Dependence of acceleration on speed in the general relativistic Galileo experiment

RADI KHRAPKO[(a)]

Moscow Aviation Institute - Volokolamskoe shosse 4, 125993 Moscow, Russia

PACS 04.20.-q

Abstract –The dependence of free fall acceleration on speed in the Schwarzschild space-time is obtained. It is confirmed that gravitation mass coincides with inertial mass

Introduction

According to a biography, Galileo had dropped two balls of different masses from the Leaning Tower of Pisa with zero speed to demonstrate that their time of descent was independent of their mass. Via this method, he proved that the objects fell with the same acceleration, while according to Aristotle's theory of gravity, objects fall at speed relative to their mass. The Galileo experiment proved that the gravitational mass, m_g, which determines the gravitational force

$$F = GMm_g / r^2 \equiv m_g g\,, \tag{1}$$

coincides with the inertial mass, m_i, which determines acceleration in the case of $v = 0$,

$$F = m_i a\,. \tag{2}$$

Realy, only if $m_g = m_i$, $m_g g = m_i a$ entails $g = a$.

This coincidence is the foundation of General Relativity because this coincidence proves that a world line is determined by the space-time itself rather than by a moving test body. Note, considering of the space-time is not a throw-back to the aether, because aether is a fiction, but space-time is real.

However, a dependence of the acceleration on initial speed of the body, when place of throwing is fixed, is interesting. The dependence is absent in Newtonian theory because the gravitational force and the mass do not depend on speed. But according to the theory of relativity, light speed cannot be exceeded. So the acceleration must tend to zero if $v \to c$. We consider the dependence of the acceleration on vertical speed of the body in the frame of General Relativity using the Schwarzschild coordinate system. It is natural that the standard definitions of speed and acceleration are in use: speed = (infinitesimal length)/(infinitesimal time), acceleration is the speed of speed change. In order to have a positive speed while $dr/dt < 0$ when a body is falling, we define

$$v = -\sqrt{\frac{g_{rr}}{g_{tt}}}\frac{dr}{dt}, \qquad a = \frac{1}{\sqrt{g_{tt}}}\frac{dv}{dt}, \tag{3}$$

where g_{rr}, g_{tt} are the metric coefficients of coordinate system in use.

The dependence of the acceleration on speed, $a(v)$, shows a dependence of the gravitational mass on speed, $m_g(v)$. Realy, as well known,

$$F = \frac{m_0 a}{\sqrt{1 - v^2/c^2}^{\,3}} \tag{4}$$

if F is parallel to v, and so, using the gravitational force (1) as F here yields

(a) E-mail: khrapko_ri@hotmail.com

$$m_g g = \frac{m_0 a}{\sqrt{1 - v^2 / c^2}^3}. \tag{5}$$

Here m_0 is the invariant mass.

Calculation

The Schwarzschild space-time with coordinates t, r has metric (put $c = r_g = 2M = 1$) [1, (100,14)]

$$ds^2 = \frac{r-1}{r} dt^2 - \frac{r}{r-1} dr^2 - r^2 d\Omega^2. \tag{6}$$

Consider a radial geodesic line using t as a parameter: $\{t, r(t)\}$..

$$\frac{D}{dt}\frac{dx^i}{dt} \equiv \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = \alpha \frac{dx^i}{dt}, \quad \Gamma^t_{tr} = -\Gamma^r_{rr} = \frac{1}{2r(r-1)}, \Gamma^r_{tt} = \frac{r-1}{2r^3}. \tag{7}$$

Here Γ^i_{jk} are the Christoffel symbols, but, as opposed to [1, (87,3)], there is not zero on the right-hand side, but a quantity, which is proportional to the tangent vector, because t is not a canonical parameter. In this case a geodesicness of the line is provided by the curvature vector is directed along the line. Equation (7) gives

for $i = t$: $\Gamma^t_{tr} 2 \frac{dr}{dt} \equiv \frac{1}{r(r-1)} \frac{dr}{dt} = \alpha$,

for $i = r$: $\frac{d^2 r}{dt^2} + \Gamma^r_{rr} \left(\frac{dr}{dt}\right)^2 + \Gamma^r_{tt} \equiv \frac{d^2 r}{dt^2} - \frac{1}{2r(r-1)} \left(\frac{dr}{dt}\right)^2 + \frac{r-1}{2r^3} = \alpha \frac{dr}{dt}$.

Eliminating α yields the geodesic line equation:

$$\frac{d^2 r}{dt^2} - \frac{3}{2r(r-1)} \left(\frac{dr}{dt}\right)^2 + \frac{r-1}{2r^3} = 0. \tag{8}$$

The derivative $\frac{dr}{dt}$ is connected with the speed:

$$v = -\frac{dr}{dt}\sqrt{\frac{g_{rr}}{g_{tt}}} = -\frac{dr}{dt}\frac{r}{r-1}, \quad g_{tt} = \frac{r-1}{r}, \quad g_{rr} = \frac{r}{r-1}, \tag{9}$$

and the second derivative $\frac{d^2 r}{dt^2}$ is connecred with the acceleration:

$$a = \frac{1}{\sqrt{g_{tt}}}\frac{dv}{dt} = -\frac{1}{\sqrt{g_{tt}}}\frac{d}{dt}\left(\frac{dr}{dt}\frac{r}{r-1}\right) = -\sqrt{\frac{r}{r-1}}\left(\frac{d^2 r}{dt^2}\frac{r}{r-1} - \left(\frac{dr}{dt}\right)^2 \frac{1}{(r-1)^2}\right). \tag{10}$$

Substituting the second derivative from (8) to (10) and using the expression of the first derivative through speed (9) yields

$$a = \frac{1}{2r^2}\sqrt{\frac{r}{r-1}}(1 - v^2) = g(1 - v^2). \tag{11}$$

Here g is the general relativistic gravitational acceleration of a motionless body at coordinate r. Really, according to eq. (7) for $i = r$, the curvature of the world-line of a motionless body, $r = \mathrm{Const}$, equals $\frac{D^2 r}{dt^2} = \Gamma^r_{tt} = \frac{r-1}{2r^3}$. So, from the viewpoint of general relativity, the motionless body has acceleration

$$g = \frac{\sqrt{g_{rr}}}{g_{tt}} \frac{D^2 r}{dt^2} = \frac{1}{2r^2} \sqrt{\frac{r}{r-1}} . \tag{12}$$

This is the free fall acceleration from *our* viewpoint, g.

Remembering light speed c and substituting (11) to (5) yields

$$m_g = \frac{m_0}{\sqrt{1 - v^2 / c^2}} . \tag{13}$$

This means that gravitation mass equals inertial mass.

It is interesting that the same dependence of acceleration on speed is peculiar to a laboratory of a constant acceleration. The space-time metric of coordinates τ, ξ of such a laboratory and the connection of these coordinates with Minkowski coordinates t, x are known:

$$ds^2 = \xi^2 d\tau^2 - d\xi^2, \quad t = \xi \mathrm{sh}\tau, \quad x = \xi \mathrm{ch}\tau . \tag{14}$$

The expression for a geodesic line of a body with constant speed v_0 in τ, ξ coordinates can be obtained by a coordinate transformation:

$$x = x_0 + v_0 t, \quad \xi \mathrm{ch}\tau = \xi_0 + v_0 \xi \mathrm{sh}\tau . \tag{15}$$

Now one can obtain speed and acceleration of the body relative to the laboratory:

$$v = -\frac{d\xi}{\sqrt{g_{\tau\tau}} d\tau} = -\frac{v_0 \mathrm{ch}\tau - \mathrm{sh}\tau}{\mathrm{ch}\tau - v_0 \mathrm{sh}\tau}, \quad a = \frac{dv}{\sqrt{g_{\tau\tau}} d\tau} = \frac{1 - v^2}{\xi} .$$

---

[1] Landau L. D., Lifshitz E. M. "The Classical Theory of Fields" (Pergamon, N. Y. 1975).

22 Jan. 2015