains nevertheless true. This is suggesting the necessity of an investigation concerning triplets of four-vectors. And in general, coming back to the theorem, this is suggesting the necessity of an investigation concerning the interaction of a pair of vectors via a bi-vector with the purpose to be able to measure four dimensional volumes.

2. Building a link with the quantum chromodynamics – first part

2.1. The context of the discussion
As already recalled previously, the theory of the (E) question (TQE) denotes in fact the theory of the decomposed extended Lie products. Let me give some basic definitions again:

2.1.1. Definition 01: extended product (EP)

$$\forall (D) \in K^3, \forall u, w \in E_0(K): \bigtriangleup_{\mathcal{A}}(u, w) = \sum_{\alpha} \sum_{\beta} u^\alpha. w^\beta. A_{\alpha\beta\gamma}. e_{\gamma}$$

2.1.2. Definition 02: extended Lie product (ELP)

$$\forall (D) \in K^3, \forall u, w \in E_0(K): [u, w]_{\mathcal{A}} = \bigtriangleup_{\mathcal{A}}(u, w) - \bigtriangleup_{\mathcal{A}}(w, u) = \sum_{\alpha} \sum_{\beta} (u^\alpha. w^\beta - w^\alpha. u^\beta). A_{\alpha\beta\gamma}. e_{\gamma}$$

2.1.3. Remark 06: Inner product
Staying at a formal level, it's easy to state that any EP/ELP is an inner product, precisely a function linking some pair in $E_0(K) \times E_0(K)$ to some element in $E_0(K)$.

2.1.4. Remark 07: Antisymmetric products and antisymmetric cubes
On the same vein, there is no difficulty to observe that any ELP is an antisymmetric product; this means that, whatever $K$ and the cube $\nabla A$ are:

$$\forall (D) \in K^3, \forall (u, w) \in E_0(K) \times E_0(K): [u, w]_{\nabla A} + [w, u]_{\nabla A} = 0$$

It is useful to avoid confusions between antisymmetric products and antisymmetric cubes. The definition for ELP built on antisymmetric cubes can be simplified as follows:

$$\forall (D) \in K^3, \forall u, w \in E_0(K): [u, w]_{\mathcal{A}} = \sum_{\alpha} \sum_{\beta} (u^\alpha. w^\beta - w^\alpha. u^\beta). A_{\alpha\beta\gamma}. e_{\gamma}$$

2.1.5. Definition 03: extended exterior product
Per definition, an extended exterior product (EEP) is the function:

$$\nabla_{\mathcal{A}} \wedge: V \times V \rightarrow K | \forall (u, w) \in V, \nabla_{\mathcal{A}} \wedge(u, w) = \sum_{\alpha} \sum_{\beta} (u^\alpha. w^\beta - u^\beta. w^\alpha). A_{\alpha\beta\gamma}. e_{\gamma}$$
As a matter of evidence:

\[ \forall (D) \land A \in K^3 \mid A_{uv}^\gamma + A_{vu}^\gamma = 0, \forall u, w \in E_0(K); \]

\[ [u, w] \land A = 2. \land A \land (u, w) \]

In these cases, the cube \( \land A \) can always be reduced but the representation of that reduction under a matrix formalism must be realized carefully because it is dimension-dependent. For example when \( D = 3 \), the cube is reduced to a (3-3) matrix denoted (per convention):

\[ [3][A] = \]

When \( D = 4 \), please see § at the beginning of that document.

2.1.6. Remark 08: The special cases of square products
Continuing the examination of elementary properties, it is relatively evident to state that the result of a square product is closely depending on the nature of \( K \). More precisely and obviously: these products systematically vanish if \( K \) is equipped with a commutative multiplication (typical examples are the sets of real, \( \mathbb{R} \), and complex numbers, \( \mathbb{C} \)); this is no more automatically the case otherwise (typical examples are the sets of quaternions, \( \mathbb{H} \), and octonions numbers, \( \mathbb{O} \)).

2.1.7. Remark 09: the reduction of the discussion to space-times
All this is in peculiar true for \( D = 4 \).

2.1.8. Remark 10: a formalism which is waking up our curiosity
Until today the TEQ has been developed in a relatively basic style. I want now to demonstrate its powerful capacities in quantum chromo-dynamics (QCD). There is seemingly a first argument encouraging me to ask if the TQE can be confronted with the QCD. Let me consider the \( U(N) \) gauge group with generators \( T^a \) as recalled in e.g. \([01; § 2.1, p.9]\). In general, these generators are elements of \( M_N(\mathbb{C}) \). As a matter of well-known facts:

\[ [T^a, T^b] = i. \sqrt{2}. f^{ab}_c. T^c \]

If, for example, I consider the case \( N = 4 \), the theory is developed on \( M_4(\mathbb{C}) \) which is a Lie algebra [07] and \( a = 1, 2..., 15 \) because \( \dim U(N) = N^2 - 1 \) which, in that case, yields \( \dim U(4) = 16 - 1 = 15 \). The formalism of that prototypical relation is appealing a comparison with the one of the extended Lie products (ELP). Indeed, as I have demonstrated it in the first part of this document (\textit{theorem 01}), \( E_4(\mathbb{C}) \) may be equipped with some ELP \( [A]^\wedge \) in such a way that \( \{E_4(\mathbb{C}), [A]^\wedge\} \) and \( \{M_4(\mathbb{C}), [...,...]\} \) are two \( \mathbb{C} \)-Lie algebras connected by morphism (an adjoint representation) coinciding with the trivial decomposition of the ELP at hand. It becomes thus meaningful to ask if and when the generators of \( U(4) \) are/can be trivial representations of these actually unknown ELP.

2.2. The GTR2 approach

2.2.1. The idea, the main results and the link with the theory of relativity
In a separate approach I have examined the construction of a GTR-like theory in considering variations of the basis vectors (the \( e_\alpha \) for \( \alpha = 0, 1, 2, 3 \)) until the second order inclusively. I called this the GTR2 approach. As already demonstrated in \([e]\), the construction of a Riemann-like tensor \([01; p. 7]\) brought an unexpected result which I may write:
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\[ t^\nu \cdot \partial_\alpha e_\nu, \partial_\beta e_\nu >_{B(4)} = F_{\alpha \beta} \]

That result is obtained for any bilinear 2-form \( B \) and after the introducing of the cube \( \nabla T \):

\[ \frac{\partial e_\nu}{\partial x^\alpha} = T_{\lambda \alpha} \cdot e_\lambda; \quad \Omega: (e_0, e_1, e_2, e_3); \quad \partial_\alpha \Omega \equiv \partial_\alpha \cdot \Omega \equiv (\partial_\alpha e_0, \partial_\alpha e_1, \partial_\alpha e_2, \partial_\alpha e_3). \]

That cube has a priori, at this stage, nothing to do with the antisymmetric cube \( \nabla A \) on which \( [A] \wedge \) is based. It is in fact just projecting the first order partial derivatives of the basis vectors. But since an antisymmetric cube is needed for the development of the TEQ, I propose to write:

\[ \nabla T = \nabla A \]

The construction of a Riemann-like tensor is possible [01; p. 13, (2.8)] and can be connected to the Riemann tensor itself [01; p. 14, (2.9)]. That procedure is thus indirectly connecting the GTR2 approach to the customized GTR. The scoop and unexpected result is the fact that that approach suggests the existence of EM fields induced by the first order variations of the basis vectors. The result can be summed up in writing:

\[ \forall \alpha, \beta = 0, 1, 2, 3 \]

\[ t^\nu \cdot \partial_\alpha e_\nu, \partial_\beta e_\nu >_{B(4)} = F_{\alpha \beta} \]

\[ \nabla T = \{ \frac{1}{2} (\partial_\alpha \Omega \cdot \partial_\beta \Omega - \partial_\beta \Omega \cdot \partial_\alpha \Omega) \} \]

2.3. A quantum of mathematics for the classification and for the pedagogy

2.3.1. The uninteresting symmetric forms

The nature (symmetric, antisymmetric, any) of the 2-form will have consequences on the properties and on the effective formalism of that type of EM field. For example, if \( B \) is totally symmetric, the matrix \( T_{\alpha \beta} \) is symmetric too and a transposition of its elements has no influence. In that precise example, the EM fields of that theory vanish. This is evidently pushing the theory into the direction of antisymmetric 2-forms. In any case, the mathematics demonstrates that:

\[ \forall (\theta, \, 2 \theta) \in V^N \times V^N \]

\[ T_2(\ldots, \ldots) (\theta, \, 2 \theta) - T_2(\ldots, \ldots) (2 \theta, \, \theta) = [\ldots \theta_\alpha \theta_\beta, \theta_\mu \theta_\nu \theta_\gamma \theta_\delta \ldots] = [\ldots b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]

\[ = [\ldots \sum_{\alpha \beta} b_{\alpha \beta} \cdot (\theta_\alpha \theta_\beta - 2 \theta_\alpha \theta_\beta) \ldots] \]
2.3.2. Remark 11: a tiny but crucial difference
Note that the following calculation differs totally from the previous one and will be reconsidered later because it feats optimally with the one of the EM fields of the theory:

\[ \forall (\theta, 2\theta) \in V^n \times V^n \]

\[ T_2(<\ldots, \ldots>\theta)(\theta, 2\theta) - T_2(<\ldots, \ldots>\theta)(2\theta, 1\theta) = [... <\ldots, \ldots>\theta(\theta, 2\theta) >_{\theta} - <\ldots, \ldots>\theta(2\theta, 1\theta) >_{\theta} ...] = [... b_{\alpha\beta}(\theta^\alpha \cdot 2\theta^\beta - 2\theta^\alpha \cdot 1\theta^\beta) ...] \]

The reader can convince himself of that effective difference in considering the \( b_{\alpha\alpha} \) terms in both cases. Terms appearing in § 2.3.1 are the \( b_{\alpha\alpha} \cdot (\theta^\alpha \cdot 2\theta^\beta - 2\theta^\alpha \cdot 1\theta^\beta) \) and those appearing here are the \( b_{\alpha\alpha} \cdot (\theta^\alpha \cdot 2\theta^\beta - 2\theta^\alpha \cdot 1\theta^\beta) \). A reduction of the analysis of the differences to the case of a square product yields the confrontation between \( b_{\alpha\alpha} \cdot (\theta^\alpha \cdot 1\theta^\beta - 1\theta^\alpha \cdot \theta^\beta) = 0 \) in all situations (I mean whatever \( K \) is) and \( b_{\alpha\alpha} \cdot (\theta^\alpha \cdot 1\theta^\beta - 1\theta^\alpha \cdot \theta^\beta) \), the result of which is now clearly depending on the nature of \( K \); in extenso: if \( K \) is a equipped with a commutative multiplication (e.g.: \( R \) or \( C \)), these \( b_{\alpha\alpha} \) terms vanish too. Otherwise (e.g.: \( K = H \) or \( O \)), they don’t automatically vanish.

2.3.3. Remark 12
Note that:

\[ (\theta^\alpha \wedge 2\theta^\beta) + (\theta^\mu \wedge 2\theta^\lambda) = (\theta^\alpha \cdot 2\theta^\beta - 2\theta^\alpha \cdot 1\theta^\beta) + (\theta^\mu \cdot 2\theta^\lambda - 2\theta^\mu \cdot 1\theta^\lambda) \]

Only if \( K = R \) or \( C \):

But if \( K = H \) or \( O \), supplementary elements must be added to the green and the blue terms (see explanations in the next §).

2.3.4. Remark 13: classifying the antisymmetric 2-forms
If the 2-form \( B \) is totally antisymmetric (which is corresponding to the interesting situations), three subcases can occur.

A) First subcase: the representation of \( B \) is a matrix with a vanishing diagonal; which means:

\[ B = \begin{bmatrix} 0 & b_{01} & b_{02} & b_{03} \\ -b_{01} & 0 & b_{12} & b_{13} \\ -b_{02} & -b_{12} & 0 & b_{23} \\ -b_{03} & -b_{13} & -b_{23} & 0 \end{bmatrix} \]

If \( K = R \) or \( C \), this is yielding:

\[ T_2(<\ldots, \ldots>\theta)(\theta, 2\theta) - T_2(<\ldots, \ldots>\theta)(2\theta, 1\theta) = T_2(\wedge B)(\theta, 2\theta) + T_2(\wedge B)(\theta, 2\theta)^t \]
Here \( \wedge B \) obviously denotes a deformation of the classical exterior product \( \wedge \) and that deformation is due to the antisymmetric (alternate) 2-form \( B \) which is, in some way, weighting the exterior product:

\[
\wedge B : V \times V \rightarrow K \mid \forall (u, w) \in V, \wedge B(u, w) = \sum_{\alpha < \beta} (u^\alpha \cdot w^\beta - u^\beta \cdot w^\alpha). b_{\alpha\beta}
\]

\( V \) is some vector space supposedly built on some leaf \( K \); the nature of which is primordial as explained previously.

**B) Second subcase:** the representation of \( B \) is a matrix with a vanishing trace; which means:

\[
B = \begin{bmatrix}
  b_{00} & b_{01} & b_{02} & b_{03} \\
  -b_{01} & b_{11} & b_{12} & b_{13} \\
  -b_{02} & -b_{12} & b_{22} & b_{23} \\
  -b_{03} & -b_{13} & -b_{23} & b_{33}
\end{bmatrix}
\]

with \( \sum_\alpha b_{\alpha\alpha} = 0 \)

In that subcase, the 2-form is amazingly reduced to 9 independent components which we may regroup in some (3-3) matrix (a topic left for later).

If \( K = \mathbb{R} \) or \( \mathbb{C} \), because the result obtained in the first part of the remark 12 holds true, this is yielding:

\[
T_2(<..., ...>)(1\theta, 2\theta) - T_2(<..., ...>)(2\theta, 1\theta) = \sum_\alpha b_{\alpha\alpha} \cdot [... (1\theta_\alpha^\mu \cdot 2\theta_\mu^\alpha - 2\theta_\lambda^\alpha \cdot 1\theta_\lambda^\mu)] + T_2(\wedge B)(1\theta, 2\theta) + T_2(\wedge B)(2\theta, 1\theta)
\]

Consequently:

- There is one supplementary linear combination of matrices in the r.h.t.; namely:

\[
\sum_\alpha b_{\alpha\alpha} \cdot [... (1\theta_\alpha^\mu \cdot 2\theta_\mu^\alpha - 2\theta_\lambda^\alpha \cdot 1\theta_\lambda^\mu)]
\]

- These new terms are antisymmetric matrices:

\[
(\mu, \lambda) \text{ term: } (1\theta_\mu^\alpha \cdot 2\theta_\lambda^\alpha - 2\theta_\mu^\alpha \cdot 1\theta_\lambda^\mu) = - (2\theta_\mu^\alpha \cdot 1\theta_\lambda^\mu - 2\theta_\lambda^\alpha \cdot 1\theta_\mu^\alpha) = - (1\theta_\mu^\alpha \cdot 2\theta_\mu^\alpha - 2\theta_\lambda^\alpha \cdot 1\theta_\lambda^\mu); \quad (\lambda, \mu) \text{ term}
\]

- They have a vanishing diagonal:

\[
(\lambda, \lambda) \text{ term: } (1\theta_\lambda^\alpha \cdot 2\theta_\lambda^\alpha - 2\theta_\lambda^\lambda \cdot 1\theta_\lambda^\lambda) = 0,
\]

- Consequently they have a vanishing trace.

**Otherwise, if \( K = \mathbb{H} \) or \( \mathbb{O} \),** these new terms are more difficult to characterize.

For the quaternions:

\[
\forall x^1, x^2 \in \mathbb{H}, (x^1 \cdot x^2 - x^2 \cdot x^1) = 2. i. (x^1_2, x^2_3 - x^3_2, x^3_2) + 2. j. (x^1_3, x^2_1 - x^1_3, x^2_1) + 2. k. (x^1_1, x^2_2 - x^1_2, x^2_2)
\]
For the octonions:

\[ \forall x^\mu \in \mathbb{O} \iff \forall \omega \in I_8: x^\mu = \sum_\omega i_{\omega} x^\mu_{\omega} \text{ and } x^\mu_{\omega} \in \mathbb{C} \]

After a little bit algebra we may state that (warning the Greek symbol \( \omega \) must not be confused with the zero 0):

\[
\forall x^1, x^2 \in \mathbb{O}, (x^1 \cdot x^2 - x^2 \cdot x^1) = (\sum_\omega i_{\omega} x^1_{\omega}) \cdot (\sum_\pi i_{\pi} x^2_{\pi}) - (\sum_\omega i_{\omega} x^2_{\omega}) \cdot (\sum_\pi i_{\pi} x^1_{\pi})
\]

\[
\downarrow \quad (x^1 \cdot x^2 - x^2 \cdot x^1) = \sum_\omega \sum_\pi i_{\omega} \cdot i_{\pi} \cdot (x^1_{\omega} \cdot x^2_{\pi} - x^2_{\omega} \cdot x^1_{\pi})
\]

Since the \( \{\ldots, i_\omega, \ldots\} \) define a Clifford’s algebra, we know that:

\[
\forall \omega, \pi \in I_8: i_{\omega} \cdot i_{\pi} + i_{\pi} \cdot i_{\omega} = 0 \text{ and } i_0 = 1, \quad (i_0)^2 = 1, \forall \omega \neq 0: (i_\omega)^2 = -1
\]

Furthermore, per hypothesis all \( (x^1_{\omega} \cdot x^2_{\pi} - x^2_{\omega} \cdot x^1_{\pi}) \) are complex numbers. So that, realizing a partition of the sum:

\[
(x^1 \cdot x^2 - x^2 \cdot x^1) = \sum_{\omega < \pi} \sum_{\pi} i_{\omega} \cdot i_{\pi} \cdot (x^1_{\omega} \cdot x^2_{\pi} - x^2_{\omega} \cdot x^1_{\pi})
\]

\[
\downarrow \quad (x^1 \cdot x^2 - x^2 \cdot x^1) = 2 \cdot \sum_{\omega < \pi} \sum_{\pi} i_{\omega} \cdot i_{\pi} \cdot (x^1_{\omega} \cdot x^2_{\pi} - x^2_{\omega} \cdot x^1_{\pi})
\]

We state that each term \( i_{\omega} \cdot i_{\pi} \) annihilates the \( i_{\pi} \cdot i_{\omega} \) term and that all others appear twice:

\[
\forall x^1, x^2 \in \mathbb{O}, (x^1 \cdot x^2 - x^2 \cdot x^1) = 2 \cdot \sum_{\omega < \pi} \sum_{\pi} i_{\omega} \cdot i_{\pi} \cdot (x^1_{\omega} \cdot x^2_{\pi} - x^2_{\omega} \cdot x^1_{\pi})
\]

This result generalizes the one obtained for the quaternions and can easily be itself once more time generalized to any Clifford’s super-algebra. It can now be applied to the \( b_{\alpha\alpha} \) terms of that approach.

C) And third subcase: the representation of B is a matrix with any trace; which means:

\[
B = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ -b_{01} & b_{11} & b_{12} & b_{13} \\ -b_{02} & -b_{12} & b_{22} & b_{23} \\ -b_{03} & -b_{13} & -b_{23} & b_{33} \end{bmatrix}, \quad \forall \text{ Trace } (B) = \sum_{\alpha} b_{\alpha\alpha}
\]

This case can easily be understood as a tiny generalization of the former.

2.3.5. Three applications for the first subcase on \( \mathbb{R} \) or \( \mathbb{C} \)

A) A first and direct application of the results is immediate in writing:

\[
N = 4, V = E_4(\mathbb{C}), (\partial_\alpha \Omega, \partial_\beta \Omega) = (\partial_\alpha \Omega, \partial_\beta \Omega)
\]
This implies:

$$T_2(\langle..., ..., \rangle_b(\vec{c}_\alpha \Omega, \vec{c}_\beta \Omega) - T_2(\langle..., ..., \rangle_b(\vec{c}_\beta \Omega, \vec{c}_\alpha \Omega) = T_2(\land B)(\vec{c}_\alpha \Omega, \vec{c}_\beta \Omega) + T_2(\land B)(\vec{c}_\beta \Omega, \vec{c}_\alpha \Omega)^t$$

B) There is another application in writing:

$$N = 4, V = E_4(\omega), 1 \theta = 2 \theta = (\text{grad}_x e_0, \text{grad}_x e_1, \text{grad}_x e_2, \text{grad}_x e_3) = \text{grad}_x \Omega$$

This implies:

$$T_2(\langle..., ..., \rangle_b(\langle 1 \theta, 2 \theta \rangle) - T_2(\langle..., ..., \rangle_b(\langle 2 \theta, 1 \theta \rangle) = T_2(\land B)(\langle 1 \theta, 2 \theta \rangle) + T_2(\land B)(\langle 2 \theta, 1 \theta \rangle)^t$$

$$
\begin{align*}
&= T_2(\land B)(\text{grad}_x \Omega, \text{grad}_x \Omega) - T_2(\langle..., ..., \rangle_b)\text{grad}_x \Omega, \text{grad}_x \Omega) \\
&= T_2(\land B)(\text{grad}_x \Omega, \text{grad}_x \Omega) + T_2(\land B)(\text{grad}_x \Omega, \text{grad}_x \Omega)^t
\end{align*}
$$

C) Where is the link with the GTR2? In fact the two previous applications are quite interesting but also quite too general for the followed purpose. Let me consider a third application:

$$N = 1, V = E_4(\omega), \forall \nu = 0, 1, 2, 3: 1 \theta = 2 \theta = \text{grad}_x (e_\nu)$$

This is yielding for each value of $\nu$:

$$T_2(\langle..., ..., \rangle_b)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) - T_2(\langle..., ..., \rangle_b)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) = T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) + T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu))^t$$

Consequently:

$$\exists. t^\nu. \{T_2(\langle..., ..., \rangle_b)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) - T_2(\langle..., ..., \rangle_b)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) = T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) + T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu))^t\}$$

This is introducing a first concrete example of “square products” (see the remark 08). At a first glance, I may think that the l.h.t. vanishes. Is it true? Whatever $1 \theta = 2 \theta$ is, $[... < 1 \alpha, 1 \beta >_b - < 2 \alpha, 2 \beta >_b ...] = 0$; that’s true. This is evidently limiting the attraction for that subcase which is only telling:

$$T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) + T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu))^t = 0$$

Furthermore, whatever $1 \theta = 2 \theta$ is, $[... < 1 \alpha, 1 \beta >_b - < 1 \alpha, 1 \beta >_b ...] = 2. [... < 1 \alpha, 1 \beta >_b ...]$. So that this kind of antisymmetric 2-form yields:

$$T_2(\land B)(\text{grad}_x (e_\nu), \text{grad}_x (e_\nu)) = 0$$

This is a set of ten vanishing extended products:

$$\land B(\vec{c}_\alpha e_\nu, \vec{c}_\beta e_\nu) = \sum_{\alpha < \beta} (\vec{c}_\alpha e_\nu, \vec{c}_\beta e_\nu - \vec{c}_\beta e_\nu, \vec{c}_\alpha e_\nu). b_{\alpha\beta} = 0$$
2.3.6. Application to the EM fields of the GTR2

For an application to the EM fields predicted by the GTR2, it would be more pertinent to end the calculation started § 2.3.2:

\[ \forall (\theta, 2\theta) \in V^N \times V^N \]

\[ T_2(<..., ..., >)(\theta, 2\theta) - T_2(\theta, 1\theta) \]

\[ = [... <\theta_{\mu}, 2\theta_{\mu} >_B - <2\theta_{\mu}, 1\theta_{\mu} >_B ...] \]

\[ = [... b_{\alpha\beta} <(\theta_{\alpha} \wedge 2\theta_{\beta} - 2\theta_{\beta} \wedge 1\theta_{\alpha}) ...] \]

\[ = [... \sum_\alpha b_{\alpha\alpha} (\theta_{\alpha} \wedge 2\theta_{\alpha} - 2\theta_{\alpha} \wedge 1\theta_{\alpha} ...] + [... \sum_{\alpha < \beta} b_{\alpha\beta} (\theta_{\alpha} \wedge 2\theta_{\beta} - 2\theta_{\beta} \wedge 1\theta_{\alpha}) ...] \]

\[ = [... \sum_\alpha b_{\alpha\alpha} (\theta_{\alpha} \wedge 2\theta_{\alpha} - 2\theta_{\alpha} \wedge 1\theta_{\alpha} ...] + [... \wedge B(\theta_{\alpha} \wedge 2\theta_{\beta} - 2\theta_{\beta} \wedge 1\theta_{\alpha}) ...] + T_2(\wedge B)(\theta, 2\theta) - T_2(\wedge B)(\theta, 1\theta) \]

Let me remark that whatever \( K \) is:

\[ (\theta_{\alpha} \wedge 2\theta_{\alpha})^{\alpha\beta} - (2\theta_{\alpha} \wedge 1\theta_{\alpha})^{\alpha\beta} = (\theta_{\alpha} \wedge 2\theta_{\alpha} - 2\theta_{\alpha} \wedge 1\theta_{\alpha})^{\alpha\beta} - (2\theta_{\alpha} \wedge 1\theta_{\alpha})^{\alpha\beta} \]

With that statement, it is possible to write:

\[ \forall (\theta, 2\theta) \in V^N \times V^N \]

\[ T_2(<..., ..., >)(\theta, 2\theta) - T_2(\theta, 1\theta) \]

\[ = [... \sum_\alpha b_{\alpha\alpha} (\theta_{\alpha} \wedge 2\theta_{\alpha} - 2\theta_{\alpha} \wedge 1\theta_{\alpha} ...] + [... \sum_{\alpha < \beta} b_{\alpha\beta} (\theta_{\alpha} \wedge 2\theta_{\beta} - 2\theta_{\beta} \wedge 1\theta_{\alpha}) ...] \]

\[ = [... \sum_\alpha b_{\alpha\alpha} (\theta_{\alpha} \wedge 2\theta_{\alpha} - 2\theta_{\alpha} \wedge 1\theta_{\alpha} ...] + [... \wedge B(\theta_{\alpha} \wedge 2\theta_{\beta} - 2\theta_{\beta} \wedge 1\theta_{\alpha}) ...] + T_2(\wedge B)(\theta, 2\theta) - T_2(\wedge B)(\theta, 1\theta) \]

Let me now reduce that general result to the special case:

\[ N = 1, V = E_4(\mathbb{C}), \forall \nu = 0, 1, 2, 3: 3\theta = 2\theta = \text{grad}_4(e_\nu) \]

This is the best way to recover the formalism of the EM fields predicted by the GTR2; of course:

\[ [... F_{\mu\nu} ...] \]

\[ = \forall: T_2(<..., ..., >)(\text{grad}_4(e_\nu), \text{grad}_4(e_\nu)) - T_2(\theta, 1\theta)(\text{grad}_4(e_\nu), \text{grad}_4(e_\nu)) \]

Within the GTR2 approach there a priori no indication concerning how many 2-forms \( B \) are potentially able to generate an EM field. Consequently, in an attempt to rebuild the EM fields of the QCD with those of the GTR2, I may (why not?) try to consider either 3 (for a link with SU(2)) or 8 (for
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a link with SU(3)) or 15 (for a link with SU(4)) different GTR2-fields. At the level of principles, this can
be symbolized with:

\[ \ldots [\ldots F_{\lambda\mu}^a \ldots] = \frac{1}{2} \cdot \llbracket T_2(\wedge B^3)(\text{grad}_4(e_\nu), \text{grad}_4(e_{\nu})) - T_2(\wedge B^3)(\text{grad}_4(e_{\nu}), \text{grad}_4(e_{\nu})) \rrbracket \]

Since, coming back to the (important) details, that kind of hypothesis implies e. g.:

\[ \wedge B^2: E_4(K) \times E_4(K) \rightarrow K \mid \forall (u, w) \in E_4(K), \wedge B^2(u, w) = \sum_{\alpha < \beta} (u^\alpha \cdot w^\beta - u^\beta \cdot w^\alpha) \cdot b_{\alpha\beta}^a, \]

the notion of exterior extended product (EEP – see definition 03) may eventually be reintroduced here. But the idea must be manipulated carefully and, for this purpose, it is crucial to count how many terms are present in an antisymmetric cube. The answer is evidently dimension-dependent and, as already evoked at the very beginning of that document, if \( D = 4 \), an antisymmetric cube \((4) \) is reduced to a set of 24 components which can be disposed in diverse manners. When \( D = 4 \), each given \( \wedge B^2(u, w) \) brings 6 components and it can easily be checked that one of these manners is realized in considering a set of four \( \wedge B^2(u, w) \) for \( a = 0, 1, 2, 3 \). At a first glance, this statement annihilates the hope of some reasonable junction with one of the three evoked options: SU(2), SU(3) or SU(4). I am convinced that it is just an illusion due to a lack of analysis. Let me try to explain.

If of course, in our customized understanding of a four dimensional space-time, there can only be one tetrad at a given place and at a given instant, nothing forbids the interaction of several physical objects at this place and at this instant; even if immediately after that instant the diverse objects follow separate destinies. This is in fact exactly what occurs when two or more particles interact together. With that viewpoint, it becomes meaningful to look for diverse partitions of 24 components. I shall come back a little bit later on that strategic topic.

2.3.7. Remark 14: Justifying a reexamination of E.B. Christoffel’s work (1869) applied to the EM fields
For now, I would just do a seemingly meaningless remark. Let me calculate the expression:

\[ \langle \text{d}x \mid \ldots [\ldots F_{\lambda\mu} \ldots] \rangle \]

Here \text{d}x is an ordinary variation of “position” in space and time. Adopting the classical \((2, 0)\) tensorial representation for the EM field, there is no difficulty to state that:

\[ \forall \text{d}x, \forall F : \langle \text{d}x \mid \ldots [\ldots F_{\lambda\mu} \ldots] \rangle \rangle = 0 \]

This quantity is invariant and must be preserved after any change of frame. This is one of the argument because I am pushed to reconsider the E. B. Christoffel’s work (1869) when the latter is applied, not to the usual metric measuring distances, but to a local connection preserving the fundamental quantum state (= the realization of the HUP at the quantum limit). That local connection is actually an unknown one which is written \( \nabla X \). That idea has been exposed in [f].

2.3.8. Remark 15: Constraining the formalism of the GTR2-EM Fields
Let me calculate:

\[ \forall \nu, \forall a, \forall \Omega, \forall \text{d}x: \langle \text{d}x \mid \ldots [\ldots T_2(\wedge B^3)(\text{grad}_4(e_\nu), \text{grad}_4(e_{\nu})) - T_2(\wedge B^3)(\text{grad}_4(e_{\nu}), \text{grad}_4(e_{\nu})) \rangle \text{d}x \rangle \]
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Since the B^a are such that:

\[ \bigwedge B^a : E_4(K) \times E_4(K) \to K \mid \forall \text{grad}_e(e_v) \in E_4(K), \]
\[ \bigwedge B^a(\partial_\lambda e_v, \partial_\mu e_v) = \sum_{\alpha < \beta} (\partial_\lambda e_v^\alpha \partial_\mu e_v^\beta - \partial_\lambda e_v^\beta \partial_\mu e_v^\alpha) \cdot b_{\alpha\beta}^a, \]

And since the B^a are bilinear 2-forms:

\[ dx^\lambda \cdot dx^\mu \cdot \bigwedge B^a(\partial_\lambda e_v, \partial_\mu e_v) = (dx^\lambda)^2 \cdot \sum_{\alpha < \beta} (\partial_\lambda e_v^\alpha \partial_\mu e_v^\beta - \partial_\lambda e_v^\beta \partial_\mu e_v^\alpha) \cdot b_{\alpha\beta}^a \]
\[ \downarrow \]
\[ dx^\lambda \cdot dx^\mu \cdot \bigwedge B^a(\partial_\lambda e_v, \partial_\mu e_v) = \sum_{\alpha < \beta} (\partial_\lambda e_v^\alpha \partial_\mu e_v^\beta \cdot dx^\lambda \cdot dx^\mu - \partial_\lambda e_v^\beta \partial_\mu e_v^\alpha \cdot dx^\lambda \cdot dx^\mu) \cdot b_{\alpha\beta}^a \]

Let me remark that:

\[ (de_v^\alpha). (de_v^\beta) = (\partial_\lambda e_v^\alpha \cdot dx^\lambda). (\partial_\mu e_v^\beta \cdot dx^\mu) \]
\[ \frac{de_v^\alpha}{dx^\lambda} = T_{v_\lambda}^\pi. e_\pi \leftrightarrow \partial_\lambda e_v^\alpha = T_{v_\lambda}^\pi. e_\pi^\alpha \]

This is allowing:

\[ dx^\lambda \cdot dx^\mu \cdot \bigwedge B^a(\partial_\lambda e_v, \partial_\mu e_v) = \sum_{\alpha < \beta} (de_v^\alpha \cdot de_v^\beta - de_v^\beta \cdot de_v^\alpha) \cdot b_{\alpha\beta}^a \]

And I may write it:

\[ \bigwedge B^a(de_v, de_v) = \sum_{\alpha < \beta} (de_v^\alpha \cdot de_v^\beta - de_v^\beta \cdot de_v^\alpha) \cdot b_{\alpha\beta}^a \]

There is no better way to understand the influence of K on that expression. It is obviously a vanishing expression each time K is equipped with a commutative multiplication, whatever the reduced cube \( \nabla B \) is! This is per se not embarrassing since that vanishing is the expected result.

But the exploration would not be complete if it wouldn’t include the cases for which \( K = \mathbb{Q} \) or \( O \). I have demonstrate in § 2.3.4 (above remark 13) that in the latters, factors in front of the components of the reduce cube \( \nabla B \) don’t systematically vanish. The relation: \( < dx \mid \ldots F_{\lambda\mu} \ldots \mid dx > = 0 \) consequently systematically needs the intervention of a subtle balance between the \( t^\nu \) and the \( b_{\alpha\beta}^a \).

\[ \forall a \in \text{Idim}(SU(N)) : \frac{1}{2} \sum_0^3 t^\nu \cdot \bigwedge B^a(de_v, de_v) = 0 \]

Please just also remark that if the four \( t^\nu \) would be \textit{invariant quantities} such that:

\[ t^\nu = (x^\nu)^2 \]

then the relation would become:

\[ \forall a \in \text{Idim}(SU(N)) : \sum_0 \bigwedge B^a(dx, dx) = 0 \]
2.4. Fundamental proposition: A link “QCD-Gravitation”
This formalism is evidently evoking the so-called gravitational term appearing in the Lorentz-Einstein law (LEL). I have recently presented an exploration in that direction for the triads ($[g]$ – involving the quaternions) and for the tetrads ($[h]$ – involving the octonions). The exploration (personal work) is yet under development. My intuition is that the gluons of the QCD are closely related to gravitational phenomenon.

As the reader may state by himself in surfing through the recent literature, the idea is far to be meaningless at low energies; e.g.: see references [07].

3. C* algebras for the theory

3.1. Epimorphisms, intern derivations
I want to scrutinize now the properties of the newly discovered adjoint representation. A lot of simple remarks can be done.

3.1.1. Remark 16
The image via $[A] \Phi$ of the zero of $\{E_4(\mathbb{C}), [A] \wedge \}$ is the zero of $M_4(\mathbb{C})$.

$$0 \in \{E_4(\mathbb{C}), [A] \wedge \} ; [A] \Phi ([4]0) = [4]0$$

The converse is not necessarily true.

$$[A] \Phi ([4]a) = [4]0 \Rightarrow A_{\alpha \beta}^\alpha . a^\gamma = 0$$

This is only a set of 16 linear combinations of the four $a^\gamma$. In writing the trivial decomposition of the $[A] \wedge ([4]a, [4]…)$ extended Lie’s products in extenso, this is:

$$\nabla A \Phi ([4]a) = \begin{bmatrix} A_{\gamma 0}^0 . a^\gamma & A_{\gamma 1}^0 . a^\gamma & A_{\gamma 2}^0 . a^\gamma & A_{\gamma 3}^0 . a^\gamma \\ A_{\gamma 0}^1 . a^\gamma & A_{\gamma 1}^1 . a^\gamma & A_{\gamma 2}^1 . a^\gamma & A_{\gamma 3}^1 . a^\gamma \\ A_{\gamma 0}^2 . a^\gamma & A_{\gamma 1}^2 . a^\gamma & A_{\gamma 2}^2 . a^\gamma & A_{\gamma 3}^2 . a^\gamma \\ A_{\gamma 0}^3 . a^\gamma & A_{\gamma 1}^3 . a^\gamma & A_{\gamma 2}^3 . a^\gamma & A_{\gamma 3}^3 . a^\gamma \end{bmatrix}$$

3.1.2. The kernel of $\varepsilon \Phi$
As special case for a double reduced cube $[4] \nabla A$, the 64 components of $[4] \nabla A$ are now reduced to a generic element of $E_4(\mathbb{C})$ which I shall write per convention:

$$\begin{array}{c}
\mathbb{C} \times \mathbb{C} \times \mathbb{C} \xrightarrow{A_{\alpha \beta}^\gamma + A_{\mu \nu}^\gamma = 0} \mathfrak{su}(5) \xrightarrow{A_{\alpha \beta}^\gamma + A_{\alpha \nu} = 0} E_4(\mathbb{C}) \\
[4] \nabla A \xrightarrow{R} \{4] : (A_{12}^3, - A_{23}^0, - A_{31}^0, - A_{12}^0) \\
\bigcirc \xrightarrow{\Delta} \bigcirc ; \forall (4][a] \in E_4(\mathbb{C}) ; c^0 \Phi ([4]a) \in \mathfrak{R}(E_4(\mathbb{C}))^{12} \end{array}$$

Here $\mathfrak{R}(E_4(\mathbb{C})^{12})$ is some representation of the exterior algebra $E_4(\mathbb{C})^{12}$ of $E_4(\mathbb{C})$ because the trivial divisor of a double reduced cube writes:
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\[ \Phi((4)a) = \begin{bmatrix}
0 & A_{21}^0.a^2 + A_{31}^0.a^3 & A_{12}^0.a^1 + A_{32}^0.a^3 & A_{13}^0.a^1 + A_{23}^0.a^2 \\
A_{20}^1.a^2 + A_{30}^1.a^3 & 0 & A_{02}^0.a^0 + A_{32}^1.a^3 & A_{03}^1.a^0 + A_{23}^1.a^2 \\
A_{10}^2.a^2 + A_{30}^2.a^3 & A_{01}^2.a^0 + A_{31}^2.a^3 & 0 & A_{03}^2.a^0 + A_{13}^2.a^1 \\
A_{10}^3.a^3 + A_{20}^3.a^2 & A_{01}^3.a^0 + A_{21}^3.a^2 & A_{02}^3.a^0 + A_{12}^3.a^1 & 0
\end{bmatrix} \]

This obviously is the representation of some bi-vector (evident demonstration because of the properties of a double reduced cube). In that case, the vanishing of that trivial split is equivalent to a set of six relations from which it may easily be inferred that, probably, all components of a non-vanishing \((4)a\) are proportional to each other’s in diverse ratios furnished by the presupposed non-vanishing four components of \((4)c\). Indeed taking the relations in positions (2, 3), (1, 3) and (1, 2):

\[ a^1 = -\frac{A_{23}^0}{A_{12}^3}.a^0; a^2 = -\frac{A_{13}^0}{A_{12}^3}.a^0; a^3 = -\frac{A_{12}^0}{A_{12}^3}.a^0 \]

This fortunately induces the three remaining relations in positions (0, 1), (0, 2) and (0, 3). Consequently for a double reduced cube, the relation \(\Phi((4)a) = (4)[0]\) induces the existence of a line with the main direction:

\[ (1, -\frac{A_{23}^0}{A_{12}^3}, -\frac{A_{13}^0}{A_{12}^3}, -\frac{A_{12}^0}{A_{12}^3}) \]

This means that: as long as the cube doesn’t vary, that direction stays invariant. What is quite more funny with that viewpoint is the fact that the double reduced cube induced the inverse image of any vanishing \(\Phi((4)a)\). With other words: \((4)c\) may be interpreted as the kern of \(\Phi\).

\[ \text{Kern } \Phi = \{(4)a \in E_4(\mathbb{C}) | (4)a = a^0. (4)c \text{ and } a^0 \in \mathbb{C} \} \]

As a matter of facts \((4)0 \in \text{Kern } \Phi\) but is not reduced to it. At this stage, the reader may be lost and he may also ask where the red path for this document is. The fact is that I am looking for the conditions allowing considering that \(\Phi\) is an epimorphism. Per definition this is obtained if, for at least one non-empty subset \(\zeta\) of \(M_4(\mathbb{C})\), I can define a family of elements in \(E_4(\mathbb{C})\), say \((4)s\), such that:

\[ \forall (4)[M] \in \zeta \subseteq M_4(\mathbb{C}), \exists! (4)s \in E_4(\mathbb{C}) | \zeta \neq \emptyset \text{ and } |(4)\Phi^{-1}(4)[M]| = (4)s \]

Here “\(\exists!\)” means: “There exists at least one”.  

3.1.3. Going a little bit further

The fact that any “point” belonging to the line generated by a reduced cube is projected on a fixed element of \(M_4(\mathbb{C})\), here namely: \((4)[0]\), may also suggest that \(\Phi\) defines perhaps fibers when it projects vectors not belonging to its kernel. But nothing is so evident and I shall only remark, coming back to \([02] \text{§ 4, pp. } 12-15\), that the \(\mathbb{C}\)-Lie algebra \(\mathfrak{sl}(5, \mathbb{C})\) can always be decomposed into two parts \([02] \text{§ 4, corollary } 4.5, (4.9)\); the first one is a Cartan sub-algebra \(\mathfrak{h}\) and the second one is defined in \([02] \text{§ 4}\); I shall write it \(X\) for simplicity:

\[ \mathfrak{sl}(5) = \mathfrak{h} \oplus X \]
This fact is inspiring another idea concerning the description of the kernel of \( \phi \). With other words if one of the five elements of the diagonal matrix \( H_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} \) is systematically zero per convention whilst the four others are coinciding with the components of the double reduced cube, - in extenso if, e.g.: \( H_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} = H_{0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5} \) it may also be though that that kernel is perhaps related to that natural partition.

\[
\begin{bmatrix}
\sum \text{[A]} & \in & \mathfrak{sl}(5) \\
\sum \text{[X]} & \in & \mathcal{X}
\end{bmatrix}
\]

\[\forall (4) a: \sum \phi (\text{[a]}), \sum \phi (\text{[X]})\]

### 3.1.4. History
In \([a]\) I have introduced notations and basic mathematical operations exhibiting clearly the idea that any ELP may be understood as a deformed Pythagorean table. There is a pleasant coincidence between that fact and the fact that E. Cartan did already introduce a long time ago (1937) these tables in \([05]; \S\S \text{14 and 15, pp. 15 -17}\). His idea was to consider a set of \( D \) vectors, e.g.: \( e_0 \) (and this could have been the \( D \) basis vectors of a canonical basis \( \Omega \)), more precisely: the contravariant coordinates of these vectors and to put them inside a table of which one calculates the determinant:

\[
\Delta = \ldots (e_A)^\alpha \ldots \text{ with } A, \alpha = 0, 1, 2, \ldots, D - 1
\]

On the same vein, he formed the table with the covariant coordinates of the same vectors:

\[
\Delta' = \ldots (e_B)_\beta \ldots \text{ with } B, \beta = 0, 1, 2, \ldots, D - 1
\]

In a next step, he then calculated the product of the determinants and remarked that:

\[
\Delta \Delta' = \ldots (e_A)^\alpha (e_B)_\beta \ldots = \ldots < e_A, e_B > \text{ \_ \_}_G \ldots
\]

is the volume of the “hyper-parallelepiped” formed by the multi-vector \( (\ldots, e_A, \ldots) \equiv \Omega \); here: \( < \ldots, \ldots >_G \) denotes a so-called “scalar product” on \( E_0(K) \) built with the help of \( [G] \).

### 3.1.5. Definition 04: Pythagorean table (basic definition)
The simplest version of a Pythagorean table is a function linking two objects via an operation of composition. Example given: the source is a pair of multi-vectors. The image is a matrix.

### 3.1.6. Proposition
The volume of the “hyper-parallelepiped” formed by the multi-vector \( (\ldots, e_A, \ldots) \equiv \Omega \) introduced by E. Cartan in \([05]\) can be translated in the language of Pythagorean tables.
3.1.7. Demonstration
Up to now, because I want to focalize on physical applications of the theory, I consider that $\Omega$ is a canonical basis for $E_0(R)$. The demonstration is evident and as a matter of facts:

$$[G] = T_2(<\ldots, \ldots>)(\Omega, \Omega) \in M_0(R) \text{ and } V \text{ (for volume) } = \Delta, \Delta' = |T_2(<\ldots, \ldots>)(\Omega, \Omega)| \in R$$

3.1.8. Remark 17: Pythagorean matrix built with the exterior product
Instead of the scalar product, I could have involved the exterior product $\wedge$ and I would have obtained another kind of Pythagorean matrices; namely:

$$T_2(\wedge)(\Omega, \Omega) = \begin{pmatrix} \ldots & e_A \wedge e_B & \ldots \\ \ldots & 0 & \ldots \\ -e_A \wedge e_B & \ldots & 0 \end{pmatrix} \in M_0(E_0(R)^2)$$

Taking care of the basic properties of the exterior product (anti-symmetry), it is obvious that that matrix is an anti-symmetric one with a vanishing diagonal:

$$T_2(\wedge)(\Omega, \Omega) = \begin{pmatrix} 0 & \ldots & e_A \wedge e_B \\ \ldots & 0 & \ldots \\ -e_A \wedge e_B & \ldots & 0 \end{pmatrix} \in M_0(E_0(R)^2)$$

It is a square matrix containing $\frac{1}{2} (D^2 - D)$ vectors of $E_0(R)$.

3.1.9. Remark 18: Pythagorean matrix built with the extended Lie’s product
In the same vein I could have built a Pythagorean table with the extended Lie’s product (ELP) and I would have obtained (the cube $[A] \wedge$ is anti-symmetric per construction/definition of the ELP):

$$T_2([A] \wedge)(\Omega, \Omega) = \begin{pmatrix} 0 & \ldots & A_{\alpha\beta}^{\gamma} \cdot \{(e_A)^{\alpha}.(e_B)^{\beta} - (e_A)^{\beta}.(e_B)^{\alpha}\}.e_{\gamma} \\ \ldots & 0 & \ldots \\ -A_{\alpha\beta}^{\gamma} \cdot \{(e_A)^{\alpha}.(e_B)^{\beta} - (e_A)^{\beta}.(e_B)^{\alpha}\}.e_{\gamma} & \ldots & 0 \end{pmatrix} \in M_0(E_0(R))$$

Exactly as $T_2(\wedge)(\Omega, \Omega)$, because of the natural anti-symmetry of $\wedge$, this is also an anti-symmetric (D-D) matrix with a totally vanishing diagonal.

3.1.10. Remark 19: Guessing a representation for $X$ – first approach concerning a $C^*$-algebra
Are $T_2(\wedge)(\Omega, \Omega)$ or $T_2([A] \wedge)(\Omega, \Omega)$ good candidates for a representation of $X$ when $D = 5$? I shall come back later on that question and only remark that each matrix $T_2(\wedge)(\Omega, \Omega)$ or $T_2([A] \wedge)(\Omega, \Omega)$ is a set of $\frac{1}{2} (D^2 - D) = 10$ vectors while $[5]X$ is $[5,5]$ matrix with $(D^2 - D) = 20$ elements in $C$. Consequently there are two problems if an answer to the asked question is wanted.

First technical problem: an element of $E_0(C)$ is not a complex number, in extenso: an element of $C$. At this stage, an isomorphism between $C$ and $E_0(C)$ is missing. Fortunately there is the “Gelfand-Mazur” theorem [ii], [06].
Second, even if such an isomorphism would be accessible within the TEQ (see my comments below), it would be a necessity to work with two matrices $T_2(\wedge)(\Omega, \Omega)$, or $T_2(\wedge)(\Omega, \Omega)$, each of them being an element of $M_0(E_0(C))$. This can be mechanically obtained if the theory is developed on $M_0(E_0(C))$; indeed, this spontaneously furnishes a real and an imaginary part, each of them being a $(5-5)$ matrix with $\frac{1}{2}$. $(D^2 - D) = 10$ vectors.

$$T_2(\wedge)(\Omega, \Omega) = \begin{pmatrix} 0 & \cdots & A_{\alpha\beta}^\gamma \cdot \{(e_A)^\alpha \cdot (e_B)^\beta - (e_A)^\beta \cdot (e_B)^\alpha\} \cdot e_\gamma \\ \cdots & \ddots & \cdots \\ -A_{\alpha\beta}^\gamma \cdot \{(e_A)^\alpha \cdot (e_B)^\beta - (e_A)^\beta \cdot (e_B)^\alpha\} \cdot e_\gamma & \cdots & 0 \end{pmatrix} \in M_0(E_0(C))$$

This means that $A_{\alpha\beta}^\gamma \cdot \{(e_A)^\alpha \cdot (e_B)^\beta - (e_A)^\beta \cdot (e_B)^\alpha\} \in C$. Is the set $M_0(E_0(C))$ a $C^*$-algebra?

3.1.11. Remark 20: a simplified notation

Prior considerations have showed:

$$\forall \{D\} \forall A \in C^3 | A_{\alpha\beta}^\gamma + A_{\beta\alpha}^\gamma = 0, \forall u, \forall w \in E_0(C); [u, w]_{\wedge A} = 2 \cdot \Box_{\wedge A}(u, w)$$

In the reduction of that theory to the cases $D = 3$ and $D = 4$ that identity writes:

$$\forall \{D\} \forall A \in C^3 | A_{\alpha\beta}^\gamma + A_{\beta\alpha}^\gamma = 0, \forall u, \forall w \in E_0(C); [u, w]_{[A]} = 2 \cdot A_{\wedge}[u, w]$$

The application of that identity to the above matrix yields:

$$T_2(\wedge)(\Omega, \Omega) = \frac{1}{2} \begin{pmatrix} 0 & \cdots & [e_A, e_B]_{[A]} \\ \cdots & \ddots & \cdots \\ -[e_A, e_B]_{[A]} & \cdots & 0 \end{pmatrix} \in M_0(E_0(C))$$

It is now a little bit easier to work with.

3.1.12. Remark 21: the volumes

One of the most important applications of all these extrapolations is the calculation of deformed volumes. In a three dimensional space ($D = 3$), “things” may be easily represented and that’s why we shall start with. Precisely, we work with a $(3-3)$ square matrix $^{(3)}[A]$ and:

$$T_2(\wedge)(\Omega, \Omega) = \frac{1}{2} \begin{pmatrix} 0 & [e_1, e_2]_{[A]} & [e_1, e_3]_{[A]} \\ -[e_1, e_2]_{[A]} & 0 & [e_2, e_3]_{[A]} \\ -[e_1, e_3]_{[A]} & -[e_2, e_3]_{[A]} & 0 \end{pmatrix} \in M_3(E_3(C))$$

For the classical (= Euclidian) configuration we know that:

$$[A] = [J]$$

As a matter of facts nothing forbids the calculation of:
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3.1.13. Definition 05: extern slide acting on the left side of $M_0(E_0(R)^{-2})$

The classical definition of an exterior product is such that, for each tensorial component:

$$(e_A \wedge e_B)^{\alpha\beta} = (e_A)^{\alpha} \cdot (e_B)^{\beta} - (e_A)^{\beta} \cdot (e_B)^{\alpha},$$

At a first glance, the extended Lie’s product seems to be a deformed exterior product. That sensation can perhaps be formalized with the concept of “extern slide acting on the left side” of matrices (in a similar way than the one I have used in [a]). Once more time, the greatest attention must be given to the details in realizing such idea. Each component of $T_2([A] \wedge \Omega, \Omega)$ is a vector of $E_0(R)$ because the reduced cube $\Box \nabla A$ in some way “projects” the exterior products $e_A \wedge e_B$, elements of the exterior algebra $E_0(R)^{-2}$, in $E_0(R)$. On the other hand, each component of $T_2(\wedge \Omega, \Omega)$ is an element of $E_0(R)^{-2}$. This statement shows that the concept of extern slide acting on the left side must be applied very carefully. Here, I may write for convenience:

3.2. Context

In 2008, I have explored the possibility to get a C*algebras structure for $\{E_4(C), \triangledown \nabla A\}$ (see [b], [c] and [d]) where $\nabla A$ was any cube; with other words: not necessarily an anti-symmetric one. There were two main difficulties: 1°) the definition of a unit element; 2°) the definition of a norm. My purpose is to restart this exploration for $\{E_4(C), [A] \wedge \bullet\}$. 

3.3. Associativity

3.3.1. Definition

$$[A] \wedge ([A] \wedge (x, y, z)) = [A] \wedge ([A] \wedge (x, y), z)$$

3.3.2. Consequence

Since $\{E_4(C), [A] \wedge \bullet\}$ is supposed to be a Lie algebra, the Jacobi’s relation holds true and:

$$[A] \wedge ([A] \wedge (y, z)) + [A] \wedge ([A] \wedge (z, x)) + [A] \wedge ([A] \wedge (x, y)) = 0$$

The associativity implies in peculiar that:

$$[A] \wedge ([A] \wedge (x, y), z)) + [A] \wedge ([A] \wedge (y, [A] \wedge (z, x))) + [A] \wedge ([A] \wedge (z, [A] \wedge (x, y))) = 0$$

Since the extended Lie’s products are anti-commutative, the first term annihilates the third term:

$$[A] \wedge ([A] \wedge (y, [A] \wedge (z, x))) = 0$$

This is an unavoidable consequence of the associativity. More precisely:

$$\sum_e \sum_{\lambda < \mu} A_{eA}^{\lambda}. (y^\lambda \cdot [A] \wedge ([A] \wedge (z, x))^{\alpha} - y^\mu \cdot [A] \wedge (z, x)^{\alpha}) = 0$$

$$\sum_e \sum_{\lambda < \mu} A_{eA}^{\lambda}. (y^\lambda \cdot \sum_e \sum_{\alpha < \beta} A_{eA}^{\alpha \beta}. (z^{\alpha} \cdot x^{\alpha} - z^{\beta} \cdot x^{\beta})) - y^\mu \cdot \sum_e \sum_{\alpha < \beta} A_{eA}^{\alpha \beta}. (z^{\alpha} \cdot x^{\alpha} - z^{\beta} \cdot x^{\beta})) = 0$$

$$\sum_e \sum_{\alpha < \beta} \sum_e \sum_{\lambda < \mu} (A_{eA}^{\lambda} \cdot y^\lambda \cdot A_{eA}^{\alpha \beta}. \cdot A^{\alpha \beta}^{\lambda} \cdot (z^{\alpha} \cdot x^{\alpha} - z^{\beta} \cdot x^{\beta})) = 0$$
The associativity is not easily and systematically obtained. I mean with this sentence: “The associativity doesn’t hold true for every triplet \((x, y, z)\) in \(\{E_4(C), [A] \wedge \}\) and the territories where that property is correct must be discovered and precisely defined.” There is at least a subset of \(\mathcal{P}(E_4(C)), [A] \wedge \) - where \(\mathcal{P}(\ldots)\) is the set of all parts of \(\ldots\) - for which that property is evident; namely:

\[ \mathcal{A} = \{ (x, y, z) \in \{E_4(C), [A] \wedge \}^3 \mid y = \pm [A] \wedge (z, x) \subset \{E_4(C), [A] \wedge \} \} \]

One of the three elements of the triplet \((x, y, z)\), more exactly the second one, must be plus or minus time the extended Lie product of the third one by the first one. Remark that any element of \(\{E_4(C), [A] \wedge \}\) is in some way generating an associative triplet. Despite of these precisions, all has not yet been said on that topic.

3.3.3. Vanishing extended Lie product

The previous discussion shows the importance to know when an ELP vanishes or not. The identity:

\[ [A] \wedge (z, x) = 0 \]

writes in extenso and per definition:

\[ \sum_{i \alpha < \rho} \sum_{i} A_{i \alpha}^{\lambda} \cdot (z^{\alpha} \cdot x^{\rho} - z^{\rho} \cdot x^{\alpha}) = 0 \]

From this, it is trivial to get:

\[ z = x \Rightarrow [A] \wedge (z, x) = 0 \]

Unfortunately, the converse is false because sums of six non vanishing terms may finally vanish.

Conclusion

This personal work is still in development and, unfortunately, no definitive conclusion can be yet given. But proofs of the existence of a deep correspondence between gluons and gravitational effects are accumulated.
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