On some analytical results for 3-cycles of the logistic map

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January 8, 2015

Abstract

In the present work new analytical results for the 3-cycles of the logistic map are obtained.

1 Introduction

The logistic map:

$$x_{n+1} = f_r(x_n), \quad f_r(x) \overset{df}{=} rx(1-x), \tag{1}$$

where r is a parameter, is perhaps the most famous example of one-dimensional discrete-time dynamical system. This simple system has very complicated dynamics, for example period doubling route to chaos, intermittent chaos near periodic windows, and crises [1, 2]. These dynamical modes, universal for some classes of mappings of form $x_{n+1} = f(x_n)$ [3, 4], were observed in several experiments [1, 5].

In this short note we attempt to obtain new analytical results for the 3-cycles of the map (1); see [6] for a survey of rigorous results for the logistic map.

2 Solving equations for the 3-cycles

Values of the 3-cycle are given by [6]:

$$p_r(x) \overset{df}{=} \frac{f_r(f_r(f_r(x))) - x}{f_r(x) - x} = 0, \tag{2}$$

where

$$\begin{aligned} p_r(x) &= r^6x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \quad (3) \\ c_5 &= -3r^6 - r^5,\ c_4 = 3r^6 + 4r^5 + r^4,\ c_3 = -r^6 - 5r^5 - 3r^4 - r^3, \\ c_2 &= 2r^5 + 3r^4 + 3r^3 + r^2,\ c_1 = -r^4 - 2r^3 - 2r^2 - r,\ c_0 = r^2 + r + 1. \end{aligned}$$

We shall solve the sextic equation $p_r(x) = 0$, with coefficients depending on parameter r (note that in [6] there is wrong sign in the second term in c_5).

Of course, due to the Galois theory, there are no general solutions by radicals of an arbitrary sextic equation. However, numerical and analytical computations show that for $r \geq 1 + 2\sqrt{2}$ there is a stable 3-cycle, subject to period doubling bifurcations (this cascade of period doubling ends at $r \cong 3.8496$), and an unstable 3-cycle, both cycles coinciding at $r = 1 + 2\sqrt{2}$. Therefore, six roots of the polynomial (3) split into two 3-cycles and this means additional symmetry. Hence, we shall try to factorize the polynomial $p_r(x)$ into product of two cubic polynomials.

We thus write:

$$p_r(x) = \left(r^3x^3 + a_2x^2 + a_1x + a_0\right)\left(r^3x^3 + b_2x^2 + b_1x + b_0\right), \tag{4}$$

with $p_r(x)$ given by (3).

The coefficients $a_2,\ a_1,\ a_0,\ b_2,\ b_1,\ b_0$ were computed, using Maple from Scientific WorkPlace 5.5, as:

$$\left.\begin{array}{l} a_2 = -r^2\left(\frac{3}{2}r + \frac{1}{2} - \frac{1}{2}\sqrt{\Delta}\right), \\ a_1 = \frac{1}{2}r\left(r^2 + 2r - 1\right) - \frac{1}{2}r\left(r+1\right)\sqrt{\Delta}, \\ a_0 = -\frac{1}{2}r^2 + \frac{1}{2}r + 1 + \frac{1}{2}r\sqrt{\Delta}, \\ b_2 = -r^2\left(\frac{3}{2}r + \frac{1}{2} + \frac{1}{2}\sqrt{\Delta}\right), \\ b_1 = \frac{1}{2}r\left(r^2 + 2r - 1\right) + \frac{1}{2}r\left(r+1\right)\sqrt{\Delta}, \\ b_0 = -\frac{1}{2}r^2 + \frac{1}{2}r + 1 - \frac{1}{2}r\sqrt{\Delta}, \end{array}\right\} \tag{5}$$

where $\Delta = r^2 - 2r - 7$.

It follows that, for real r, the coefficients $a_2,\ a_1,\ a_0,\ b_2,\ b_1,\ b_0$ are real for $\Delta \geq 0$, i.e. for $r \geq 1 + 2\sqrt{2}$ or $r \leq 1 - 2\sqrt{2}$. These necessary conditions for existence of 3-cycles are also sufficient. Accordingly, one pair of 3-cycles (one stable and another unstable) is born at $r = r_{cr}^{(1)} = 1 + 2\sqrt{2}$ and another pair at $r = r_{cr}^{(2)} = 1 - 2\sqrt{2}$.

There are thus two cubic equations for values of 3-cycles:

$$r^3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{6a}$$
$$r^3x^3 + b_2x^2 + b_1x + b_0 = 0, \tag{6b}$$

which can be easily solved via, for example, Cardano formulae. Numerical computations show that the stable 3-cycle is given by (6a) while (6b) yields the unstable 3-cycle.

Bifurcations diagrams displaying both stable 3-cycles are shown in Fig. 1.

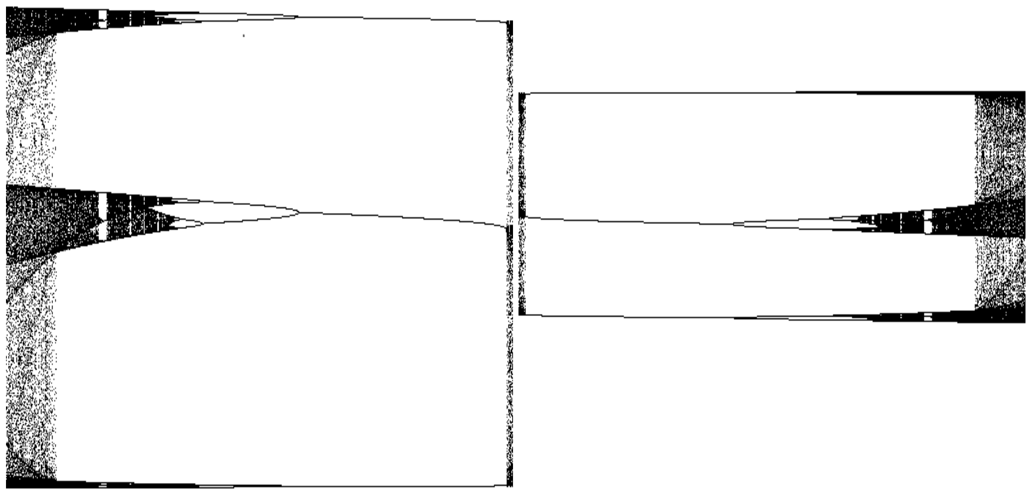

Figure 1: Bifurcation diagrams. Left figure: $r \in [-1.680,\ 1.828]$, right figure: $r \in [3.828,\ 3.860]$.

3 Closing remarks

Several generalizations come to mind. First of all, equations for 3-cycles can be solved for other quadratic maps. Secondly, equations for other n-cycles of quadratic maps should factorize into products of polynomials of order n. Finally, equations for n-cycles should factorize as well for higher-order polynomial maps.

Similar method was used by Kulkarni [7] who solved the sextic equation by factorizing it into two cubic equations as in the present report (it was, of course, necessary to add one condition for the coefficients, not fulfilled by the coefficients of polynomial (3)).

Acknowledgement 1 *These results constitute part of the communication: Andrzej Okniński, Chaos-order transitions in 2D maps, presented at the conference Heraeus-Stiftung Conference on Chaos in Dissipative Systems, Trassenheide, Germany, 8-11 April 1992.*

References

[1] May, Robert M., Simple mathematical models with very complicated dynamics, Nature **261**, 5560 (1976) 459-467.

[2] Robinson, Rex Clark, *An introduction to dynamical systems: continuous and discrete.* Vol. 19. American Mathematical Soc., 2012.

[3] Feigenbaum, Mitchell J., Quantitative universality for a class of nonlinear transformations, Journal of Statistical Physics **19** (1978) 25-52.

[4] Feigenbaum, Mitchell J., The universal metric properties of nonlinear transformations, Journal of Statistical Physics **21** (1979) 669-706.

[5] Schuster, Heinz Georg, and Just, Wolfram, *Deterministic chaos: an introduction,* John Wiley & Sons, 2006.

[6] Weisstein, Eric W., Logistic Map, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LogisticMap.html.

[7] Kulkarni, Raghavendra G., Solving Sextic Equations, Atlantic Electronic J. Math. **3** (2008) 56-60.