About Stability–Behavior of linear Structures in Space–Time Transitions.

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1. Abstract.

Authors like D.CAMPBELL[1], J.GUCKENHEIMER, P.HOLMES[4], H.HAKEN[6], H_O.PEITGEN, H.JÜRGENS, D.SAUPE[9] and many others told, that various solutions of nonlinear dynamical systems (like stationary points or inner cycles e.g.) can change their stability behavior, as soon as certain system parameters are modified appropriately. Caused by these so called bifurcations very often the topology of the original structure may be changed significantly. Similar effects in connection with linear structures could not be observed so far. But for specific spacetime transformations of linear structures similar phenomena will become existent too.

Changes in stability behavior of linear structures due to transformations in spacetime will be the subject of this article. The investigations are motivated:

- Because it is of main interest to realize that structural instabilities can happen even for linear structures and to understand the reasons for such a behavior.
- Because such insights may be helpful in observations of cosmic events that occurred under completely different spacetime conditions.

2. Introduction.

We start with the general class of dynamical systems, which mathematically can be described by:

$$d\underline{U}(t)/dt \;=\; f(\underline{U}(t),t) \;\Leftarrow\; \langle\, \underline{U} \in \mathbf{R}^3 \;,\; f(\underline{U},t) \subseteq \mathbf{R}^3{\times}\mathbf{R} \,\rangle.$$

This kind of physical systems relate the change in time $\langle\, t \,\rangle$ of a vector $\langle\, \underline{U} \,\rangle$ to a non_ linear function $\langle\, f \,\rangle$ that depends on $\langle\, t \,\rangle$ and $\langle\, \underline{U} \,\rangle$ as well. But for our further investigations it's enough to specialize on the group of autonomous systems:

$$d\underline{U}(t)/dt \;=\; f(\underline{U}(t)),$$

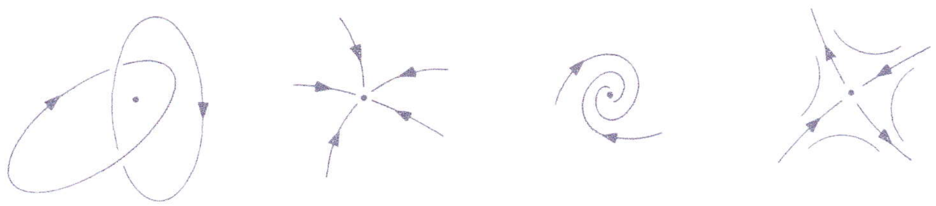
where the general function $\langle\, f \,\rangle$ is no more explicitly dependent on $\langle\, t \,\rangle$.

From an autonomous system's dynamical equation with specific starting_conditions a sequence of system_states can be obtained. These states can be represented as points in a space appropriately related to the system and afterwards connected by curves named orbits or trajectories. Trajectories according to various starting_conditions build up a solution_structure for the appropriate development_equation. It has to be noted that these structures of autonomous systems are constant in time.

In connection with a dynamical structure we are interested especially in its characteristic components like e.g. critical points or closed curves and what their stabilities under various system_conditions are concerned. An important class of solutions for:

$$d\underline{U}(t)/dt \;=\; f(\underline{U}(t))$$

are equilibrium or steady_state solutions represented by single points in the states_ space associated with the dynamical system, they correspond to critical points in the field $\langle\, f \,\rangle$. A steady_state solution $\langle\, \underline{U}_0 \,\rangle$ cannot be reached by $\langle\, \underline{U}(t) \,\rangle$ in a finite time, so it is isolated from the rest of the structure. If orbits $\langle\, \underline{U}(t) \,\rangle$ passing through points in the neighborhood of $\langle\, \underline{U}_0 \,\rangle$ and remain close to it for $\langle\, t\rightarrow\infty \,\rangle$, then $\langle\, \underline{U}_0 \,\rangle$ is called stable, otherwise it is called unstable. Example of stable or instable fixed_points are:



For small deviations $\langle\, \underline{X} \,\rangle$ from the fixed_point $\langle\, \underline{U}_0 \,\rangle$ the field $\langle\, f(\underline{U}) \,\rangle$ can be developed into a TAYLOR_series and stopped after the linear term:

$$\underline{U} \;=\; \underline{U}_0+\underline{X} \;\Rightarrow\; f(\underline{X}) \;=\; [\partial f(\underline{U}_0)/\partial \underline{X}]d\underline{X}+\ldots \;=\; \mathbb{P}d\underline{X},$$

– with $\langle\, \mathbb{P} \,\rangle$ as JACOBI_matrix – and thus the development_equation becomes:

$$dX \;=\; \mathbb{P}d\underline{X}.$$

Saddles, nodes (sources or sinks), foci, star_shaped nodes, inflected nodes as isolated points in a structure have JACOBIANs $\langle\, \mathbb{P} \,\rangle$ with eigenvalues $\langle\, \lambda \,\rangle$:

$$\mathrm{Re}(\lambda) \;\neq\; 0,$$

and the HARTMAN–GROBMAN theorem:

- If the JACOBIAN has no eigenvalues with zero real part, then the family of trajectories near the singular point of a nonlinear system and those of a locally linearized system have same topological structure, which means in a neighborhood of the isolated point there exists a homeomorphic mapping which maps trajectories of the nonlinear system into trajectories of the linear system,

becomes applicable for them.

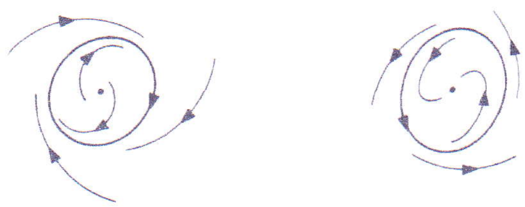
Another class of solutions for:

$$d\underline{U}(t)/dt \;=\; f(\underline{U}(t))$$

are of periodic character, like:

$$\underline{U}'(t+T) \;=\; \underline{U}'(t)$$

with time_period $\langle\, T \,\rangle$. Periodic solutions of dynamical systems represent closed orbits of their trajectory_structures. If no other closed path exists near a close orbit, the orbit is called a limit_cycle. If all nearby paths approach the cycle for $\langle\, t\rightarrow\infty \,\rangle$ it is stable, otherwise it is unstable. Specific examples for both types are:



The following maybe helpful to understand, why closed curves are solutions for our system_ equations.

$$d\underline{U}'(t)/dt \;=\; a(t)\underline{U}'(t).$$

If the function ⟨ a(t) ⟩ remains unchanged after period ⟨ T ⟩:

$$a(t+nT) \;=\; a^n(t) \;=\; a(t),$$

then ⟨ a(t) ⟩ must be a periodic function and we may further obtain:

$$[d\underline{U}'(t+T)/dt \;=\; a(t)\underline{U}'(t+T)] \quad , \quad [d\underline{U}'(t+nT)/dt \;=\; a(t)\underline{U}'(t+nT)].$$

This leads to:

$$\underline{U}'(t+T) \;=\; \kappa\underline{U}'(t) \quad , \quad \underline{U}'(t+nT) \;=\; \kappa^n\underline{U}'(t),$$

with ⟨ κ ⟩ as a constant factor. The last equation can be solved by an onset:

$$\underline{U}'(t) \;=\; e^{\lambda t}\underline{W}(t)$$

which results in:

$$\begin{aligned} e^{\lambda(t+T)}\underline{W}(t+T) &= \kappa e^{\lambda t}\underline{W}(t) \\ e^{\lambda T}\underline{W}(t+T) &= \kappa\underline{W}(t) \\ e^{\lambda T} &= \kappa \\ \underline{W}(t+T) &= \underline{W}(t). \end{aligned}$$

and shows ⟨ $\underline{W}(t)$ ⟩ as a function with period ⟨ T ⟩ and:

$$\underline{U}'(t) \;=\; e^{\lambda T}\underline{W}(t).$$

as the general expression for a closed paths. A center in the nonlinear system satisfies:

$$\mathrm{Re}(\lambda) \;=\; 0.$$

Despite HARTMAN–GROBMAN theorem is no more applicable, a restrained mapping between structures of nonlinear into linear systems is still possible. Although such mappings may result in significant changes of the structure, certain properties will still be preserved. For instance singular points will again be mapped into singular points and closed curves into closed curves. Thus we can be sure, that a center from a nonlinear orbit_structure will also become a center in its linear homöomorphic structure.

In order to get a quick insight to the subsequent formalism, a list of terms and definitions will be anticipated:

- $\underline{x},\underline{\omega}$: Small underlined letters symbolize vectors in spacetime.
- $\underline{X},\underline{\Omega}$: Underlined capital_letters symbolize space_related vectors.
- $\mathbb{P},\mathbb{Q}$: Double_printed capital_letters symbolize (2•2)_matrices.
- a,i : Small latin letters used as indices will range 1→3.
- ξ,ς : Small greek letters used as indices will range 0→3.
- A,I : Large latin capital_letters used as indices will be valued 1∨2∨3.
- $x^k\underline{e}_k$: EINSTEIN's convention is valid: Over indices appearing more than once in a term will be summed up.
- c : Value of speed of light.
- $g_{ab}(\underline{x})$: Tensor of metrics at position $\underline{x}$ in spacetime.

3. Transforming Orbit_Structures between Inertial_Frames.

Inertial_frames are tied to geodetic_lines in a flat spacetime, they move uniformly on straight lines. For observations of succeeding events from two such frames $\{x^p\}$ and $\{y^q\}$ moving relatively against each other with a constant velocity of:

$$\underline{V} \;=\; \{V^\nu\}$$

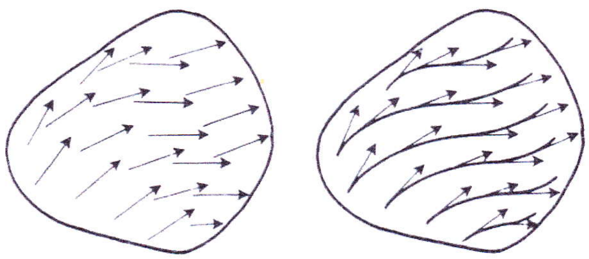
LORENTZ_transformations (see for example R.D'INVERNO[7] or C.W.MISNER, K.S.THORNE, J.A. WHEELER[8]) are applicable in the form:

$$\begin{aligned}
d\underline{Y} &= \beta(d\underline{X}-\underline{V}d\varrho) \\
&= dY^\iota \underline{\hat{e}}_\iota \\
d\rho &= \beta(d\varrho-\underline{V}\cdot d\underline{X}/c^2) \\
d\underline{X} &= dX^\alpha \underline{e}_\alpha \\
\beta &= (1-\underline{V}^2/c^2)^{-1/2} \\
d\varrho &= dx^{[0]} \\
d\rho &= dy^{[0]}.
\end{aligned}$$

From this we will obtain:

$$\begin{aligned}
d\underline{Y}/d\rho &= (d\underline{X}-\underline{V}d\varrho)/(d\varrho-\underline{V}\cdot d\underline{X}/c^2) \\
&= ([d\underline{X}/d\varrho]-\underline{V})/(1-\underline{V}\cdot[d\underline{X}/d\varrho]/c^2) \\
&= [d\underline{X}/d\varrho]-\underline{V} \qquad \Leftarrow \quad [\,\underline{V}\cdot[d\underline{X}/d\varrho] \;\ll\; c^2\,].
\end{aligned}$$

Tangent_vector $\langle\, d\underline{X}/d\varrho \,\rangle$ in $\{X^\alpha\}$ has to be translated by the constant velocity $\{\underline{V}\}$ in order to get transformed into $\langle\, d\underline{Y}/d\rho \,\rangle$ of $\{Y^\iota\}$. Because of:

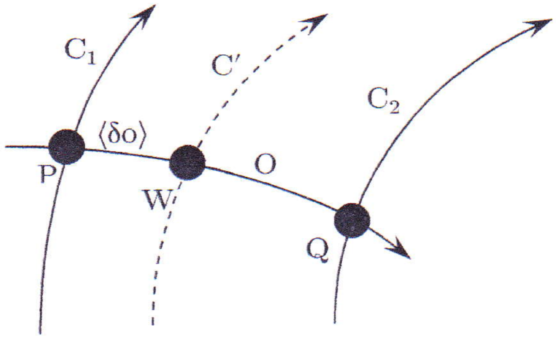


the trajectory_ structures in $\{X^\alpha\}$ and $\{Y^\iota\}$ therefore will be similar. Stability of both structures must be the same as their topology as well.

4. Transformation of linear Structures in curved Spacetime.

Next we consider a surface ⟨ S ⟩ in curved spacetime formed by a congruence of time_like geodetic_curves ⟨ C ⟩. By a congruence of curves we understand a family of lines with the property, only one of them passes through a point of ⟨ S ⟩.

Within this set of curves for unforced, suspended motions of masses in curved space_time we start our consideration with points ⟨ P ⟩ on ⟨ C_1 ⟩ and ⟨ W ⟩ on ⟨ C' ⟩ and keep in mind an additional curve ⟨ C_2 ⟩ beyond ⟨ C' ⟩. We specify a second congruence of orbits running more or less perpendicular to the geodetic_curves ⟨ C ⟩. One of these orbits ⟨ O ⟩ starting from ⟨ P ⟩ on ⟨ C_1 ⟩ passes at a distance ⟨ δo ⟩ through ⟨ W ⟩ and keeps further running through ⟨ Q ⟩ on ⟨ C_2 ⟩



In addition a tensor_field ⟨ $T^{ab}(\underline{u})$ ⟩ is introduced between ⟨ C_1 ⟩ and ⟨ C_2 ⟩ changing with its position ⟨ $\underline{u}$ ⟩ in spacetime. A relation between ⟨ $T^{ab}(P)$ ⟩ and ⟨ $T^{ab}(W)$ ⟩ is obtained by a so called infinitesimal transportation of ⟨ $T^{ab}(P)$ ⟩ by dragging (or active transforming) it along ⟨ δo ⟩ into ⟨ W ⟩ as ⟨ $T^{ab}(W)_Đ$ ⟩ and compare it subsequently with ⟨ $T^{ab}(W)$ ⟩. For ⟨ P ⟩ and ⟨ W ⟩ we may write:

$$(P : \underline{x} = \{x^q\}) \quad \wedge \quad (W : \underline{w} = \{w^r\})$$

and in point ⟨ P ⟩ we have a vector ⟨ $\underline{t}$ ⟩ tangential to orbit ⟨ O ⟩:

$$\underline{t} \;=\; d\underline{x}/do.$$

We assume, that the distance between ⟨ P ⟩ and ⟨ W ⟩ is small enough for:

$$\underline{w} \;=\; \underline{x}+\delta o\bullet\underline{t} \;\Rightarrow\; \partial w^a/\partial x^b \;=\; \delta^a{}_b+\delta o\bullet(\partial t^a/\partial x^b).$$

The relation ⟨ $T^{ab}(P) \to T^{ab}(W)_Đ$ ⟩ in the contravariant tensor_field is then given by:

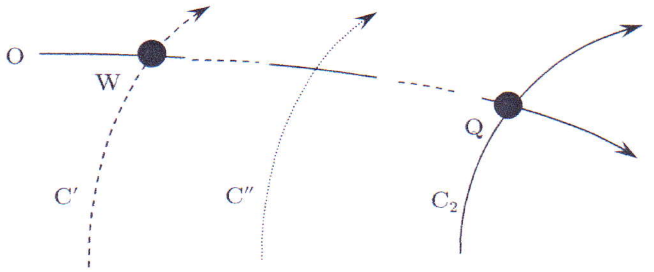
$$\begin{aligned} T^{ab}(\underline{w})_Đ \;&=\; (\partial w^a/\partial x^c)(\partial w^b/\partial x^d)T^{cd}(\underline{x}) \\ &=\; [\delta^a{}_c+\delta o(\partial t^a/\partial x^c)][\delta^b{}_d+\delta o(\partial t^b/\partial x^d)]T^{cd}(\underline{x}) \\ &=\; T^{ab}(\underline{x})+ \\ &\quad\; [(\partial t^a/\partial x^c)T^{cb}(\underline{x})+(\partial t^b/\partial x^d)T^{ad}(\underline{x})]\delta o+ \\ &\quad\; (\partial t^a/\partial x^c)(\partial t^b/\partial x^d)T^{cd}(\underline{x})\delta o^2 \end{aligned}$$

and thus⟨ $T^{ab}(P)$ ⟩ and ⟨ $T^{ab}(W)_Đ$ ⟩ must be of same type and order. The same will be true for ⟨ $T^{ab}(W)_Đ$ ⟩ and ⟨ $T^{ab}(W)$ ⟩ because LIE_derivation of ⟨ $T^{ab}(\underline{x})$ ⟩ with respect to ⟨ $\underline{t}$ ⟩ is (e.g. due to R.D'INVERNO[7]) built by:

$$L_{\underline{t}} \;=\; \lim_{\delta o\to 0}\{[T^{ab}(\underline{w})-T^{ab}(\underline{w})_Đ]/\delta o\}$$

where the subtracted tensors must be comparable. Therefore, as a first result we obtain so far, type and order of tensor ⟨ $T^{ab}(P)$ ⟩ will be preserved in ⟨ $T^{ab}(W)$ ⟩ when it becomes infinitesimally transported in the just described way.

The distance between ⟨ C_1 ⟩ and ⟨ C_2 ⟩ may be bridged by a sequence of infinitesimal transportations of the specified type from ⟨ W ⟩ of ⟨ C′ ⟩ via ⟨ C″ ⟩ into ⟨ Q ⟩ of ⟨ C_2 ⟩



We can extrapolate from preceding discussions, that type and order of ⟨ $T^{ab}(P)$ ⟩ step by step must be preserved in ⟨ $T^{ab}(Q)$ ⟩.

The tensors ⟨ $T^{ab}(P)$ ⟩ and ⟨ $T^{ab}(Q)$ ⟩ now will become (2•2)_matrices:

$$T^{ab}(P) = T^{ab}(\underline{x}) = \mathbb{H}(\underline{x}) = \mathbb{P}$$
$$T^{ab}(Q) = T^{ab}(\underline{y}) = \mathbb{H}(\underline{y}) = \mathbb{Q}$$

during the following discussions. The coordinate_systems ⟨ $\{x^a\}$ ⟩ and ⟨ $\{y^b\}$ ⟩ in general are different from each other and relations:

$$dy^j = (\partial y^j/\partial x^a)dx^a$$

must hold between them under the conditions:

$$g_{ab}(\underline{x})[dx^a/d\varrho][dx^b/d\varrho] = 1 = g_{jk}(\underline{y})[dy^j/d\rho][dy^k/d\rho],$$

with ⟨ $g_{ab}(\underline{x}), g_{jk}(\underline{y})$ ⟩ are the metric_tensors at the spacetime_positions ⟨ $\underline{x}$ ⟩ and ⟨ $\underline{y}$ ⟩, where:

$$(\varrho = x^{[0]}) \wedge (\rho = y^{[0]})$$

are the appropriate eigentimes on ⟨ C_1 ⟩ and ⟨ C_2 ⟩. Eigenvalues of ⟨ $\mathbb{P}$ ⟩ and ⟨ $\mathbb{Q}$ ⟩ are obtained from their characteristic equations:

$$\begin{aligned} \det(\mathbb{P}-\lambda) = 0 &= (P^{[11]}-\lambda)\cdot(P^{[22]}-\lambda)-P^{[12]}P^{[21]} \\ \lambda_{1,2} &= ½(P^{[11]}+P^{[22]}) \pm ½[(P^{[11]}+P^{[22]})^2-4(P^{[11]}P^{[22]}-P^{[12]}P^{[21]})]^{½} \\ &= ½tr(\mathbb{P}) \pm ½[tr(\mathbb{P})^2-4\det(\mathbb{P})]^{½} \end{aligned}$$

and:

$$\begin{aligned} \det(\mathbb{Q}-\mu) = 0 &= (Q^{[11]}-\mu)\cdot(Q^{[22]}-\mu)-Q^{[12]}Q^{[21]} \\ \mu_{1,2} &= ½(Q^{[11]}+Q^{[22]}) \pm ½[(Q^{[11]}+Q^{[22]})^2-4(Q^{[11]}Q^{[22]}-Q^{[12]}Q^{[21]})]^{½} \\ &= ½tr(\mathbb{Q}) \pm ½[tr(\mathbb{Q})^2-4\det(\mathbb{Q})]^{½}. \end{aligned}$$

Between ⟨ $\mathbb{P}$ ⟩ and ⟨ $\mathbb{Q}$ ⟩ the following relations are valid:

$$Q^{ij} = [\partial z^i/\partial y^a][\partial z^j/\partial y^b]P^{ab}.$$

From:

$$\{dy^j = (\partial y^j/\partial x^a)dx^a\} \wedge \{g_{ab}(\underline{x})[dx^a/d\varrho][dx^b/d\varrho] = 1 = g_{jk}(\underline{y})[dy^j/d\rho][dy^k/d\rho]\}$$

we will obtain:

$$
\begin{aligned}
dy^q &= [\partial y^q/\partial x^p]dx^p \\
dy^j dy^k &= [\partial y^j/\partial x^a][\,\partial y^k/\partial x^b]dx^a dx^b \\
dx^a dx^b &= (d\varrho)^2 g^{ab}(\underline{x}) \\
dy^j dy^k &= (d\rho)^2 g^{jk}(\underline{y}) \\
(d\rho)^2 g^{jk}(\underline{y}) &= [\partial y^j/\partial x^a][\,\partial y^k/\partial x^b](d\varrho)^2 g^{ab}(\underline{x}) \\
(d\rho/d\varrho)^2 g^{jk}(\underline{y}) g_{ab}(\underline{x}) &= [\partial y^j/\partial x^a][\,\partial y^k/\partial x^b]
\end{aligned}
$$

and thus will get from above:

$$Q^{jk} = (d\rho/d\varrho)^2 g^{jk}(\underline{y}) g_{ab}(\underline{x}) P^{ab}.$$

With these preparations some of the planar orbit_structures from above can be investigate with regard to their stability–behavior during transformations in curved spacetime.

4.1. Bifurcation of a Fixpoint in curved Spacetime.

From discussions of P.G.BAKKER[1] we know, that the linearized orbit_structures near a planar sink in an appropriate reference_ frame:

$$\underline{X} = \{X^{[1]}, X^{[2]}\}$$

on geodetic_line $\langle C_1 \rangle$ can be expressed by the following equation:

$$d\underline{X}/d\varrho = \mathbb{P}\underline{X}$$

where the eigenvalues of the (2•2)_matrix $\langle \mathbb{P} \rangle$ obey the following conditions:

$$[\lambda_1 \neq \lambda_2] \;\wedge\; [\mathrm{Im}(\lambda_1, \lambda_2) = 0] \;\wedge\; [\lambda_1 \bullet \lambda_2 > 0] \;\wedge\; [\lambda_1 + \lambda_2 < 0].$$

The eigenvalues are obtained from the characteristic equation as:

$$½\mathrm{tr}(\mathbb{P}) \pm ½\{\mathrm{tr}(\mathbb{P})^2 - 4\det(\mathbb{P})\}^{½} = \lambda_{1,2}$$

and therefore:

$$\lambda_1 + \lambda_2 < 0 \;\Rightarrow\; \{(\mathrm{tr}(\mathbb{P}) = P^{[11]} + P^{[22]}) < 0\} \;\wedge\; \{P^{[11]}P^{[22]} > 0\}$$

must hold, as well:

$$
\begin{aligned}
\lambda_1 \bullet \lambda_2 > 0 &\Rightarrow \mathrm{tr}(\mathbb{P})^2/4 - \mathrm{tr}(\mathbb{P})^2/4 + \det(\mathbb{P}) > 0 \\
\det(\mathbb{P}) > 0 &\Rightarrow P^{[11]}P^{[22]} - P^{[12]}P^{[21]} > 0 \Rightarrow P^{[12]}P^{[21]} \gtreqless 0 \\
& \qquad P^{[12]}P^{[21]} \gtreqless 0 \Rightarrow P^{[12]} + P^{[21]} \gtreqless 0
\end{aligned}
$$

as:

$$
\begin{aligned}
\mathrm{Im}(\lambda_1, \lambda_2) = 0 \Rightarrow \mathrm{tr}(\mathbb{P})^2 - 4\det(\mathbb{P}) &= [P^{[11]} + P^{[22]}]^2 \\
&\quad - 4P^{[11]}P^{[22]} + 4P^{[12]}P^{[21]} > 0 \\
&= (P^{[11]})^2 + 2P^{[11]}P^{[22]} + (P^{[11]})^2 \\
&\quad - 4P^{[11]}P^{[22]} + 4P^{[12]}P^{[21]} > 0
\end{aligned}
$$

$$= (P^{[11]})^2 - 2P^{[11]}P^{[22]} + (P^{[11]})^2 + 4P^{[12]}P^{[21]} > 0$$
$$= (P^{[11]} - P^{[22]})^2 + 4P^{[12]}P^{[2} > 0$$

From the last condition, we finally may follow:

$$(P^{[11]} - P^{[22]})^2 + 4P^{[12]}P^{[21]} > 0 \Rightarrow P^{[12]}P^{[21]} \gtrless 0 \Rightarrow P^{[12]} + P^{[21]} \gtrless 0.$$

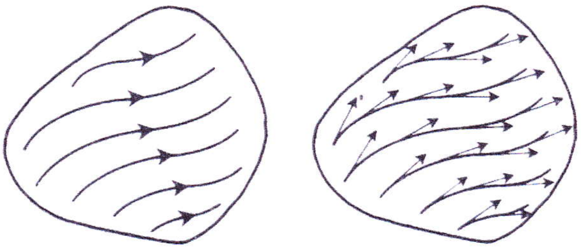
The matrix $\langle \mathbb{P} \rangle$ on geodetic_line $\langle C_1 \rangle$ will be transformed by a series of infinitesimal transportations into matrix $\langle \mathbb{Q} \rangle$ of the same type and order on geodetic_line $\langle C_2 \rangle$. Thus in a reference_frame:

$$\underline{Y} = \{Y^{[1]}, Y^{[2]}\}$$

a vector_field $\langle \mathbb{Q}\underline{Y} \rangle$ can be defined whose tangential_field in each point is given by:

$$d\underline{Y}/d\rho = \mathbb{Q}\underline{Y},$$

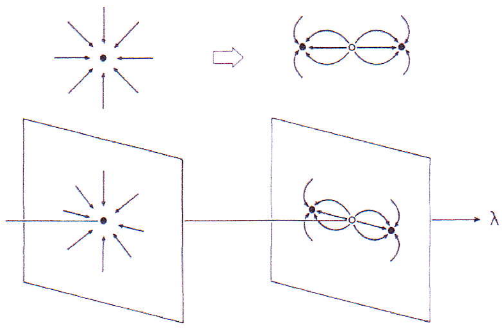
the relation between $\langle \underline{Y} \rangle$ and $\langle \mathbb{Q}\underline{Y} \rangle$ can be visualized in the following way:



If the matrix $\langle \mathbb{Q} \rangle$ gets real eigenvalues $\langle \mu_1, \mu_2 \rangle$ with different signs:

$$\mu_1 \cdot \mu_2 < 0$$

then due to P.G.BAKKER[1] the sink from $\langle \{X^{[1]}, X^{[2]}\} \rangle$ must have been converted into an unstable saddle in $\langle \{Y^{[1]}, Y^{[2]}\} \rangle$ on its way from $\langle C_1 \rangle$ to $\langle C_2 \rangle$



That this may happen under conditions from above, will be shown subsequently.

From matrix $\langle \mathbb{Q} \rangle$ one gets:

$$\mu = \tfrac{1}{2}[\mathrm{tr}(\mathbb{Q}) \pm \{\mathrm{tr}(\mathbb{Q})^2 - 4\det(\mathbb{Q})\}^{1/2}] \Leftarrow \mathrm{Im}(\mu) = 0$$
$$\mu_1 \cdot \mu_2 = \tfrac{1}{4}[\mathrm{tr}(\mathbb{Q})^2 - \mathrm{tr}(\mathbb{Q})^2 + 4\det(\mathbb{Q})] < 0$$

and thus one obtains:

$$\det(\mathbb{Q}) = Q^{[11]}Q^{[22]} - Q^{[12]}Q^{[21]} < 0.$$

Because of the general relations:

$$Q^{jk} = (d\rho/d\varrho)^2 g^{jk}(\underline{y}) g_{ab}(\underline{x}) P^{ab}$$

between $\langle \mathbb{Q} \rangle$ and $\langle \mathbb{P} \rangle$ holds and the following due to R.D'INVERNO[7] is valid:

$$[(g^{pq} = g^{qp}) > 0] \wedge [(g_{pq} = g_{qp}) > 0],$$

one may conclude:

$$\begin{aligned} \det(\mathbb{Q}) &= Q^{[11]}Q^{[22]} - Q^{[12]}Q^{[21]} && < 0 \\ &= (d\rho/d\varrho)^2 \\ &\quad \{g^{[11]}(\underline{y})g^{[22]}(\underline{y}) - [g^{[12]}(\underline{y})]^2\} \\ &\quad \{g_{[11]}(\underline{x})P^{[11]} + g_{[22]}(\underline{x})P^{[22]} + g_{[12]}(\underline{x})[P^{[12]} + P^{[21]}]\} && < 0 \end{aligned}$$

This can obviously be fulfilled, if:

$$\{g^{[11]}(\underline{y})g^{[22]}(\underline{y}) - [g^{[12]}(\underline{y})]^2 > 0\} \wedge \{P^{[12]} + P^{[21]} \lesseqgtr 0\} \wedge \{g_{[11]}(\underline{x})P^{[11]} + g_{[22]}(\underline{x})P^{[22]} \gtreqless 0\}$$
$$\{g^{[11]}(\underline{y})g^{[22]}(\underline{y}) - [g^{[12]}(\underline{y})]^2 < 0\} \wedge \{P^{[12]} + P^{[21]} > 0\} \wedge \{g_{[11]}(\underline{x})P^{[11]} + g_{[22]}(\underline{x})P^{[22]} > 0\}.$$

According to HARTMAN–GROBMAN, for a saddle_point of:

$$d\underline{Y}/d\rho = \mathbb{Q}\underline{Y}$$

with:

$$\mathrm{Re}(\mu_1, \mu_2) \neq 0$$

no topological change of the trajectory_structure will be observed, if a smooth non_ linearity is added to the equation. The flow may become different if compared to the undisturbed case, but its main properties will be preserved. Thus singular points or closed curves will remain objects of same kind or two points $\langle \underline{Y}_1 \rangle$ and $\langle \underline{Y}_2 \rangle$ of the same curve will again be parts of its picture_curve. Therefore we assume, to the linear system:

$$d\underline{Y}/d\rho = \mathbb{Q}\underline{Y}$$

is added:

$$\varepsilon f(\underline{Y}) \Leftarrow [\varepsilon \in \mathbf{R} \;,\; f(\underline{Y}) = \text{differentiable}]$$

which will change the system into the nonlinear one:

$$d\underline{Y}/d\rho = \mathbb{Q}\underline{Y} + \varepsilon f(\underline{Y}).$$

The singular point $\langle \underline{Y}_0 \rangle$ of the latter system:

$$\mathbb{Q}\underline{Y}_0 + \varepsilon f(\underline{Y}_0) = 0 \Rightarrow \underline{Y}_0 = -\varepsilon \mathbb{Q}^{-1} f(\underline{Y}_0).$$

for:

$$\varepsilon < |\mathrm{Re}(\mu_1, \mu_2) \neq 0|$$

will keep the topology of the linear system unchanged. According to the HARTMAN–GROBMAN theorem:

$$[d\underline{Y}/d\rho = \mathbb{Q}\underline{Y}] \;\&\; [d\underline{Y}/d\rho = \mathbb{Q}\underline{Y} + \varepsilon f(\underline{Y})]$$

will have equivalent orbit_structures from a topological point of view with the same eigenvalues $\langle \mu_1,\mu_2 \rangle$ for the matrices $\langle \mathbb{Q}+\varepsilon f(\underline{Y}) \rangle$ and $\langle \mathbb{Q} \rangle$. In a specific case for:

$$q \Leftarrow \underline{Z} = \underline{Y}-\underline{Y}_0$$

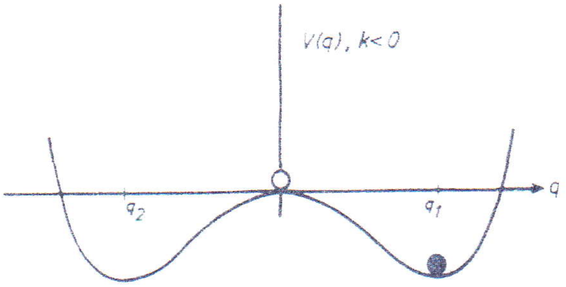
a 1_dimensional component of deviations from $\langle \underline{Y}_0 \rangle$, the dynamical equation:

$$dq/d\rho = -kq-k_1q^3$$

may become valid, its solution is then obtained as:

$$V(q) = -kq-k_1q^3 \Leftarrow [k < 0] \wedge [k_1 > 0],$$

which schematically can be drawn as:



Stable points at $\langle q_1 \rangle$ and $\langle q_2 \rangle$ of $\langle V(q) \rangle$ will coexist with the unstable saddle at $\langle q{=}0 \rangle$ and in essence, the former sink from the original linear structure bifurcates into an unstable saddle with additional stable points.

Thus one may finally conclude, as far as certain conditions for:

$$\mathbb{P} \quad , \quad g_{ab}(\underline{x}) \quad , \quad g_{ij}(\underline{y})$$

are fulfilled in curved spacetime, a linear stable fixed_point can bifurcate into an unstable fixed_point with additional stable points. Because no kind of affine mappings afterwards will make the structures on $\langle C_1 \rangle$ and $\langle C_2 \rangle$ matching again, this kind of bifurcation causes a significant topological change from original and to final structure.

4.2. HOPF_Bifurcation in curved Spacetime.

From discussions of P.G.BAKKER[1] we know again, that specific stable fixed_points linearly approximated in a plane may have a JACOBIAN_matrix $\langle \mathbb{P} \rangle$ with conjugate_ complex eigenvalues:

$$\lambda_1 = \psi+i\omega$$
$$\lambda_2 = \psi-i\omega$$

and negative real_ parts:

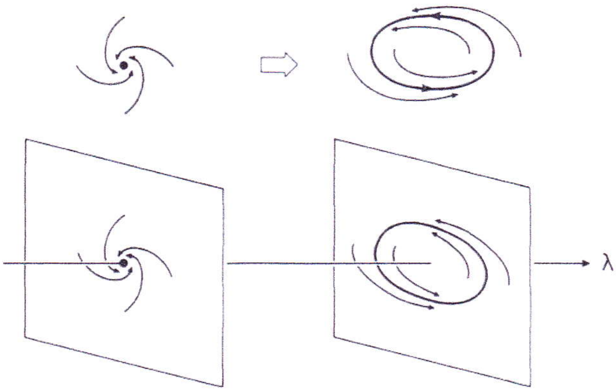
$$\psi < 0.$$

If a spacetime_transformation of the fixed_point from reference_frame $\langle \{X^{[1]},X^{[2]}\} \rangle$ on geodetic_curve $\langle C_1 \rangle$ into reference_frame $\langle \{Y^{[1]},Y^{[2]}\} \rangle$ on geodetic_curve $\langle C_2 \rangle$ causes:

$$\mathrm{Re}(\mu(\mathbb{Q})) \rightarrow \geq 0$$

to cross the imaginary axis, then due to H.HAKEN[7] an oscillation starts. In other words, a former fixed_point on $\langle C_1 \rangle$ bifurcates into a cyclic motion on $\langle C_2 \rangle$, a so called

HOPF_bifurcation takes place



In order to get insight to the mechanism, we start considering:

$$
\begin{array}{llllll}
\lambda_{1,2} & = & \tfrac{1}{2}[\mathrm{tr}(\mathbb{P})\pm\{\mathrm{tr}(\mathbb{P})^2-4\det(\mathbb{P})\}^{\frac{1}{2}}] & = & \psi\pm i\omega & \\
\psi & = & \mathrm{tr}(\mathbb{P}) & < & 0 & \Rightarrow\ P^{[11]}+P^{[22]} < 0 \\
\omega^2 & = & 4\det(\mathbb{P})-\mathrm{tr}(\mathbb{P})^2 & > & 0 & \\
 & = & 4P^{[11]}P^{[22]}-4P^{[12]}P^{[21]} & & & \\
 & & -(P^{[11]})^2-2\,P^{[11]}P^{[22]}-(P^{[22]})^2 & > & 0 & \\
 & = & 2P^{[11]}P^{[22]}-4P^{[12]}P^{[21]} & & & \\
 & & -(P^{[11]})^2-(P^{[22]})^2 & > & 0 & \\
 & = & -4P^{[12]}P^{[21]} & & & \\
 & & -(P^{[11]})^2+2P^{[11]}P^{[22]}-(P^{[22]})^2 & > & 0 & \\
 & = & -4P^{[12]}P^{[21]}-(P^{[11]}-P^{[22]})^2 & > & 0 & \Rightarrow\ P^{[12]}P^{[21]} < 0 \\
 & & P^{[12]}P^{[21]} & < & 0 & \Rightarrow\ P^{[12]}+P^{[21]} \gtrless 0.
\end{array}
$$

After the transformation we expect for matrix $\langle\, \mathbb{Q} \,\rangle$:

$$
\begin{array}{lll}
\mu_{1,2} & = & \tfrac{1}{2}[\mathrm{tr}(\mathbb{Q})\pm\{\mathrm{tr}(\mathbb{Q})^2-4\det(\mathbb{Q})\}^{\frac{1}{2}}] \\
\mathrm{tr}(\mathbb{Q}) & = & Q^{[11]}+Q^{[22]} \qquad \geq\ 0.
\end{array}
$$

Because general relations:

$$Q^{ij} \;=\; (d\rho/d\varrho)^2 g^{ij}(\underline{y})g_{ab}(\underline{x})P^{ab},$$

are applicable between $\langle\, \mathbb{P} \,\rangle$ and $\langle\, \mathbb{Q} \,\rangle$, we will further obtain:

$$
\begin{array}{lll}
\mathrm{tr}(\mathbb{Q}) & = & (d\rho/d\varrho)^2 \\
 & & \{g^{[11]}(\underline{y})+g^{[22]}(\underline{y})\} \\
 & & \{g_{[11]}(\underline{x})P^{[11]}+ g_{[22]}(\underline{x})P^{[22]}+g_{[12]}(\underline{x})[P^{[12]}+P^{[21]}]\} \ \geq\ 0.
\end{array}
$$

We know, a metric is always represented by a symmetric and positive_definite tensor, therefore:

$$g^{[11]}(\underline{y})+g^{[22]}(\underline{y}) \;>\; 0$$

is always valid. If the combinations between $\langle\, g_{ab}(\underline{y}) \,\rangle$ and $\langle\, P^{ab} \,\rangle$ are appropriately be chosen from the alternatives above, it becomes possible to get:

$$g_{[11]}(\underline{y})P^{[11]}+ g_{[22]}(\underline{y})P^{[22]}+g_{[12]}(\underline{y})[P^{[12]}+P^{[21]}] \;>\; 0.$$

Thus finally we may resume, a transformation in curved spacetime may cause transitions

$$\mathrm{tr}(\mathbb{P}) \;<\; 0 \;\rightarrow\; \mathrm{tr}(\mathbb{Q}) \;>\; 0$$

and therefore a HOPF_bifucation of a fixed_point in linearized structure will be possible.

Because the cycle resulting from the appropriate transformation can be contracted by affine mapping to a single point in the trajectory_structure, a HOPF_bifurcation will not change the topology of the original structure.

4.3. Bifurcation of a Cycle in curved Spacetime.

If in a linearized planar structure in a reference_frame $\langle \{X^{[1]},X^{[2]}\} \rangle$ on geodetic curve $\langle C_1 \rangle$ will enclose a center with a stable closed curve, the JACOBIAN_matrix $\langle \mathbb{P} \rangle$ of the appropriate dynamic system will have – according to P.G. BAKKER[1] – pure imaginary eigenvalues $\langle \lambda_1,\lambda_2 \rangle$:

$$\begin{aligned} \mathrm{Re}(\lambda_1, \lambda_2) &= 0 \\ \lambda_1 \cdot \lambda_2 &> 0. \end{aligned}$$

Eigenvalues again are obtained from the characteristic equation in its general form:

$$\begin{array}{rcllrcl} \mathrm{tr}(\mathbb{P})/2 \pm i\{\det(\mathbb{P}) - \mathrm{tr}(\mathbb{P})^2/4\}^{1/2} & = & \lambda_{1,2} & \Rightarrow & \mathrm{tr}(\mathbb{P}) = P^{[11]}+P^{[22]} & = & 0 \\ \pm i\{\det(\mathbb{P})\}^{1/2} & = & \lambda_{1,2} & & & & \\ \det(\mathbb{P}) & > & 0 & \Rightarrow & -(P^{[11]})^2 - P^{[12]}P^{[21]} & > & 0 \\ & & & \Rightarrow & -P^{[12]}P^{[21]} & > & 0 \\ -P^{[12]}P^{[21]} & > & 0 & \Rightarrow & P^{[12]}+P^{[21]} & \gtrless & 0. \end{array}$$

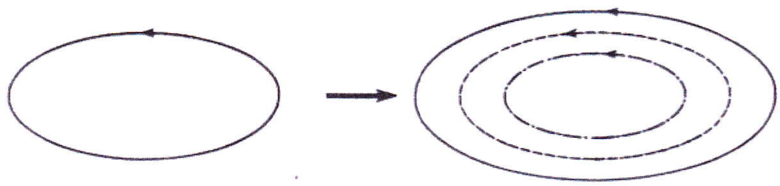
The eigenvalues of $\langle \mathbb{Q} \rangle$ may generally be written as:

$$\begin{array}{rcl} \mathrm{tr}(\mathbb{Q})/2 \pm i\{\det(\mathbb{Q}) - \mathrm{tr}(\mathbb{Q})^2/4\}^{1/2}] & = & \mu_{1,2} \\ Q^{[11]}+Q^{[22]} & = & \mathrm{Sp}(\mathbb{Q}) \\ Q^{[11]}Q^{[22]} - Q^{[12]}Q^{[21]} & = & \det(\mathbb{Q}). \end{array}$$

If the transformation of $\langle \mathbb{P} \rangle$ into $\langle \mathbb{Q} \rangle$ will result in:

$$\mathrm{tr}(\mathbb{Q}) \;>\; 0,$$

then – according to H.HAKEN[7] – a former stable cycle on geodetic curve $\langle C_1 \rangle$ has bifurcated into an unstable cycle on $\langle C_2 \rangle$ associated with other stable cycles



As we already know, between $\langle \mathbb{P} \rangle$ and $\langle \mathbb{Q} \rangle$ relations:

$$Q^{ij} \;=\; (d\rho/d\varrho)^2 g^{ij}(\underline{y}) g_{ab}(\underline{x}) P^{ab},$$

must hold and therefore we may further write:

$$
\begin{aligned}
\mathrm{tr}(\mathbb{Q}) &= (d\rho/d\varrho)^2\{g^{[11]}(\underline{y})+g^{[22]}(\underline{y})\}\{g_{[11]}(\underline{x})P^{[11]}+g_{[22]}(\underline{x})P^{[22]}+g_{[12]}(\underline{x})[P^{[12]}+P^{[21]}]\} \\
&= (d\rho/d\varrho)^2\{g^{[11]}(\underline{y})+g^{[22]}(\underline{y})\}\{P^{[11]}[g_{[11]}(\underline{x})-g_{[22]}(\underline{x})]+g_{[12]}(\underline{x})[P^{[12]}+P^{[21]}]\}
\end{aligned}
$$

Taking in mind again, that metric is always a symmetric tensor with positive_definite components, whose values at $\langle\, \underline{x} \,\rangle$ and $\langle\, \underline{y} \,\rangle$ can be chosen according to the conditions above, and with the proper restrictions for $\langle\, \mathbb{P} \,\rangle$, one will obtain:

$$
\begin{array}{rclcrcl}
g^{[11]}(\underline{y})+g^{[22]}(\underline{y}) & > & 0 & & & & \\
g_{[11]}(\underline{x})-g_{[22]}(\underline{x}) & > & 0 & \Rightarrow & P^{[11]}[g_{[11]}(\underline{x})-g_{[22]}(\underline{x})] & > & 0 \\
P^{[12]}+P^{[21]} & > & 0 & \Rightarrow & g_{[12]}(\underline{x})[P^{[12]}+P^{[21]}] & > & 0,
\end{array}
$$

and this finally results in:

$$
\mathrm{tr}(\mathbb{Q}) \;>\; 0.
$$

This makes it obvious, a transformation from $\langle\, C_1 \,\rangle$ to $\langle\, C_2 \,\rangle$ in curved spacetime may cause a stable cycle of a linear structure to bifurcate into an unstable cycle with simultaneous generation of other stable cycles. Because each sample of cycles created by the appropriate transformation can be contracted through affine mappings onto one cycle again, this kind of bifurcation will not change the topology of the original structure.

5. Summary.

From the group of dynamical systems specified by a first order time_derivation of the space_variable in a nonlinear field, which normally is explicitly dependent on the space_ and time_variable, we extracted in a first step the class of autonomous ones. A system of this type has a field_function, which is no more explicitly time_dependent what leads to its solution_structure as depending on the space_variable only. Next we concentrated on those autonomeous systems, which have planar orbit_structures containing fixed_point solutions and/or cycles both constant in time as the flows surrounding them as well. Field_functions of the dynamical system may be linearized for small deviations from the just mentioned objects in good agreement with the original, nonlinear ones by expanding them into TAYLOR_series and neglect all terms of higher than linear order. Afterwards these structures were tested whether they change stabilities of their fixed points and/or cycles and probably their topologies as well during transformations in spacetime.

In cases where environments are transformed between reference_frames bounded to geodetic curves in flat spacetime (so_called inertial_frames), the original and final structures keep similar and thus do not change topology, as long as they move relatively to each other slower than speed of light.

However, if linear structures between suspended frames on different geodetic curves in a curved spacetime are transformed, stabilities of fixed_points or cycles may change and the topologies of their orbit_flows as well. These factual situations we have demonstrated by three typical examples:

- In the first one it could be shown, that a sink may possibly convert into a saddle while new stable points are generated simultaneously. The topology of the flow_ structure changed during this bifurcation because afterwards no affine mappings could match the image_flow with the original one again.
- By another example could be shown, how during spacetime_transformation a stable fixed_point may convert to a cycle in performing a so called HOPF_bifurcation. In contrary to the former example, this operation will not change the topology of the orbit_structure.
- Finally it became obvious by a further example, that during a spacetime_ transformation an initially stable cycle can convert to an unstable one and give rise simultaneously for the creation of further stable cycles. Again no change in topology between the original_ and image_structure will occur, because simple contraction may cause them to match again.

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