ON THE COMPLETELY INTEGRABLE CALOGERO TYPE DISCRETIZATIONS OF NONLINEAR LAX INTEGRABLE DYNAMICAL SYSTEMS AND THE RELATED MARKOV TYPE CO-ADJOINT ORBITS

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Abstract. The Calogero type matrix discretization scheme is applied to constructing the Lax type integrable discretizations of one wide enough class of nonlinear integrable dynamical systems on functional manifolds. Their Lie-algebraic structure and complete integrability related with co-adjoint orbits on the Markov co-algebras is discussed. It is shown that a set of conservation laws and the associated Poisson structure ensue as a byproduct of the approach devised. Based on the Lie algebras quasi-representation property the limiting procedure of finding the nonlinear dynamical systems on the corresponding functional spaces is demonstrated.

1. Introduction: the discretization and related Markov algebra splitting

With a fairly generous definition, a one-dimensional real-valued discrete nonlinear dynamical system on a manifold $M \subset l_2(\mathbb{Z};\mathbb{R}^m)$ for some finite $m \in \mathbb{Z}_+$ is any evolution equation that can be written down as

$$du/dt = K[u], \tag{1.1}$$

where $t \in \mathbb{R}$ is the evolution parameter, $u \in M$ and $K : M \to T(M)$ is some smooth enough vector field [1, 2] on the manifold M. Very often such equations (1.1) can be naturally obtained as the standard discretization [3, 8, 13, 16] of a given smooth nonlinear differential dynamical system

$$d\mathrm{u}/dt = \mathcal{K}[\mathrm{u}] \tag{1.2}$$

on a functional submanifold $\mathcal{M} \subset L_2(\mathbb{R};\mathbb{R}^m)$, generated by a smooth vector field $\mathcal{K} : \mathcal{M} \to T(\mathcal{M})$. Namely, there exist such mesh points $x_j \neq x_i \in \mathbb{R}$ for $i \neq j \in \mathbb{Z}$, that the corresponding vector $\{\mathrm{u}(x_j) \in \mathbb{R}^m : j \in \mathbb{Z}\} = u \in M$ and the suitable discretization of (1.2) coincides with (1.1).

Other approach to the discretization of (1.2) is based on the Calogero type [7, 8] scheme of constructing finite-dimensional quasi-representations of the infinite-dimensional Heisenberg-Weyl algebra of operators $\mathfrak{h} := \{\hat{x}, D_x, \hat{1} : x \in \mathbb{R}\}$, where $D_x := \partial/\partial x$, in some functional submanifold $\mathcal{M}_0 \subset C^\infty([a,b];\mathbb{R}) \cap L_2([a,b];\mathbb{R})$ of differentiable functions, as owing to the well known von-Neumann theorem [17, 26], there exists no exact representation of $\mathfrak{h}$ in a finite-dimensional functional subspace $\mathcal{M}_0^N \subset \mathcal{M}_0$ for any $N \in \mathbb{Z}_+$. For example, any smooth scalar function $f \in \mathcal{M}_0$ on an interval $[a,b] \subset \mathbb{R}$ can be interpolated [26, 7] in a polynomial form as follows:

$$f(x) \to f_N(x) := \sum_{j=1}^{N} (f(x_j)\rho_j^{-1}) e_j(x), \tag{1.3}$$

$$e_j(x) := \prod_{i=\overline{1,N}, i\neq j} (x - x_i), \rho_j := \prod_{i=\overline{1,N}, i\neq j}^{N} (x_j - x_i)$$

and its derivative, respectively, as

$$D_x f(x) \to D_x f_N(x) = \sum_{i,j=\overline{1,N}} Z_{ij}(f(x_j)\rho_j^{-1})\ e_i(x)), \tag{1.4}$$

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where $f_N(x) \in \mathcal{M}_0^N$ subject to the polynomial basis $\{e_j(x) \in \mathcal{M}_0^N : j = \overline{1,N}\}$. Then the well known Calogero type quasi-representation [7, 18, 14, 10, 11] of the Heisenberg-Weyl algebra $\mathfrak{h}$ is obtained as

$$
\begin{aligned}
&End(\mathcal{M}_0) \ni \hat{x} \to X := diag\{x_1, x_2, ..., x_N\} \in End\ l_2(\mathbb{Z}_N; \mathbb{R}), \\
&End(\mathcal{M}_0) \ni \hat{1} \to I := diag\{1, 1, ..., 1\} \in End\ l_2(\mathbb{Z}_N; \mathbb{R}) \\
&End(\mathcal{M}_0) \ni D_x \to Z := \{Z_{ij} := (x_i - x_j)^{-1}, i \neq j = \overline{1,N}; \\
&\qquad Z_{ii} := \sum_{j=1, j\neq i}^{N} (x_i - x_j)^{-1} : i = \overline{1,N}\} \in End\ l_2(\mathbb{Z}_N; \mathbb{R}),
\end{aligned}
\tag{1.5}
$$

where interpolating mesh points $x_i \neq x_j \in \mathbb{R}, i \neq j = \overline{1,N}$, are chosen to be different and satisfying in a suitably defined finite-dimensional Hilbert space $l_2(\mathbb{Z}_N; \mathbb{R})$ the strong as $N \to \infty$ limiting canonical Lie algebra relationship

$$
\lim_{N\to\infty}([Z, X] - I) = 0. \tag{1.6}
$$

The matrix quasi-representations (1.5) make it possible to construct easily a *naive* matrix discretization of the nonlinear dynamical system (1.2) as follows:

$$
dU^{(m)}/dt = \mathcal{K}(U^{(m)}, [Z^{(m)}, U^{(m)}], [Z^{(m)}, [Z^{(m)}, U^{(m)}]], ...[\underbrace{Z^{(m)}, ..., [Z^{(m)}}_{(p\text{-}times)}, U^{(m)}]]), \tag{1.7}
$$

where the matrices

$$
U^{(m)} := u(X^{(m)}) = diag(u(x_1), u(x_2), ..., u(x_N)) \in End\ l_2(\mathbb{Z}_N; \mathbb{R})^{\otimes m}, \tag{1.8}
$$

$$
Z^{(m)} := \underbrace{Z \otimes Z \otimes ... \otimes Z}_{(m\text{-}times)} \in End\ l_2(\mathbb{Z}_N; \mathbb{R})^{\otimes m}
$$

belong to the tensor product matrix space

$$
Endl_2(\mathbb{Z}_N; \mathbb{R})^{\otimes m} := Endl_2(\mathbb{Z}_N; \mathbb{R}) \otimes Endl_2(\mathbb{Z}_N; \mathbb{R}) \underset{(m\text{-}times)}{\otimes ... \otimes} Endl_2(\mathbb{Z}_N; \mathbb{R}). \tag{1.9}
$$

When deriving the matrix equation (1.7), we took into account that $\mathcal{K}[\mathrm{u}] := \mathcal{K}(u, D_x\mathrm{u}, D_x^2\mathrm{u}, ...D_x^p\mathrm{u})$ for some fixed $p \in \mathbb{Z}_+$, and for arbitrary operator mapping $\varphi_N(\hat{x}) : \mathcal{M}_0^N \to \mathcal{M}_0^N$ we made use of the Calogero type quasi-representation property:

$$
End(\mathcal{M}_0) \ni (D_x^n \varphi_N)(\hat{x}) \to \underbrace{[Z, [Z, [Z, ..., [Z}_{(n\text{-}times)}, \varphi(X)]]]...] \in End\ l_2(\mathbb{Z}_N; \mathbb{R}), \tag{1.10}
$$

which holds for arbitrary operator derivatives $(D_x^n \varphi_N)(\hat{x}) : \mathcal{M}_0^N \to \mathcal{M}_0^N, n \in \mathbb{Z}_+$.

Now we will take into account the observation [15] that the quasi-representations (1.5) belong, respectively, to the Markov direct sum splitting of the general Lie algebra $gl(N; \mathbb{R}) := \mathfrak{g} = \mathrm{M}(\mathfrak{g}) \oplus \mathrm{E}(\mathfrak{g})$:

$$
I, X \in \mathrm{E}(\mathfrak{g}), \quad Z \in \mathrm{M}(\mathfrak{g}), \tag{1.11}
$$

where, by definition, the linear subspaces

$$
\begin{aligned}
&\mathrm{M}(\mathfrak{g}) := \{\mathrm{M}(A) \in \mathfrak{g} : \mathrm{M}(A) = A - diag(eA)\} \\
&\mathrm{E}(\mathfrak{g}) := \{\mathrm{E}(A) \in \mathfrak{g} : \mathrm{E}(A) = diag(eA),\ e := (1, 1, ..., 1) \in l_2(\mathbb{Z}_N; \mathbb{R})^*,
\end{aligned}
\tag{1.12}
$$

are Lie subalgebras of the Lie algebra $\mathfrak{g}$. Introduce now, by definition, projections $\mathrm{P}_{\mathrm{M}(\mathfrak{g})} : \mathfrak{g} = \mathrm{M}(\mathfrak{g}) \subset \mathfrak{g}$, $\mathrm{P}_{\mathrm{E}(\mathfrak{g})} : \mathfrak{g} = \mathrm{E}(\mathfrak{g}) \subset \mathfrak{g}$. Then within the standard R-matrix approach [4, 5, 12, 23, 25] the expression

$$
[X, Y]_{\mathrm{R}} := [\mathrm{R}X, Y] + [X, \mathrm{R}Y],\ \mathrm{R} := 1/2(\mathrm{P}_{M(\mathfrak{g})} - \mathrm{P}_{E(\mathfrak{g})}), \tag{1.13}
$$

for arbitrary $X, Y \in \mathfrak{g}$ defines on $\mathfrak{g}$ a new Lie commutator structure, generating on the space $\mathcal{D}(\mathfrak{g})$ the deformed Lie-Poisson bracket

$$
\{\gamma, \eta\}_{\mathrm{R}}(\alpha) :=< \alpha, [\nabla\gamma(\alpha), \nabla\eta(\alpha)]_{\mathrm{R}} >=< \alpha, [\mathrm{P}_{M(\mathfrak{g})}\nabla\gamma(\alpha), \mathrm{P}_{M(\mathfrak{g})}\nabla\eta(\alpha)] > \tag{1.14}
$$

for $\gamma, \eta \in \mathcal{D}(\mathfrak{g})$ and any $\alpha \in \mathfrak{g}^*$, generalizing the classical Lie-Poisson bracket

$$
\{\gamma, \eta\}(\alpha) :=< \alpha, [\nabla\gamma(\alpha), \nabla\eta(\alpha)] >=< \alpha, [\nabla\gamma(\alpha), \nabla\eta(\alpha)] > \tag{1.15}
$$

on $\mathfrak{g}$. Here the bi-linear trace-functional on the $\mathfrak{g}$

$$< X, Y >:= \mathrm{tr}\ (XY) \tag{1.16}$$

for $X, Y \in \mathfrak{g}$ is nondegenerate and Ad-invariant. Taking into account that with respect to this trace-functional (1.16) the Lie algebra $\mathfrak{g} \simeq \mathfrak{g}^*$, the Poisson bracket (1.14) generates for any Hamiltonian function $H \in \mathcal{D}(\mathfrak{g})$ the following dynamical system on arbitrary $\alpha \in \mathfrak{g}$:

$$d\alpha/dt = \mathrm{P}_{\mathrm{E}(\mathfrak{g})^\perp}[\mathrm{P}_{\mathrm{M}(\mathfrak{g})}\nabla H(\alpha), \alpha], \tag{1.17}$$

where we took into account that projections $\mathrm{P}^*_{\mathrm{M}(\mathfrak{g})} \simeq \mathrm{P}_{\mathrm{E}(\mathfrak{g})^\perp}$ and $\mathrm{P}^*_{\mathrm{E}(\mathfrak{g})} \simeq \mathrm{P}_{\mathrm{M}(\mathfrak{g})^\perp}$. This construction becomes more simpler in the case when the Hamiltonian function $H \in \mathrm{I}(\mathfrak{g})$ is taken to be a Casimir one with respect to the classical Lie-Poisson bracket (1.15), satisfying the condition

$$[\alpha, \nabla H(\alpha)] = 0 \tag{1.18}$$

for any $\alpha \in \mathfrak{g}$.

Amongst nonlinear differential dynamical systems (1.2) there exist a wide class of nonlinear evolution equations which are Lax type [8, 16, 19, 12] integrable and whose discretizations are often very important for their numerical analysis and diverse applications. Yet in general, the presented above directly discretized matrix dynamical system (1.7) does not *a priori* inherits the Lax type integrability of (1.2). Thus, a natural question arises: *how to construct a priori Lax type integrable matrix discretization of a given Lax type integrable nonlinear dynamical system (1.2)?*

As the Lax type representations of the presumably integrable dynamical systems (1.2) depend on an arbitrary spectral parameter $\lambda \in \mathbb{C}$, it motivates us to study their corresponding matrix Lax type quasi-representations, also depending on the spectral parameter $\lambda \in \mathbb{C}$, as well as depending on the basis matrix representation operators (1.5), belonging, respectively, to the Markov direct sum splitting of the general Lie algebra $gl(N;\mathbb{R}) := \mathfrak{g} = \mathrm{M}(\mathfrak{g}) \oplus \mathrm{E}(\mathfrak{g})$. This can be done effectively by means of introducing the notion of the metrized loop [4, 12, 23, 5] algebra $\tilde{\mathfrak{g}} := \mathfrak{g}\otimes\mathbb{C}[[\lambda,\ \lambda^{-1}]]$ and the related Lax type integrable Poisson flows on it.

Below I present a solution to this posed above question in the case of a special class of nonlinear Lax type integrable dynamical systems on functional manifolds making use of the Calogero type discretization scheme and the analysis of the Markov type co-adjoint orbits by means of the related Lie-algebraic techniques.

2. The Lie algebraic setting and the Calogero type linear matrix spectral problems

There is introduced a metrized loop algebra $\tilde{\mathfrak{g}} := \mathfrak{g}\otimes\mathbb{C}[[\lambda,\ \lambda^{-1}]]$, generated by the Lie algebra $\mathfrak{g} := gl(N;\mathbb{R})$ and the related Laurent series

$$\tilde{\mathfrak{g}} :=\{X(\lambda) = \sum_{j\ll\infty} X_j\lambda^j : X_j \in \mathfrak{g},\quad \mathbb{Z} \ni j \ll \infty, \lambda \in \mathbb{C}\}. \tag{2.1}$$

It is endowed with the standard matrix Lie bracket

$$[X(\lambda), Y(\lambda)] := \sum_{s\ll\infty} \lambda^s (\sum_{j+k=s} [X_j, Y_k]), \tag{2.2}$$

defined for any $X(\lambda), Y(\lambda) \in \tilde{\mathfrak{g}}$ and the simplest Ad-invariant nondegenerate scalar product

$$< X(\lambda), Y(\lambda) >_{-\mathrm{I}} = \mathrm{Tr}(X(\lambda)Y(\lambda)) := res\ tr(X(\lambda)Y(\lambda)), \tag{2.3}$$

satisfying for all $X(\lambda), Y(\lambda)$ and $Z(\lambda) \in \tilde{\mathfrak{g}}$ the condition

$$< X(\lambda), [Y(\lambda), Z(\lambda)] >_{-\mathrm{I}} = < [X(\lambda), Y(\lambda)], Z(\lambda) >_{-\mathrm{I}}. \tag{2.4}$$

Consider now the introduced above Markov splitting (1.12) of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} := \mathrm{M}(\mathfrak{g}) \oplus \mathrm{E}(\mathfrak{g}), \tag{2.5}$$

where for any $X \in \mathfrak{g}$

$$\mathrm{M}(X) := X - diag(eX), \quad \mathrm{E}(X) := diag(eX),\ e := (1, 1, ..., 1) \in l_2(\mathbb{Z}_N)^*, \tag{2.6}$$

and the components $\mathrm{M}(\mathfrak{g}) \subset \mathfrak{g}$ and $\mathrm{E}(\mathfrak{g}) \subset \mathfrak{g}$ are suitable matrix Lie subalgebras, that is $[\mathrm{M}(\mathfrak{g}), \mathrm{M}(\mathfrak{g})] \subset \mathrm{M}(\mathfrak{g})$ and $[\mathrm{E}(\mathfrak{g}), \mathrm{E}(\mathfrak{g})] = 0 \in \mathrm{E}(\mathfrak{g})$.

The following observation is crucial for our next analysis: the loop algebra $\tilde{\mathfrak{g}}$ inherits the Markov type splitting (2.5) into two Lie subalgebras:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_-, \tag{2.7}$$

where the projection $P_+\tilde{\mathfrak{g}} :=\tilde{\mathfrak{g}}_+$,

$$\tilde{\mathfrak{g}}_+ := \{X(\lambda) = \sum_{j\in\mathbb{Z}_+} X_j\lambda^j : X_0 \in \mathrm{M}(\mathfrak{g}),\quad X_j \in \mathfrak{g},\quad \mathbb{N}\ni j \ll \infty, \lambda \in \mathbb{C}\} \tag{2.8}$$

and the projection $\mathrm{P}_-\tilde{\mathfrak{g}} :=\tilde{\mathfrak{g}}_-$,

$$\tilde{\mathfrak{g}}_- := \{X(\lambda) = \sum_{j\in\mathbb{Z}_-} Y_j\lambda^j : Y_0 \in \mathrm{E}(\mathfrak{g}),\quad Y_j \in \mathfrak{g},\quad j \in \mathbb{Z}_-\backslash\{0\}\ , \lambda \in \mathbb{C}\} \tag{2.9}$$

satisfy the Lie commutator relationships $[\tilde{\mathfrak{g}}_+, \tilde{\mathfrak{g}}_+] \subset \tilde{\mathfrak{g}}_+$ and $[\tilde{\mathfrak{g}}_-, \tilde{\mathfrak{g}}_-] \subset \tilde{\mathfrak{g}}_-$. It is easy to calculate the adjoint spaces $\tilde{\mathfrak{g}}_+^*$ and $\tilde{\mathfrak{g}}_-^*$ to Lie subalgebras (2.8) and (2.9):

$$\tilde{\mathfrak{g}}_+^* \simeq \tilde{\mathfrak{g}}_-^\top = \{Y(\lambda) = \sum_{j\in\mathbb{Z}_-} Y_j\lambda^j : Y_0 \in \mathrm{E}(\mathfrak{g})^\perp,\quad Y_j \in \mathfrak{g},\quad j \in \mathbb{Z}_-\backslash\{0\}\ , \lambda \in \mathbb{C}\}, \tag{2.10}$$

$$\tilde{\mathfrak{g}}_-^* \simeq \tilde{\mathfrak{g}}_+^\top = \{X(\lambda) = \sum_{j\in\mathbb{Z}_+} X_j\lambda^j : X_0 \in \mathrm{M}(\mathfrak{g})^\perp,\quad X_j \in \mathfrak{g},\quad \mathbb{N}\ni j \ll \infty, \lambda \in \mathbb{C}\},$$

where we took into account that

$$\mathrm{M}(\mathfrak{g})^* \simeq \mathrm{E}(\mathfrak{g})^\perp = \{Y \in \mathfrak{g} :\ diag(Y) = 0\}, \tag{2.11}$$

$$\mathrm{E}(\mathfrak{g})^* \simeq \mathrm{M}(\mathfrak{g})^\perp = \{X \in \mathfrak{g} :\quad X\ = q\otimes e,$$

$$e := (1,1,...,1)\ \in l_2(\mathbb{Z}_N;\mathbb{R})^*, q \in l_2(\mathbb{Z}_N;\mathbb{R})\}.$$

The splitting (2.7) makes it possible to define a related classical R-structure on the loop algebra $\tilde{\mathfrak{g}}$: for any $X(\lambda), Y(\lambda) \in \tilde{\mathfrak{g}}$ the commutator

$$[X(\lambda), Y(\lambda)]_{\mathrm{R}} := \quad ([\mathrm{R}X(\lambda), Y(\lambda)] +\ [X(\lambda), \mathrm{R}Y(\lambda)]), \tag{2.12}$$

satisfies the Lie algebra commutator property, where the linear space homomorphism $\mathrm{R} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is defined for an arbitrary $X(\lambda) \in \tilde{\mathfrak{g}}$ as

$$\mathrm{R}X(\lambda) :=\ 1/2(\mathrm{P}_+X(\lambda) - \mathrm{P}_-X(\lambda)). \tag{2.13}$$

The following important classical (see, for instance, [4, 5, 23]) theorem holds.

Theorem 2.1. *(Adler-Kostant-Souriau) Let smooth functionals $\gamma, \eta : \tilde{\mathfrak{g}}^* \to \mathbb{R}$ be Casimir ones subject to the Lie bracket (2.2), that is*

$$[\nabla\gamma(l(\lambda)), l(\lambda)] = 0 = [\nabla\eta(l(\lambda)), l(\lambda)]. \tag{2.14}$$

for any $l(\lambda) \in \mathfrak{g}^*$. Then their modified Lie-Poisson bracket

$$\{\gamma, \eta\}\ :=< l(\lambda), [\nabla\gamma(l(\lambda)), \nabla\eta(l(\lambda))]_{\mathrm{R}} >_{-1} \tag{2.15}$$

vanishes: $\{\gamma, \eta\}\ = 0$ on the whole space $\tilde{\mathfrak{g}}^*$.

Based on the splitting (2.7) one can easily to calculate the actions of adjoint operators $\mathrm{P}_+^* : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_-^\top \subset \tilde{\mathfrak{g}}_-$ and $\mathrm{P}_-^* : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_+^\top \subset \tilde{\mathfrak{g}}_+$. Namely, owing to the identification $\tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{g}}^*$ one finds that the following equalities

$$\mathrm{P}_+^* = \mathrm{P}_{\tilde{\mathfrak{g}}_-^\top} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_-^\top \subset \tilde{\mathfrak{g}}_- \tag{2.16}$$

and

$$\mathrm{P}_-^* = \mathrm{P}_{\tilde{\mathfrak{g}}_+^\top} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_+^\top \subset \tilde{\mathfrak{g}}_+ \tag{2.17}$$

hold.

Theorem 2.1 above and the equalities (2.16) and (2.17) make it possible to construct a wide class of Liouvile integrable dynamical systems [5, 23] on the Markov matrix subspace $E(\mathfrak{g})$, if to reduce the related Hamiltonian vector field

$$\frac{d}{dt}\alpha(\lambda) := \{H, \alpha(\lambda\} = [\mathrm{P}_+\nabla H(\alpha(\lambda)), \alpha(\lambda)], \tag{2.18}$$

on the element $\alpha(\lambda) \in \tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{g}}^*$, generated by a specially chosen smooth Casimir functional $H \quad : \tilde{\mathfrak{g}} \to \mathbb{R}$. The latter is considered with respect to the standard Lie-Poisson bracket

$$\{\gamma, \eta\}_{Lie} \ :=< l(\lambda), [\nabla\gamma(l(\lambda)), \nabla\eta(l(\lambda))] \ >_{-1} \tag{2.19}$$

for any smooth functionals $\gamma, \eta \in \mathcal{D}(\tilde{\mathfrak{g}})$.

Consider the following smooth functionals $\gamma_n^{(k)} : \tilde{\mathfrak{g}}^* \to \mathbb{R}$, $n, k \in \mathbb{Z}_+$, where

$$\begin{aligned} \gamma_n^{(k)} \ &:= 1/(n+1) < (\alpha(\lambda)\lambda^{-|\alpha(\lambda)|})^{n+1}, \lambda^{k+|\alpha(\lambda)|} \ >_{-1}= \\ &= 1/(n+1)Tr(\lambda^{k+|\alpha(\lambda)|} \ (\alpha(\lambda)\lambda^{-|\alpha(\lambda)|})^n). \end{aligned} \tag{2.20}$$

They are, evidently, Casimir ones for the Poisson bracket (2.19), whose gradients equal

$$\nabla\gamma_n^{(k)}(\alpha(\lambda)) = (\alpha(\lambda)\lambda^{-|\alpha(\lambda)|})^n\lambda^k. \tag{2.21}$$

Taking, for example, $n = 2$ and $k = 4$, the corresponding value of the element $\mathrm{P}_+\nabla\gamma_2^{(4)}(\alpha(\lambda)) \in \tilde{\mathfrak{g}}_+$ at the element $\alpha(\lambda) = \lambda^3 I + \lambda^2 U \ + \lambda V + Z$ then equals

$$\begin{aligned} &\mathrm{P}_+\nabla\gamma_2^{(4)}(\lambda^3 I + \lambda^2 U \ + \lambda V + Z) = \lambda^4 I + 2\lambda^3 U + \\ &+\lambda^2(2V + U^2) + \lambda(2Z + UV + VU) + \mathrm{M}(ZU + UZ) \ . \end{aligned} \tag{2.22}$$

From the commutator relationship (2.18) one obtains that the following matrix equation

$$\begin{aligned} &\tfrac{d}{dt}(\lambda^2 U \ + \lambda V + Z) = [\lambda^3 I + 3\lambda^2 U/2 + \lambda(3Z/2 - 3U^2/8) - \\ &-U^3/16 - 3(ZU + UZ)/8 \ , \lambda^2 U \ + \lambda V + Z] \end{aligned} \tag{2.23}$$

holds for any parameter $\lambda \in \mathbb{C}$. The trace-functionals $H_m := tr(\lambda^3 I + \lambda^2 U \ + \lambda V + Z)^{m/3}$, $m \in \mathbb{Z}_+$, are all nontrivial and involutive with respect to the Poisson bracket (2.15) conservation laws of the Riemann type discrete matrix dynamical system

$$\begin{aligned} &dU/dt = -2[Z, V], \\ &dV/dt = -[Z, VU + UZ] + [UZ + ZU, V], \end{aligned} \tag{2.24}$$

easily following from (2.23). Moreover, the discrete matrix dynamical system (2.24), as follows from is a completely integrable discrete approximation of the corresponding limiting as $N \to \infty$ Riemann type equations in partial derivatives

$$du/dt = \ -2v_x, \qquad dv/dt = uv_x - vu_x. \tag{2.25}$$

Thus, concerning the limiting dynamical system (2.25) one can formulate the following proposition.

Proposition 2.2. *The dynamical system (2.25) allows for any $N \in \mathbb{Z}_+$ the completely integrable matrix discretization (2.24), which allows the Lax type representation (2.23).*

As a simple consequence of Proposition 2.2, the obtained above limiting dynamical system (2.25) is also Lax type integrable. More applications of the devised above Calogero type discrete approximation approach to different Lax type integrable nonlinear dynamical systems are under preparation.

3. Conclusion

An observation, that the Calogero type discretization of the Heisenberg-Weyl algebra is related to the Markov type splitting of the general Lie algebra $gl(N;\mathbb{R}), N \in \mathbb{Z}_{+}$, proved to be both interesting and useful for analytical constructing completely integrable discretizations of the Lax type integrable nonlinear dynamical systems on functional manifolds. They are represented as co-adjoint orbits on the Markov type co-algebras and analyzed by means of the modern Lie-algebraic techniques. It is shown that a set of conservation laws and the associated Poisson structure ensue as a byproduct of the approach devised. Based on the Lie algebras quasi-representation property the limiting procedure of finding the nonlinear dynamical systems on the corresponding functional spaces is demonstrated.

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