Dirac-Wu-Yang Monopoles, Gauge Symmetry, Orientation-Entanglement & Twist, Quantum Topology, Atomic Structure, Electric / Magnetic Symmetry Breaking, and the Fractional Quantum Hall Effect

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Abstract: The purpose of this paper is to explain the pattern of fill factors observed in the Fractional Quantum Hall Effect (FQHE) to be restricted to odd-integer denominators as well as the sole even-integer denominator of 2. The method is to use the mathematics of gauge theory to fully develop Dirac monopoles without strings as originally taught by Wu and Yang, while accounting for topological orientation-entanglement and related "twistor" relationships between spinors and their environment in the physical space of spacetime. We find that the odd-integer denominators are permitted and the even-integer denominators excluded if FQHE only displays electrons of identical orientation-entanglement "version," i.e., only electrons separated by 4π not 2π. We also find that the even-integer denominator of 2 is permitted if entangled electrons can pair into boson states, and that all other even-integer denominators are excluded because bosons are not subject to the same Exclusion statistics as are fermions. Because this proposed relation between the Dirac monopoles and the FQHE presupposes an electric / magnetic duality near 0K, and because magnetic monopoles are certainly not observed at higher temperatures, we also find how to break this duality symmetry with the consequence that the low-temperature Dirac monopoles are replaced by a "thermal residue" at higher temperatures. We conclude that the observed FQHE fill factor pattern can be fundamentally explained using nothing other than the mathematics of gauge theory in view of orientation, entanglement and twist, with proper breaking of the low-temperature electric / magnetic duality. An unanticipated bonus is that the quantum topology emerging from this analysis appears to map precisely to the electronic orbital structure of atoms. This provides the basis for proposed experiments to closely observe the FQHE quasiparticles to seek correlations to the angular momentum observed in atomic electron shells, and to boson spin states.

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Contents

1. Introduction: Wu and Yang and the Dirac Monopole without Strings

The Fractional Quantum Hall Effect (FQHE) observed in two-dimensional systems of electrons at low temperatures in superconducting materials subjected to large perpendicular magnetic fields is characterized by observed filling factors $\nu = n/l$, where n and l are each integers, but where l is an *odd integer only*, with the exception that *l may also be the even integer 2*. In other words, the apparent pattern, widely reported and studied in the literature, is $\nu = n/l$ with $n = \pm 1, \pm 2, \pm 3...$ and $l = 1, 2, 3, 5, 7, 9, 11...$, see, e.g., [1], [2], [3], [4], [5], [6] generally, and for the even denominator 2, see, e.g., [6], [7] for $\nu = 1/2$, [8] for $\nu = 3/2$, [9] for $\nu = 5/2$ and [10] for $\nu = 7/2$. Two questions arise from this effect: why are the denominators in the filling factor odd but not even (including the quantization of whole unit charges with denominator l=1), and why is the even denominator l=2 an apparent exception? We show that this pattern of filling factor denominators has a fundamental explanation based on using the mathematics of $U(1)_{em}$ gauge theory to develop the Dirac Quantization Condition (DQC) for Dirac-Wu-Yang (DWY) monopoles, in view of how orientation-entanglement (OE) applies to fermion spinors but not to bosons, and also in view of a "twisting" associated with orientation-entanglement which appears to have been underreported in the literature. Along the way, we demonstrate have the electric / magnetic duality symmetry of Dirac monopoles does exist near 0K, and how that symmetry is broken at higher temperatures leaving in its stead a "thermal residue" possible responsible in a fundamental way for the very existence of heat energy in nature.

In 1931 Dirac [11] discovered that the existence of magnetic monopoles would imply that the electric charge must be quantized. While charge quantization had been known for several decades based on the experimental work of Thompson [12] and Millikan [13], Dirac was apparently the first to lay out a possible theoretical imperative for this quantization. Using a hypothesized solenoid of singularly-thin width known as the Dirac string to shunt magnetic field lines out to mathematical infinity, Dirac established that a magnetic charge strength μ would be related to the electric charge strength e according to $e\mu = 2\pi n$, where n is an integer. This became known as the Dirac Quantization Condition (DQC). This electric charge strength is the same one which, at low probe energies, is related to the running "fine structure" coupling via $4\pi\alpha = e^2/\hbar c \cong 1/137.036$, see, e.g., Witten's [14], pages 27 and 28. Subsequently, Wu and Yang used gauge potentials, which are locally- but not globally-exact, to obtain the exact same DQC without strings [15], [16]. Their approach is concisely summarized by Zee on pages 220-221 of [17] and will be briefly reviewed here, because it provides the methodological basis for understanding the pattern of filling factors observed for the FQHE. Throughout we shall use the natural units of $\hbar = c = 1$.

Using the differential one form $A = A_\mu dx^\mu$ for the electromagnetic gauge field a.k.a. vector potential and the differential two-form $F = \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, a magnetic charge μ may be *defined* as the total net magnetic flux $\mu \equiv \oiint F$ passing through a closed two-dimensional surface S^2 which for convenience and symmetry we may take to be a sphere. Differential exterior calculus in spacetime geometry teaches that the exterior derivative of an exterior derivative is zero, dd=0, which means that the three-form equation $dF = ddA = 0$. Thus, via Gauss / Stokes, $\iiint 0 = \iiint dF = \oiint F = \mu$. In classical electrodynamics prior to Dirac,

this was taken to mean that the magnetic charge μ=0. But a close consideration of gauge symmetry, which is locally but not globally exact, tells a different story:

When a spin ½ fermion wavefunction (which we shall generally regard as that of the electron) undergoes a local gauge (really, phase) transformation $\psi(x) \rightarrow \psi'(x) = e^{i\Lambda(x)}\psi(x)$, the gauge field one-form transforms under $U(1)_{em}$ as

$$A \rightarrow A' = A + e^{-i\Lambda} de^{i\Lambda} / ie \,. \tag{1.1}$$

More generally for larger non-abelian gauge groups with gauge potential G and charge g, this transformation is $G \rightarrow G' = U^{\dagger}(G + d)U / ig$ where U is a unitary matrix $U^{\dagger}U = 1$. If we represent F in polar coordinates (r, φ, θ) in the three-dimensional space of physical spacetime as $F = (\mu / 4\pi) d\cos\theta d\varphi$, then because $F = dA$ and dd=0, we can deduce that $A = (\mu / 4\pi)\cos\theta d\varphi$. However, $d\varphi$ is indeterminate on the north and south poles, which is an inherent feature of three-dimensional space as are the non-commuting rotational properties of this space when represented by SO(3) or its double covering SU(2). To remove this indeterminacy and create a smooth geometric interface, we may define north and south coordinate patches over $A_N = (\mu / 4\pi)(\cos\theta - 1) d\varphi$ and $A_S = (\mu / 4\pi)(\cos\theta + 1) d\varphi$, respectively. But at places where these patches overlap, these gauge potentials are not the same, and specifically, their difference is $A_S - A_N = (\mu / 2\pi) d\varphi$, or written slightly differently:

$$A_N \rightarrow A'_N \equiv A_S = A_N + (\mu / 2\pi) d\varphi. \tag{1.2}$$

So comparing this with (1.1), to unite the two patches we may regard $A_S \equiv A'_N$ as a gauge-transformed state A'_N of A_N, for which the gauge transformation is simply:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi. \tag{1.3}$$

We simply note for the moment that $A_S \equiv A'_N$ which yields (1.3) is actually a commonly-made *assumption* that the north and south gauge field patches differ from one another by no more than a gauge transformation and so are not observably distinct, in order to yield a smooth unbroken geometric relationship between the north and south patches. *Whether the physics we observe in the natural world agrees with this assumption is a separate question.* In section 9 a reexamination of this assumption will have important consequences for relating physics near 0K with physics at other, higher temperatures.

This differential equation (1.3) for Λ and φ in relation to e and μ is solved by:

$$\exp(i\Lambda) = \exp\left(ie\mu \frac{\varphi}{2\pi} \right), \tag{1.4}$$

as can be seen simply by plugging $e^{i\Lambda}$ from (1.4) into the left hand side of (1.3) and reducing. This relates the azimuth angle φ which is one of the three spacetime coordinates, to the local gauge (phase) angle Λ, and thereby connects rotations through φ in physical space to rotations through Λ in the gauge space in a manner that we shall now explore in detail.

In polar coordinates, the azimuths $\varphi = 0$ and $\varphi = 2\pi$ in (1.4) describe exactly the same *orientation* (but not entanglement) on the surface of S^2. So to make sense of (1.4) at like-orientations, substituting $\varphi = 0$ and $\varphi = 2\pi$ into (1.4) and equating the two terms following this substitution, we must have:

$$\exp\left(i\Lambda\right) = \exp\left(ie\mu \cdot 0\right) = 1 = \exp\left(ie\mu \cdot 1\right). \tag{1.5}$$

Specifically, this means that $\exp\left(ie\mu\right) = 1$. Mathematically, the general solution for an equation of this form is $\exp\left(i2\pi n\right) = 1$ for any integer $n = 0, \pm 1, \pm 2, \pm 3...$, which is infinitely degenerate but quantized. As a result, the solution to (1.5) is:

$$\Lambda = e\mu = 2\pi n. \tag{1.6}$$

This, of course, is the Dirac Quantization Condition (DQC), which we see may also specified in relation to the gauge (phase) parameter Λ which is seen to be an quantized integer multiple of 2π.[*] It will be immediately apparent that this equation has an electric / magnetic duality symmetry under $e \leftrightarrow \mu$ interchange. And it will be equally apparent that if magnetic charges do exist in nature, they do not seem as of yet to have ever been observed. So understanding if there is some real, observable physics to be found from the monopole in (1.6) and their derivation is an undertaking of substantial interest.

Further, (1.6) with simple rearrangement tells us that the electric charge is quantized according to:

$$e = n\frac{2\pi}{\mu} = ne_{\text{u}} = \frac{\Lambda}{\mu}, \tag{1.7}$$

where the n=1 "unit" (u) of electric charge is $e_{\text{u}} \equiv 2\pi / \mu$, defined as 2π times the inverse of the magnetic charge. The customary interpretation of $e = n\left(2\pi / \mu\right)$ in (1.7), ever since Dirac first found this relationship, is the conditional statement that *if* this magnetic charge "exists," *then* the electric charge is quantized in units of e_{u}. It is important to keep in mind that the converse of

[*] It should be noted that when we used the local angles $\varphi(x) = \varphi_0 = 0$ and $\varphi(x) = \varphi_0 + 2\pi$ in (1.4), the implicit choice of $\varphi_0 = 0$ had no special physical significance. We could have used any other $0 < \varphi_0 < 2\pi$ or indeed any φ_0 whatsoever and still ended up with the exact same DQC in (1.6); $\varphi_0 = 0$ was merely the easiest mathematical choice. This means the DQC (1.6) is invariant under local gauge symmetry, as it must be to have possible physical meaning.

this conditional is not true: the observed quantization of electric charge does *not* imply that the magnetic charges do exist. In fact, as best as is known, this DWY magnetic charge μ has *not* been observed to date, while the firmly-established quantization of electric charge is explained not on the basis of these DWY magnetic charges, but on the basis of the charge generators $Q = Y/2 + I^3$ which emerge in Yang-Mills gauge theory following the electroweak symmetry breaking of $SU(2)_W \times U(1)_Y$ down to $U(1)_{em}$. So if this DWY monopole "exists," it would have to "exist" under some very specialized set of physical conditions and it would of course be desirable to know what those conditions might be.

We may finally go back to the original definition $\mu \equiv \oiint F$ and isolate μ in (1.6), thus:

$$\oiint F = \mu = \frac{2\pi}{e} n = n\mu_{\text{u}} = \frac{\Lambda}{e}, \tag{1.8}$$

where we also define an n=1 unit of magnetic charge $\mu_{\text{u}} \equiv 2\pi/e$, similarly quantized. By appropriate local gauge transformation, and specifically by choosing n=0 which is the same as choosing the phase angle $\Lambda = 0$, the nonzero surface integral can be made to vanish, $\oiint F = 0$. But this does not invalidate (1.7) and (1.8) nor does it prevent us from seeking to draw physical conclusions from these. It simply means that n=0 with no monopoles and no electric charges is one of the permitted states. Again, the meaning of the whole range of charges $e = ne_{\text{u}}$ for $n \neq 0$ has been physically-interpreted ever since Dirac discovered this, as suggesting that the "existence" of a magnetic charge would imply electric charge quantization, with the further understanding that the converse is not true.

This is how Wu and Yang obtain Dirac monopoles and the DQC without strings.

2. Quantum Topology and Orientation-Entanglement-Twist (OET): the Observable Distinctness of Similar Geometric Orientations

If we define a reduced gauge angle $\text{Ʌ} \equiv \Lambda/2\pi$, then by (1.6) this reduced $\text{Ʌ} = n$ is a quantum number which states the number of "windings" through the complex gauge / phase space contained in the operator $e^{i\Lambda(x)} = \cos\Lambda(x) + i\sin\Lambda(x) \equiv a + bi$ of the local gauge transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$. But in the DQC, $\text{Ʌ} = n$ also becomes the charge quantum number n, and so (1.7) may be rewritten as $e = \text{Ʌ}e_{\text{u}} = ne_{\text{u}}$. Therefore, every gauge transformation $\Lambda \to \Lambda + 2\pi$ adding an angle of 2π also adds one unit of electric and magnetic charge. This is the first indication of a conceptual challenge which will occupy of fair share of attention in this paper and lead us to undertake a detailed study of *quantum topology* which will in turn reveal some unanticipated insights about the observed electronic structure of atoms. Let us now introduce this challenge.

If these DWY monopoles were to exist under some specialized set of physical conditions, *then* the electric charge would be quantized, $e = \text{Ʌ}e_{\text{u}} = ne_{\text{u}}$, and this quantum number $\text{Ʌ} = n$

would of course have to be a physical observable. In geometry however, angles measuring rotation which differ from one another by 2π, such as $\Lambda = 0, 2\pi, 4\pi, 6\pi...$, are *trigonometrically indistinguishable*. Indeed, we already used this indistinguishability of 0 and 2π to write (1.5) and then derive (1.6). Consequently, if we draw x and y axes on a sheet of paper and then draw a vector starting at the origin which points along the +x axis, we cannot state looking at the paper whether the angle between that vector and the x axis is 0, or is 2π or 4π or 6π or any other multiple of 2π. It can be *any* of these, because these are all *indistinguishable orientations*. So $\not{\Lambda} = 0, 1, 2, 3, 4...$ is not a *geometric* observable. But if the DWY monopoles were to exist under some physical conditions, then the $\not{\Lambda} = n$ in $e = \not{\Lambda} e_u = n e_u$ would have to be observable. If we observed one unit of charge, we would know that $\Lambda = 2\pi$. Observing two units of charge we would know that $\Lambda = 4\pi$, and so on. Thus comes the question: *how can* $\not{\Lambda} = \Lambda / 2\pi = n$ *be an observable when it is a charge quantum number but not be an observable when it represents the number of geometric "windings"?*

Answering this and analogous questions which will shortly arise about the azimuth angle φ in three-dimensional physical space, will require us to develop the *quantum topology* of Orientation-Entanglement (OE) and Twist, which is rooted in Misner, Thorne and Wheeler's (MTW) widely regarded review of OE at section 41.5 of [18]. What we shall find is effectively this: in *geometry*, as distinct from *topology*, *orientations* in the set of angles $\Lambda = 2\pi n$ manifest no observable features to distinguish them from one another. Orientation differences between these angles are not observable, so *n* is not a geometric observable. Entanglement, however, is an aspect of *topology* which tracks the relationship between a vector ("object") and its "environment" via sets of connecting "threads." When OE is considered, orientations which differ from one another by 2π *are* observably-different, because they have opposite *entanglements*. This is also verbalized by stating that they have opposite "versions." But vectors rotated by angles in the set $\varphi \rightarrow \varphi' = \varphi + 4\pi n$ still manifest no observable features to distinguish them from one another, because they all have the same *orientations and entanglements*. Once entangled by rotations in integer multiples of $+4\pi$, the vector can be restored to its initial disentangled state without any reverse-directional -4π rotations, via various "disentangling" manipulations of the threads. So vectors rotated to these angles are all said to have the same OE versions. So the question posed in the last paragraph has a partial answer whereby angles differing by 2π can be observably distinct, but still remains unresolved as to angles differing by 4π.

The 4π distinctness question is answered fully, by a third aspect of OE which appears to have been widely overlooked or at least underdeveloped in the literature, and that is twist: When one carefully studies OE, then depending upon how one "disentangles" the "threads" following any rotation of the "object" through some multiple of 4π, it is possible to completely restore OE by disentangling the threads from one another, yet still have observable "twists" remain in individual threads *which twists were not there at the start*. Once these twists are considered, every angle in the set $\Lambda = 4\pi n$ *is observably distinct* from every other angle in this set, because the number of twists that can observed after disentangling a 4π rotation is different from the number of twists that can be observed after disentangling an 8π rotation, and these differ from the number of twists that can be observed after disentangling each of a $12\pi, 16\pi, 20\pi...$ rotation. Thus, upon considering Orientation-Entanglement-Twist (OET), every single angle in the set

$\Lambda = 2\pi n$ is *observably topologically distinct* from every other angle in this same set. So it is on the basis of OE&T that each of the quantized states in $e = \Lambda e_u = n e_u$ gains the possibility of being physically observable, because each of the angles in the set $\Lambda = 2\pi n$ is observably topologically distinct.

A totally unanticipated bonus from this analysis, however, when it is applied to the azimuth angle φ in the three-dimensional physical space of spacetime, is that the topological rotations and twists based on OET following a $+4\pi l'$ rotation of the "object" through an integer number of double windings $l' = 0,1,2,3...$, can be summarized using quantum numbers designated as l', m_z', s_z', with $l' = 0,1,2,3...$ and $-l \le m_z' \le +l$ and $s_z' = \pm\frac{1}{2}$, where l' is the number of double rotations, m_z' and s_z' are the possible numbers of double twists after disentangling depending upon the disentangling procedure used, and the + and – signs represent the right or left handedness of these twists in reference to the axis of twist defined as +z. This topological summary of OET – and this is the bonus – maps *exactly, on a one-to-one basis*, with the angular momentum quantum numbers l, m_z, s_z observed in the electronic structure of atoms. This exact mapping raises the prospect that atomic structure (and even nuclear structure because protons and neutrons, albeit composite, are also fermions like electrons and form similar shell structures in the nucleus) can be explained strictly on the basis of quantum topology. If this were to be possible, then the quantum numbers l, m_z, s_z with $l < n$ and $-l \le m_z \le +l$ and $s_z = \pm\frac{1}{2}$ would no longer just be electronic state rules with fundamental origins unknown, but would be *topological mandates from physical space*. If such a connection can be empirically confirmed – and the FQHE experiments to be proposed here are intended to do exactly that – this would take us a step closer to the ultimate fulfillment of Wheeler's geometrodynamic program [19], [20] in the spirit of Einstein [21] and Weyl [22], [23], [24], of establishing that the entirety of the observed natural world is no more and no less than a manifestation of spacetime geometry and spacetime topology.

Now, it is time to return to the Dirac Quantization Condition.

3. The Fractional Denominators Indicated by Dirac-Wu-Yang (DWY): are they Somehow Related to the Fractional Quantum Hall Effect (FQHE)?

If we closely study the derivation by Wu and Yang summarized in section 1, we see that there are some additional quantum states indicated that have not yet been considered. Referring to (1.5), not only do $\varphi = 0$ and $\varphi = 2\pi$ describe exactly the same *orientation* (sans entanglement and twist) in the *physical space*, but so too do $\varphi = 4\pi$, $\varphi = 6\pi$, $\varphi = 8\pi$, etc. So starting with $\varphi = 2\pi$ and considering all positive (right-handed about the z-axis) rotations which we summarize by $\varphi = 2\pi l$ using a positive integer $l = 1,2,3,4,5,6...$, if we proceed solely on the basis of geometric orientation and do not concern ourselves with entanglement or twist, we may now extend (1.5) to:

$$\exp(i\Lambda) = 1 = \exp(ie\mu \cdot 1) = \exp(ie\mu \cdot 2) = \exp(ie\mu \cdot 3) = \exp(ie\mu \cdot 4) = \exp(ie\mu \cdot 5) = \exp(ie\mu \cdot 6)....(3.1)$$

Each of the above is a separate relationship of the general form $\exp\left(ie\mu\cdot l\right)=1$, where $l=1,2,3,4,5,6...=\varphi/2\pi$ is an integer *not the same* as the *n* already in use and corresponds to the number of azimuth windings. At the same time, as noted after (1.5), the general solution for an equation of this form is $\exp\left(i2\pi n\right)=1$ with this integer $n=0,\pm1,\pm2,\pm3...$, as before. Comparing $\exp\left(ie\mu\cdot l\right)=1$ with $\exp\left(i2\pi n\right)=1$ means that more generally, $e\mu\cdot l=2\pi n$ i.e. $e\mu=2\pi\left(n/l\right)$, or restated, also using $\Lambda=2\pi n$ from (1.6), that:

$$e=\frac{n}{l}\frac{2\pi}{\mu}=\frac{n}{l}e_{\text{u}}=\nu e_{\text{u}}=\frac{\Lambda}{l}\frac{1}{\mu}, \tag{3.2}$$

where we may define a "filling factor"

$$\nu\equiv\frac{n}{l};\quad n=0,\pm1,\pm2,\pm3...;\quad l\equiv\varphi/2\pi=1,2,3,4,5,6.... \tag{3.3}$$

This electric charge is both quantized *and fractionalized*. Likewise, for the magnetic charge defined as in section 1,

$$\mu=\frac{n}{l}\frac{2\pi}{e}=\frac{n}{l}\mu_{\text{u}}=\nu\mu_{\text{u}}=\frac{\Lambda}{l}\frac{1}{e} \tag{3.4}$$

is also quantized and fractionalized. If we define a reduced azimuth $\cancel{\varphi}\equiv\varphi/2\pi=l$, then with the reduced gauge angle $\cancel{\Lambda}\equiv\Lambda/2\pi=n$ as before, we can rewrite this fill factor (3.3) as:

$$\nu=\frac{n}{l}=\frac{\cancel{\Lambda}}{\cancel{\varphi}}=\frac{\Lambda}{\varphi};\quad \cancel{\Lambda}=0,\pm1,\pm2,\pm3...;\quad \cancel{\varphi}=1,2,3,4,5,6.... \tag{3.5}$$

This is simply to ratio of gauge-space windings to physical-space windings, and equivalently, the ratio of the gauge angle to the spatial azimuth angle.

The conditional statement we may make based on the above is the following: *If* these DWY monopoles exist under some specialized set of physical conditions, *then* not only are the electric and magnetic charges quantized, but each unit of electric charge e_{u} or magnetic charge μ_{u} is also *fractionalized* into $\nu=n\cdot\left(1/l\right)$ quantized *n* fractions $1/l$ of itself. As with (1.6), see the related footnote, this relationship is locally gauge invariant. More generally, what we now see that that *U(1)$_{em}$ gauge theory itself* inexorably implies that *if* these DWY monopoles exist under some specific set of physical conditions, *then* electric charge is quantized, *and electric charge is also fractionalized*. In other words, based on the Dirac-Wu-Yang derivation, the DQC is really a DQFC, Dirac Quantization *and Fractionalization* Condition.

For a single rotation through $\varphi=2\pi$ where $\cancel{\varphi}=l=1$, (3.2) and (3.5) become:

$$e\mu = 2\pi n = 2\pi \cancel{\Lambda} = \Lambda\,, \tag{3.6}$$

which is identical to (1.6) and so recovers the usual Dirac Quantization Condition without fractionalized charges, as a special case. If we further specialize this to a single winding $\cancel{\Lambda} = 1$ a.k.a. $\Lambda = 2\pi$ in the gauge space, then:

$$e = \frac{2\pi}{\mu} = e_{\mathrm{u}}\,, \tag{3.7}$$

which is the unit of electric charge for which, as noted in the introduction [14], $4\pi\alpha = e^2$ in natural units. The conditional statement we can make based on this specialization to $\varphi = 2\pi$ and $\Lambda = 2\pi$ is that *if* these DWY monopoles exist under some specialized set of physical conditions, *then* the special case in which $\varphi = 2\pi$ and $\Lambda = 2\pi$ has an electric charge equal to that of a single electron. So the $\cancel{\varphi} = \cancel{\Lambda} = 1$ electric charge state of a DWY monopole with a single winding in both the three-dimensional physical space and the two-dimensional gauge space is the same as the state of a single electron with charge $e = 2\pi / \mu = \sqrt{4\pi\alpha}$.

This has a consequence of immediate interest. If a single winding $\varphi = 2\pi$ about the azimuth yields the equation $e = (2\pi/\mu)n = ne_{\mathrm{u}}$ which describes the quantization of an unfractionalized electron, then according to (3.5) each additional winding about φ will fractionalize the charge in proportion to $n/\cancel{\varphi}$. If we consider orientation *and entanglement*, then not all of these states can be disentangled. The only electron states which can be disentangled are those which differ from $\varphi = 2\pi$ by a 4π rotation, i.e., for which $\varphi = 2\pi + 4\pi l$. Consequently, the set of DWY electric and magnetic charge states which can be disentangled is restricted to those in which the fractional denominator $\cancel{\varphi} = 1 + 2l = 1, 3, 5, 7...$ is *an odd integer*. All of the even-integer charge states are entangled states which cannot be disentangled.

The conditional statement we can now make based on this observation, is the following: *If* the DWY monopoles exist under certain physical conditions, *then* the set of electric and magnetic charge states which can exist *in a disentangled state* are given by:

$$\nu = \frac{n}{1+2l} = \frac{\cancel{\Lambda}}{\cancel{\varphi}} = \frac{\Lambda}{\varphi}; \quad \cancel{\Lambda} = 0, \pm 1, \pm 2, \pm 3...; \quad \cancel{\varphi} = 1 + 2l = 1, 3, 5, 7, 9...; \quad l = 0, 1, 2, 3, 4..., \tag{3.8}$$

that is, they are fractionalized such that the fractional denominator $\cancel{\varphi} = 1 + 2l = 1, 3, 5, 7, 9...$ is *an odd integer*. With the exception of the even denominator 2, all of the observed states of the FQHE also have odd denominators. So the refinement if this conditional statement is this: *If* the DWY monopoles exist under certain physical conditions, *then* the set of disentangled electric charges is precisely the same as what is observed in the FQHE, except for $\cancel{\varphi} = 2$.

This raises the central questions to be studied in the paper and eventually answered affirmatively with some proposed avenues for experimental validation or contradiction: Are the odd-integer denominators observed in FQHE a physical consequence of the odd-integer

fractionalization of *disentangled* electric and magnetic charge states in (3.8)? And, is the even FQHE denominator of 2 also a physical consequence of (3.8) based on what would be a $\varphi = 2$ *entangled* state in (3.8)? Finally, does (3.8) also provide an explanation for why all other even denominators are *not* observed in FQHE?

4. Three Questions to Consider: Observable Topological Distinctness of Angles with 4π Orientation Separation, Three-Dimensional DWY $U(1)_{em}$ Gauge Analysis versus Two-Dimensional FQHE Electronic Configurations, and Low-Temperature Electric / Magnetic Duality Symmetry Breaking

For the fractionalization in (3.5) of Dirac-Wu-Yang monopoles based on $U(1)_{em}$ gauge theory – or the odd-integer fractionalization (3.8) assuming the even integer $\varphi = 2$ can also be understood – to specify a valid connection between the fractionalized DWY charge states and the fractionalized quasiparticle states of FQHE found in the empirical data [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], there are at least three questions which would need to be answered:

The first question arises because in geometry absent consideration of topology, angles which differ from one another by 2π orientations are geometrically, trigonometrically indistinguishable. This applies to the azimuth angle φ, just as it applied to the gauge angle Λ as discussed in section 2. If we take into account topological orientation *and entanglement* (OE), then angles which differ by 2π do become topologically distinct and thus have the possibility to be observably distinct because they have opposite entanglements. But angles differing by 4π appear to remain indistinct because they have the same topological orientation and entanglement.

Now, because the fractional denominators in FQHE are empirically observed, this means that *if* the denominator $\varphi = 1 + 2l$ a.k.a. $\varphi = 2\pi + 4\pi l$ in (3.8) was to actually be the odd denominator of the FQHE, *then* $\varphi = 2\pi + 4\pi l$ would have to be observable. But even with OE considered, angles in the set $\varphi = 2\pi + 4\pi l$ are indistinct from one another. So we now ask the same question about the azimuth angle $\varphi = 1 + 2l = 1, 3, 5, 7...$ that we asked in section 2 about the gauge angle $\Lambda = n$: If $\varphi = 1 + 2l$ was to be the odd-fractional FQHE denominator and thus an observable, how could this be an observable as the FQHE denominator yet not be an observable when it represents the number of azimuth windings $\varphi = 1 + 2l$ of states with identical OE?

As we shall show in the next two sections, when orientation and entanglement are very carefully analyzed, then depending upon the disentangling procedure employed, there can also be seen an observable "*twist*" of the "threads" which connect an "object" to its "environment" which twist did not exist at the outset. Therefore, even following disentangling, similarly-entangled angles in the set $\varphi = 1 + 2l$ can become topologically distinct from one another. Consequently, these observable twists make it possible for $\varphi = 2\pi + 4\pi l$ and thus the denominator in (3.8) to be a topological observable. Moreover, unexpectedly, it turns out that the pattern of this twist-resultant topological distinctness precisely mirrors the pattern of orbital and spin angular momentum observed in the electronic structure of atoms. Specifically, writing $\frac{1}{2}\varphi = l + \frac{1}{2}$, we find that with $|\xi\rangle$ representing spinor eigenstates, l relates precisely to the

Casimir operator of the orbital angular momentum in $\mathbf{L}^2\left|\xi\right\rangle = l\left(l+1\right)\left|\xi\right\rangle$ and $\frac{1}{2}$ relates precisely to Casimir operator of spin $s=\frac{1}{2}$ in $\mathbf{S}^2\left|\xi\right\rangle = s\left(s+1\right)\left|\xi\right\rangle$. And as shown in (3.6) and (3.7), the $\varphi = 2\pi$ a.k.a. $\frac{1}{2}\varphi = \frac{1}{2} = s$ state with $l=0$ has the electric charge of an unfractionalized electron. So when orientation-entanglement and twist are all considered, there is no observability problem with $\text{Ѧ} = n$ or with $\varphi = 1+2l$ in $\nu = \text{Ѧ}/\varphi$ in (3.8) because each of Ѧ and φ is a topological observable. Using "twistors" as an element of spacetime topology was originally proposed by Penrose [25] and subsequently advanced by others including Witten in [26], and will become a central feature of the development here.

The second question arises because the Dirac-Wu-Yang theoretical argument based on $U(1)_{em}$ gauge theory is developed within the three-dimensional physical space of spacetime geometry, and is understood to apply to systems of electrons, protons, and neutrons for which no fractionally charged particles and no Dirac-Wu-Yang magnetic monopoles have ever been observed. But at the level of analysis where the quasiparticle language applies, the system is fundamentally two-dimensional, because the superconducting materials used together with the ultra-low temperatures and large perpendicular magnetic fields applied to stimulate the observed FQHE, combine in some fashion to substantially remove one degree of spatial freedom from the electrons and so restrict the electrons to two space dimensions while also giving rise to superconductivity. And in some way that needs to be understood, these all synergistically coact to produce the $1,2,3,5,7,9...$ denominator pattern which is observed. Because of this apparent difference between the three-dimensional space of Dirac-Wu-Yang and the two-dimensional restricted space of FQHE, one might take the *a priori* view that there is no connection between Dirac-Wu-Yang and FQHE. So at the very least, if there is some hidden, not-yet-understood connection between these two fundamentally-different environments of Dirac-Wu-Yang and FQHE, it is important for such a connection to be carefully developed and understood.

The third question that arises stems simply from the observational data that insofar as is known, magnetic monopoles have never been observed. So if the DWY monopoles are responsible for FQHE which would means that these monopoles do exist near 0K, then it would be important to understand how this low-temperature duality symmetry between $U(1)_{em}$ electric and magnetic charges becomes broken at higher temperatures. As to this third question, we must keep in mind that while the high energies of Grand Unified Theories (GUT) have certain symmetries which are broken at lower energies, so too, *low temperatures* near absolute zero are also thought to cause displays of certain symmetries which become broken at higher temperatures, see, e.g., Volovok's [27]. This third question can also be posed in relation to the second by asking whether by tightly constraining the electrons to two rather than three dimensions, and extracting virtually all of their heat energy leaving them only with their Fermi energies, we are forcing the electrons in superconductors near 0K into some highly-constrained topological condition which forces them to reveal their entanglements and to display an electric and magnetic monopole symmetry and a charge fractionalization which they otherwise can keep hidden from observation at all higher temperatures.

Now, to address the first of these three questions, we embark in the next two sections upon a detailed study of topological orientation, entanglement and twist.

5. Twist: The Missing Ingredient from Orientation-Entanglement Analysis, and how this may lead to a Topological Understanding of Quantization

Spinors, which includes electrons, reverse sign upon a spatial rotation through an angle φ by an odd multiple of 2π. Specifically, as Misner, Thorne and Wheeler (MTW) point out in one of the most widely-regarded discussions of this topic in [18] at section 41.5, the spin matrix of a rotation $R = \cos(\varphi/2) - i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin(\varphi/2)$ (see MTW [41.48]) reverses sign upon a rotation through an odd multiple of 2π, as does the sign of a spinor under $\xi \to \xi' = R\xi$ (MTW [41.50]). This sign reversal does not, however, appear in the transformation law for a vector, $X \to X' = RXR^*$ (MTW [41.49]).

Misner, Thorne and Wheeler provide a visual, macroscopic, intuitive, essentially-topological understanding for this result by considering the orientation and entanglement of an object relative to its surrounding environment, because while orientation is restored under a 2π rotation, it takes a 4π rotation to restore the object's state of entanglement, i.e., to restore the complete "version" of the object. They do, however, at page 1148 of [18], make the statement:

> "Whether there is also a detectable difference in the physics . . . for two inequivalent versions of an object is not known."

This question of whether MTW Orientation-Entanglement (OE) brings about detectable physics in physics will be the focus of the next two sections, and in some ways, the remainder of this paper. In these next two sections, we shall find that OE with Twist provides a topological understanding of the observed electronic structure of atomic shells, and provides the basis for the physical reduced angles a.k.a. winding numbers in the sets $\mathcal{A} = n$ and $\varphi = 1 + 2l$ in the fill factor $\nu = \mathcal{A}/\varphi$ in (3.8) to be physical observables. We shall also show how ultra-low temperatures constraining electrons in superconductors to two dimensions may be represented in terms of constraints on OE and Twist, which will help us figure out how to break the electric / magnetic duality at higher temperatures.

Ross in [28] "hypothesize[s] that the OE relations are important to physics [and] represent the deep relationship between any particle or material body and its environment." He proceeds to show (reference renumbered) "that Dirac [11] magnetic monopoles do not satisfy the OE relationship" and "hypothesize[s] that this is the reason they have never been seen despite extensive searches . . . and despite having a natural and elegant theory underlying them . . . going back to the more natural symmetry of Maxwell's equations with magnetic monopole sources present. " He then states that "[s]ince all known particles satisfy the OE relations and we show that Dirac magnetic monopoles which have not been seen do not satisfy these relationships, it is hoped that this paper will stimulate further work on the OE relations themselves and their topological role in physics." This further work on the OE relationships is precisely the subject of the present development, and will lead us to understand the fractional quasiparticles of FQHE as electron states which obtain their observability due to topologically distinct OET states, and which obtain their observed angular momentum in atomic shells based on the twisting which remains after the OET threads are disentangled.

Figure 31.6 of MTW's [18] which is also posted online at [29], shows a spherical "object connected to its surroundings by elastic threads." Indeed, it is these "threads" and various configurations of these "threads" which most directly illustrate the "deep relationship between any particle or material body and its environment" mentioned by [28]. It is also these "threads" themselves which will be the focus of the present discussion. As is well-understood, it is always possible following a 720° rotation or integer multiples thereof of an object connected to its environment with "untwisted threads," to remove all entanglement from the connections of that object to its environment. But of particular importance, as we shall now develop here, the sequences of disentangling the "threads" from one another are *not unique*. Depending upon the sequence chosen, even after disentangling, *the "threads" may still each maintain individual twists*, or they may have all twisting removed and have been returned to an untwisted state. The surprise is that this twisting maps precisely to the structure of electronic shells in atoms.

To simplify this development without any loss of information, rather than use the spherical "object" and the spherical "environment" and the "threads" employed in Figure 31.6 of [18], let us employ a first "bar" or "stick" which represents the "environment" and a second "bar" or "stick" which represents the "object," and a pair of "ribbons" which represent the connections of this "object" to its "environment." These two ways of representing OE do topologically map into one another as is shown below in Figure 1, which is why we can use the "bars and ribbons" as an alternative way of representing Figure 31.6 of [18].

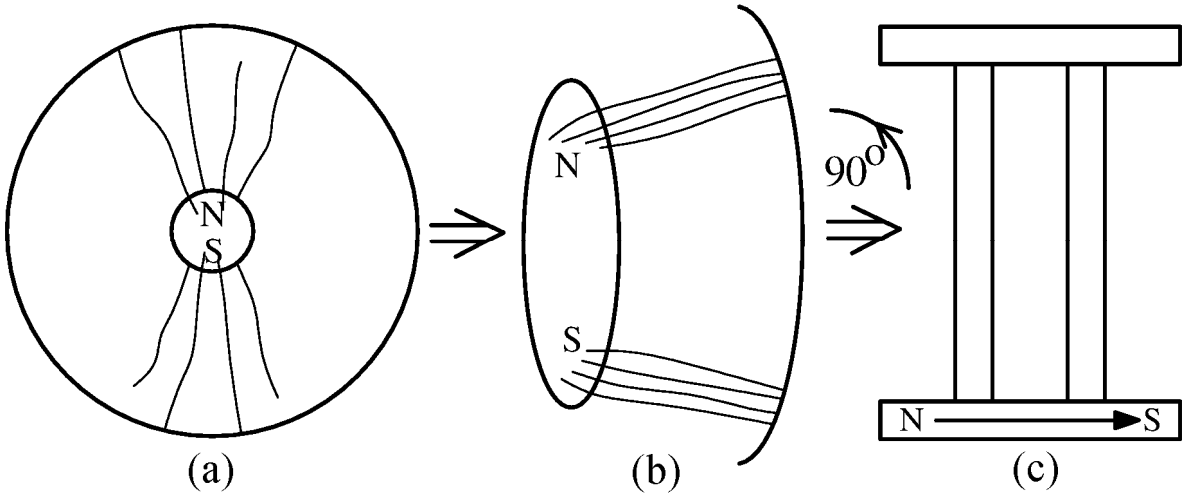

Figure 1: Topological Deformation of Figure 31.6 of [18] (MTW) into a "Bar and Ribbon" Configuration

Specifically, to verify this topological mapping which means that Figure 1(c) belongs to the same homotopy group as Figure 1(a), one may start with the OE system shown in drawing 1 from Figure 31.6 of [18], replicated in Figure 1(a) above. To maintain points of reference, we label the north N and south H hemispheres of the object as shown above, which hemispheres also have north and south "thread" connections to the environment. Then, as shown in Figure 1(b) above, one may topologically deform the object by stretching it in into a vertical elongation, while relocating the threads to the right along the northeastern and southeastern regions of the environment which can be snipped on the left side past all the threads without losing any relevant

information about the topology. Then, one can take the entire Figure 1(b) and rotate it 90° counterclockwise to arrive at Figure 1(c) above. In this final step, the *environment* is simply represented by a top "bar" or a "stick" at the top of 2(c), the *object* is represented by a bottom "bar" or a "stick" at the bottom of 2(c) which maintains the "north" and "south" labels simply as a point of reference back to Figure 1(a) (notwithstanding that these are east and west in Figure 1(c)) and now also introduces a directional vector running from north to south, and the north and south *threads* are merged together into a pair of "ribbons" which represent the *entanglement* between the object and its environment. It will be appreciated that the ribbons capture the same topological information as the threads (just think of the two lines bounding the ribbon widths as being two threads and then add a few more threads in between for good measure, see, for example, the web animation at [30]). The benefit of employing "ribbons" (or thick "threads" with discernable width) rather than thin threads abstracted to being infinitesimally thin is that it is much easier with a two-sided ribbon to illustrate and track any twisting which may occur in the course of performing OE operations, which will be very central to the ensuing discussion.

This "bar and ribbon" configuration in Figure 1(c) is often used in illustrations of the OE relationships, see, for example, an online animation at [31]. For the interested reader to follow the forthcoming development, it is easy and advisable to construct a physical apparatus resembling Figure 1(c) by taking two sticks or dowels or even pencils, and then gluing or stapling two ribbons or shoelaces or even rubber bands to the sticks in the configuration illustrated. It also helps to color each side of the ribbons differently for monitoring twists. At the web link https://jayryablon.files.wordpress.com/2014/12/figure.jpg the apparatus constructed by the author for this purpose may be viewed at in the upper-left photograph, with the other photographs showing some states of twist that we shall now review.

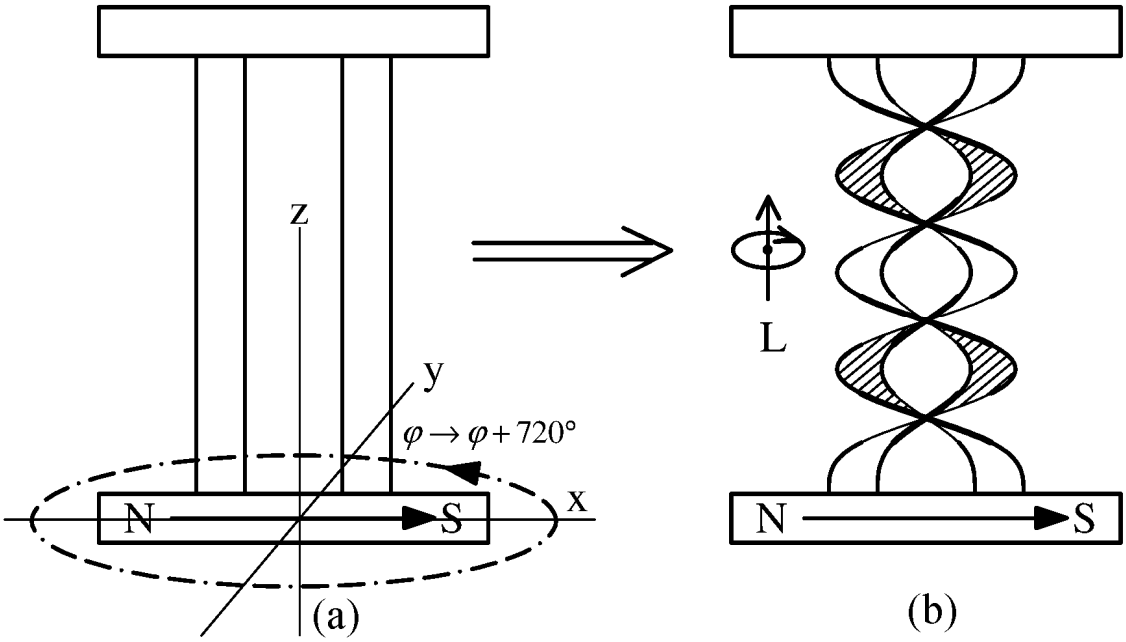

Figure 2: Environmental OE Consequences of Rotating a Vector through 720°

Starting with the bar and ribbon configuration of Figure 1(c), let us first immobilize the top "environmental" bar. This abstractly "fixes" the environment. Let us then rotate the bottom

bar – which from now on we shall simply call the bottom "vector" – by a $\varphi \to \varphi + 4\pi = \varphi + 720°$ right-handed counterclockwise rotation about the +z axis through the angle φ in the x-y plane, as shown in Figure 2(a) above, to arrive at the configuration of Figure 2(b) above. We shall wish to study the relationship of this vector to its environment through the behaviors of the ribbons under some operations to be elaborated momentarily. In using the word "vector" in the present context, we are simply referring to the orientation arrow in Figure 1(c), not to a "vector" in the sense of a spin 1 particle. The result of this $+4\pi$ rotation is shown in Figure 2(b) above.

It is worth keeping in mind that all three x, y, z dimensions are utilized in this rotating operation. By rotating the vector through φ we are utilizing the x-y plane, while the ribbons and the environmental bar are situated above this x-y plane along the z dimension. It is also worth keeping in mind for later, that for electrons frozen in two dimensions at low temperature in superconducting materials in the FQHE environment, one degree of spatial freedom is effectively removed. It is also important to keep in mind that this angle φ is an azimuth angle of rotation in three space dimensions, just as was the azimuth angle φ first introduced after (5.1) when we wrote the electromagnetic field strength as $F = (\mu / 4\pi) d\cos\theta d\varphi$. So it is appropriate to try to relate these two angles φ to one another because physically they mean the same thing.

In Figure 2(b), to provide depth perspective so it can be seen what is passing in front of and behind what else, the wider lines illustrated on each ribbon indicate a passing in front of the narrower lines illustrated on each ribbon, and diagonal hash lines are used to illustrate the opposite face of the ribbon relative to the face shown in Figure 2(a). In Figure 2(b), we reach a state in which the ribbons are entangled with one another, with the entanglement forming a left-handed helix in relation to vector pointing vertically in the +z direction, as illustrated. And in addition, each of the two individual ribbons also is twisted into a left-handed helix (L), as illustrated. Again, it is helpful for the reader to construct and use this bar and ribbon apparatus to see all of this. In addition, both the entanglement helix and the twist helixes are *double* helixes, in the sense that there are two full helix rotations of $-4\pi = -720°$, using a convention in which a right helix has a positive sign and a left helix has a negative sign. Certainly, while the original φ and the final $\varphi + 4\pi$ are indistinguishable geometrically, *they are topologically distinct*, because their relation to the "environment" as manifest by the ribbon entanglement and twist is different. Geometrically, the vector has an identical orientation in Figure 2(a) as in 2(b). Topologically, it is clear that the Figures are different, and the number and handedness of the helixes is a form of physical encoding which tells us exactly what sort of rotation has occurred to get into that entangled and twisted state. The same considerations apply when we rotate the gauge angle Λ. However, in the complex gauge space there is no actual third dimension analogous to the z axis, while for φ not only is there a z axis, but this z axis establishes coordinates in a real observed dimension of physical space.

Now let's discuss ways to disentangle these ribbons, one from the other. It must be made clear that when we talk about disentangling ribbons, we are talking about disentangling two ribbons *from one another*. That is a separate matter from removing the twists from each individual ribbon. One way to disentangle these ribbons is to simply release the N→S vector and let it "hang" from the environment bar and un-rotate, analogously to a child's swing that has had the seat twisted into Figure 2(b) and then is released to rotate back under the pull of gravity to its

ground state of Figure 2(a). Indeed, it is helpful and physically pertinent think of the unrotated, untwisted Figure 2(a) as representing a sort of topological "ground state" to which Figure 2(b) will return if the N→S vector is given freedom to rotate through the x-y plane, i.e., if it keeps its freedom in all three dimensions. If this is allowed, not only will the two ribbons naturally disentangle, but all twists in each ribbon will also naturally be removed.

But let us say we do *not* let the N→S vector rotate any more. Now that it has been rotated from $\varphi \rightarrow \varphi + 4\pi$ let us suppose that we remove its freedom to rotate through the x-y plane. Let us now mandate that the top and bottom bars are heretofore to remain locked into immobile alignment with one another with no relative rotations allowed? In other words, let us now remove the degree of freedom along the y axis and lock everything into the two-dimensional x-z plane. What happens then? Is there a way to return to the "ground state" even with the two bars locked into immobile alignment and freedom confined to the two dimensions of the x-z plane?

This is where disentangling operations are used, whereby we can return to a ground state – or as will be seen at least to a lower energy state – using only two dimensions, if we move the ribbons around the N→S vector. However, the ability to disentangle in this way is subject to an important caveat that there must at least be some very minimal freedom to use the y axis to get the ribbons around the ends of the N→S bar and past the bar, and in particular, we still must have access to the y-dimension for at least the smallest cross-sectional thickness of the ribbon itself. We make note of this now, but this will be very important to understanding how the DWY monopoles in three space dimensions connect to FQHE in two space dimensions.

Now, if the initial rotation in Figure 2 had been through only $+2\pi = +360°$, Figure 2(b) would contain all *single* left-handed helixes, both for its entanglement between the two ribbons and for the twists of each ribbon. And, as is well known, there would be no way to disentangle the two ribbons from each other with the two bars locked into immobile relative alignment using only manipulations of the ribbons. But from Figure 2(b), because of the double helix entanglement which results from the double winding rotation through $\varphi \rightarrow \varphi + 4\pi = \varphi + 720°$ a.k.a. $\varphi \rightarrow \varphi + 2$ using the reduced azimuth $\varphi \equiv \varphi / 2\pi$ earlier defined, aligned-bar disentangling is possible using only ribbon manipulations. And specifically, in order to disentangle the two ribbons using only operations of the ribbons with both the top-bar environment and the bottom bar vector remaining relatively immobile, one must perform *two* ribbon operations, and there are three choices for how these two ribbon operations may be done.

For the first choice, as shown in Figure 3 below, for the first ribbon operation, one can take the *north ribbon*, wind it in front of and past the north "pole," wind it beneath and behind the entire vector, and then wind it back above the vector in front of and past the south "pole." Then, for the second operation, one can take the *south ribbon*, wind it past the north "pole," wind it beneath the entire vector, and then wind it back above the vector past the south "pole." This can be done in either order, that is, one can use the south ribbon in the first operation and the north ribbon in the second operation and end up with the exact same result as the vice-versa operation, which, as shown in Figure 3 below, not only disentangles the two ribbons from each other, but also removes the individual twists in each ribbon. We denote this by placing the number "0" next to each ribbon to indicate that it has no residual twist. Here, with one operation

using the north ribbon and a second operation involving the south ribbon, in either order, we have restored the original OET state of Figure 2(a) in its entirety.

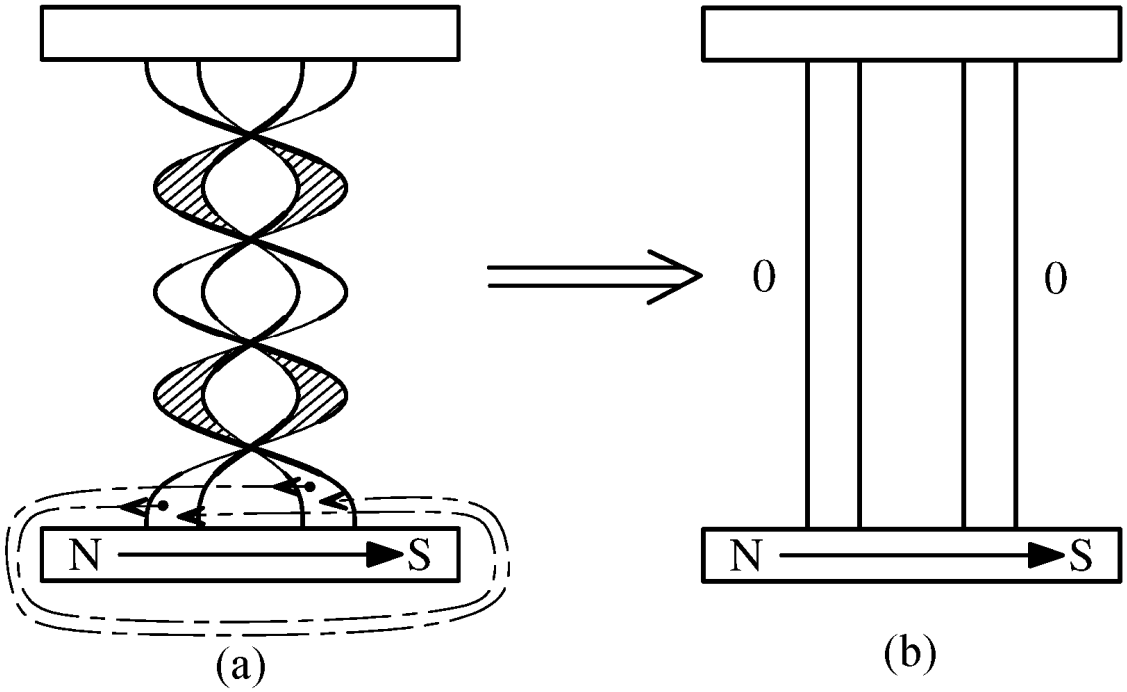

Figure 3: The Disentangling Operation $0 \rightarrow \varphi + 2 \rightarrow \{N, S\} \rightarrow 0$

In either case, however, whether the north or south ribbon is operated first, the ribbon windings *must go from north to south*, that is, the ribbons must be first brought around the north pole, then wound past the vector, then be brought back past the south pole. If the ribbons are wound from south to north, they will become even further entangled, and the net effect will be that of having performed a $\varphi \rightarrow \varphi + 8\pi$ a.k.a. a $\varphi \rightarrow \varphi + 4$ quadruple rotation starting from Figure 2(a). The question occurs why there is this apparent asymmetry in which the ribbons must be brought past the north pole first, but that is explained by the fact that the Figure 2 rotation was done counterclockwise i.e. right-handedly about the z-axis, and thus was positively signed, $\varphi \rightarrow \varphi + 4\pi$. Had the rotation been clockwise hence negative according to the customary conventions for defining angular rotation, i.e., $\varphi \rightarrow \varphi - 4\pi$ a.k.a. $\varphi \rightarrow \varphi - 2$, then disentangling would have required winding the ribbons first over the south and then over the north pole. So there is in fact an overall symmetry to these operations, and one can choose – as we now do – a convention of only doing positive, right handed rotations and then always starting disentangling about the north pole, rather than doing negative rotations then starting disentangling about the south pole.

It is also important to observe that although the disentangling operation can take place very close to the x-z plane, *at least some small incursion into the y dimension is required*, of at least the narrowest cross-sectional width of the ribbon. How do we see this? If one labels the N→S bar with N and S near the ends, then when first moving a ribbon past the N pole, it will be impossible to progress without the N becoming momentarily obscured by the ribbon. So if the narrowest cross section of the ribbon has some small length ε, the ribbon will at least pass

through $y=+\varepsilon$ to get past the N pole. The same will be true at the S pole. Further, when moving the ribbon past the length of the N→S vector, the ribbon will have to go behind the bar through $y=-\varepsilon$, and the ribbon will always visually obscure some part of the back of the bar while this disentangling is occurring. So although we have removed the y-axis from being available *for rotation* of the object bar vector, we must make a very small portion of the y axis available for passing the ribbons during disentangling. If we do not do so, then the ribbons are frozen as is – the word "frozen" being a deliberate choice in relation to FQHE near 0K – and cannot be disentangled. Now, let's turn to what happens as a result of this disentangling, if we *are* permitted a small $y=\pm\varepsilon$ ribbon incursion into the y axis, and let's develop some notational shorthand to discuss this.

We shall use the shorthand $0,0\rightarrow\varphi+2\rightarrow N/N,S/N\rightarrow 0,0$ to represent this operation in Figure 3 in which both the north and south ribbons start with no twists $0,0$, the azimuth is positively rotated through two windings $\varphi+2$, the north and then south ribbons are wound over the north pole N/N, S/N, and the disentangled state finally restores no twists $0,0$ which was the original ground state. If the reader does this operation but instead performs S/N, N/N in opposite order, it will be seen that the order of ribbon operations does not matter and the same end result is reached in either case. This means that $0,0\rightarrow\varphi+2\rightarrow S/N,N/N\rightarrow 0,0$ as well. Thus, when the initial rotation is negative $\varphi\rightarrow\varphi-2$ rather than positive $\varphi\rightarrow\varphi+2$, as we have already started to discuss, $0,0\rightarrow\varphi-2\rightarrow N/S,S/S\rightarrow 0,0$ and $0,0\rightarrow\varphi-2\rightarrow S/S,N/S\rightarrow 0,0$ are also operations which restore the initial disentangled state with no twists. And as already stated, in recognition of this symmetry, we shall work *only with positive right-handed rotations*, which means that ribbons must always go first over the north pole to achieve disentangling. So by adopting this convention, we can drop the "$/N$" from the notation because it is always to be implicitly assumed. We also keep in mind that the final configuration is invariant under the order in which the north and south ribbons are operated, i.e., under either temporal ordering (N,S) or (S,N) of the permutated ribbon set $\{N,S\}$. Thus, we can simplify the shorthand to write the Figure 3 operation as $0,0\rightarrow\varphi+2\rightarrow\{N,S\}\rightarrow 0,0$, simply indicating that either (N,S) or (S,N) over the north pole will restore a disentangled, untwisted state following a $\varphi+2$ rotation of a vector.

For the second choice to disentangle the ribbons, one can take the north ribbon and wind it *twice* past the north pole, then past the vector, then past the south pole, and the ribbons will still disentangle as before. But here, there will be a residual twist in each ribbon, as now shown below in Figure 4 below. Now, because we have used the ribbon set $\{N,N\}$ to disentangle the ribbons, the north ribbon maintains a double helix twist with right-handed parity as defined along the +z axis which we denote by 2R, while the south ribbon also has a double helix twist but with left-handed parity which we denote as 2L. The ribbons are fully disentangled, and yet, the end state in Figure 4(b) is observably, physically-distinct from the end state of Figure 3(b), based wholly on the operation that was used to disentangle the ribbons. *So even though a rotation of a vector through $\varphi\rightarrow\varphi+4\pi$ yields the exact same orientation and the exact same entanglement for that vector, the final, physical state can still be different from the starting state, wholly*

dependent upon how the disentangling operation has taken place. This means that $\varphi + 4\pi$ is topologically observably distinct from φ, even after disentangling. These two orientations separated by 4π have the same OE version and can both be disentangled, but they still may have different OET twist configurations.

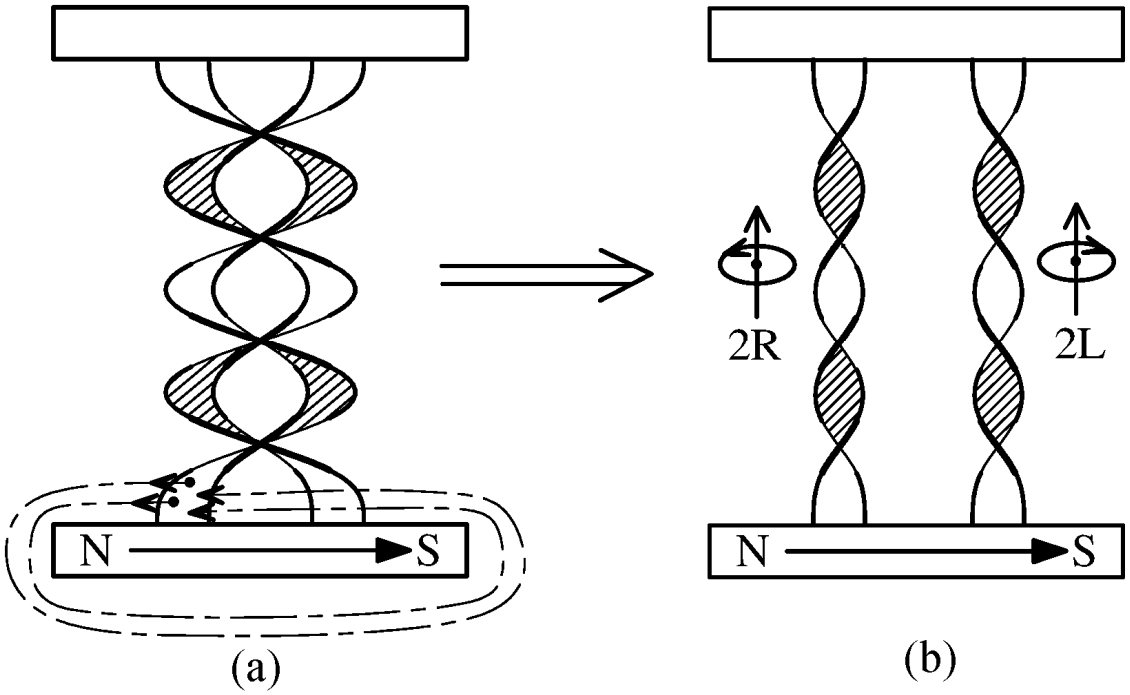

Figure 4: The Disentangling Operation $0 \to \varphi + 2 \to \{N, N\} \to 2R$

Using the notation developed above, we may use $0, 0 \to \varphi + 2 \to \{N, N\} \to 2R, 2L$ to denote the final state of Figure 4(b) in which the north ribbon ends up with a double right-handed helix and the south ribbon ends up with a double left-handed helix. The $\varphi + 2$ of the two windings gets inherited by the 2 in the two double helix twists. It will be apparent, however, that the left and right twists are offsetting, which is to say that the *net twist* of the overall system remains zero as it was when it started in Figure 2(a). In general, it turns out that this "conservation of twist" result carries through to all OET disentangling. So, if we know that the north ribbon has ended up with *2R*, then we automatically know that the south ribbon has ended up with *2L*. And if the north ribbon has a 0 twist then so too does the south ribbon. Thus, we can use the twist conservation under OET disentangling to simplify the summary of the Figure 4 operations to $0 \to \varphi + 2 \to \{N, N\} \to 2R$, showing only 0 as the initial twist and *2R* as the final twist for the north ribbon, and deducing by implication that 0 and *2L* are therefore the initial and final twists for the south ribbon. In this way, we adopt a *convention* whereby the helicity twist of the "north" ribbon is used to characterize the helicity twist of the south ribbon and therefore of the overall OET system following disentangling.

For the third and final choice to disentangle the ribbons, one can take the south ribbon and wind it *twice* about the north pole, then the vector, then the south pole, and the two ribbons

will again disentangle from one another. But here, there will be a residual twist in each ribbon oppositely to that shown in Figure 4, as now shown in Figure 5 below. Here, we have used the operational set $\{S,S\}$ to disentangle the ribbons. The north ribbon maintains a double helix twist but now with left-handed parity which we denote by 2L, while the south ribbon also has a double helix twist but with right-handed parity which we denote as 2R. Twist is still conserved, i.e., the net ribbon twist is zero, so continuing to represent the end result simply by the *2L* state of the north ribbon, $0 \to \not{\varphi} + 2 \to \{S,S\} \to 2L$ now summarizes the Figure 5 operation.

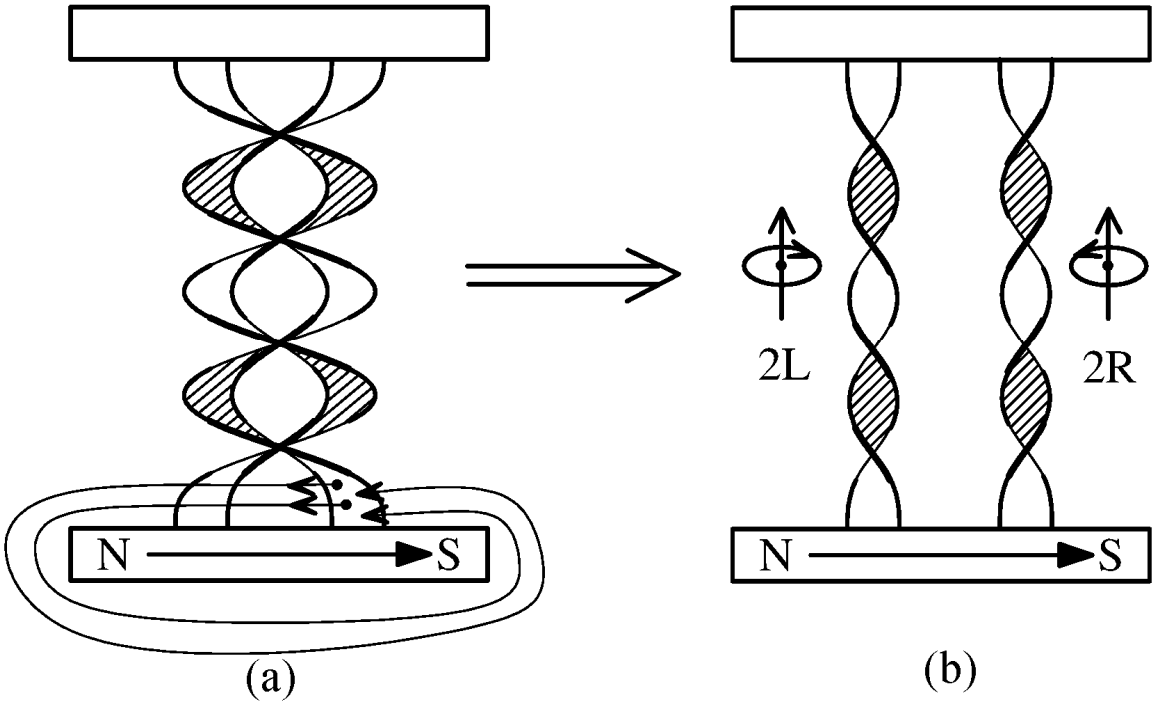

Figure 5: The Disentangling Operation $0 \to \not{\varphi} + 2 \to \{S,S\} \to 2L$

Returning to Figure 3, because we now know that twist is conserved, as already done with Figures 4 and 5, we further consolidate the summary of this operation to $0 \to \not{\varphi} + 2 \to \{N,S\} \to 0$. These are the operations shown in the captions for these three figures. They key thing we learn from all of this is that a state which starts at 0 for the north ribbon can be disentangled into one of three states: 2*R*, 0, and 2*L* depending on whether we disentangle with $\{N,N\}$, $\{N,S\}$ or $\{S,S\}$, respectively.

Now let us now make some final changes to our notation. Because the two ribbons can only be disentangled from one another in this way if there have been two rotations to begin with, and because the results 2*R* and 2*L* both have two twists, let us talk from now on about the number of *double* rotations and the number of *double* twists. So in all of the above, we started with one (1) double rotation and the states which retained non-zero twist ended with one (1) double twist. Also, because the non-zero twist end results always contain a left- or right-handed double twist, let us use the "+" sign to denote a right-handed and "-" to represent a left-handed

twist in relation to the +z axis. And since the number of rotations and the number of twists are both topologically-quantized integers, let us assign quantum numbers to these.

In all of the foregoing we first rotated the azimuth by $\varphi \to \varphi + 2$, which we rewrite as $\varphi \to \varphi + 2l'$ with $l' = +1$ denoting the number of double rotations which as discussed is always a positive number. The reason for using "primes" in the notation will momentarily become evident. So all of Figures 3, 4 and 5 can be summarized by the double rotation quantum number $l' = +1$, and instead of writing $\varphi + 2$ in our notation, we simply write $l' = +1$. After disentangling the two ribbons, depending upon the operation used, we ended up with 2*R*, 0, or 2*L*. For these let us use the respective double twist quantum numbers $m' = +1$, $m' = 0$ and $m' = -1$, and. Both l' and m' are quantized, but there is no mystery to this because they simply represent the number of double rotations and the number of double twists. So using this notation, we can consolidate all of the results from Figures 3, 4 and 5 in the following triplet of final states:

$$0 \to l' = +1 \to \begin{cases} \{N, N\} \to m' = +1 & \left(\left| l' = +1, m' = +1 \right\rangle\right) \\ \{N, S\} \to m' = 0 & \left(\left| l' = +1, m' = 0 \right\rangle\right) \\ \{S, S\} \to m' = -1 & \left(\left| l' = +1, m' = -1 \right\rangle\right) \end{cases} . \tag{5.1}$$

To the point: the triplet of final states is $\left| l' = +1, m' = +1 \right\rangle$, $\left| l' = +1, m' = 0 \right\rangle$ and $\left| l' = +1, m' = -1 \right\rangle$.

Now let's repeat everything we have just done, but instead of a single double-winding $l' = +1$, let's start with Figure 2a, and do two double-windings, $\varphi \to \varphi + 8\pi$, i.e., $\varphi \to \varphi + 4$. This is now an $l' = +2$ state, and it requires four ribbon operations. But instead of showing more drawings, let's just use the consolidated notation to represent the results. As discussed earlier, ribbons must always be brought first past the north and then past the south pole, because the rotation is a positive rotation. Doing otherwise will create further entangling, rather than disentangling. As also reviewed, the temporal order with which one operates the ribbons does not matter because as with $l' = +1$ the final twist results are invariant with respect to this order. So the five disentangling operations which can each be applied in any temporal permutation are $\{N, N, N, N\}$, $\{N, N, N, S\}$, $\{N, N, S, S\}$, $\{N, S, S, S\}$ and $\{S, S, S, S\}$. What we now have, in place of five more figures, are the five resulting states:

$$0 \to l' = +2 \to \begin{cases} \{N, N, N, N\} \to m' = +2 & \left(\left| l' = +2, m' = +2 \right\rangle\right) \\ \{N, N, N, S\} \to m' = +1 & \left(\left| l' = +2, m' = +1 \right\rangle\right) \\ \{N, N, S, S\} \to m' = 0 & \left(\left| l' = +2, m' = 0 \right\rangle\right) \\ \{N, S, S, S\} \to m' = -1 & \left(\left| l' = +2, m' = -1 \right\rangle\right) \\ \{S, S, S, S\} \to m' = -2 & \left(\left| l' = +2, m' = -2 \right\rangle\right) \end{cases} . \tag{5.2}$$

So now we start to see the pattern when Orientation-Entanglement (OE) is analyzed also with careful consideration of Twist, for which we shall use the acronym OET. In general, for OET, l' which represents the number of double rotations / windings is an integer which always has the value $l' = 0, 1, 2, 3, ...$ ($l' = 0$ is represented by Figure 2(a) and has the singlet state $m' = 0$ with no twists), and the resultant twist of the north ribbon following disentangling ranges over the integers m' for which $-l' \leq m' \leq +l'$. So, for example, if we go next to $l' = +3$ with three double windings we have seven states $m' = 0, \pm 1, \pm 2, \pm 3$. And in general, the number of end states will be equal to $2l'+1$ for any given l'. Strikingly, if we simply remove the "primes," this is the same pattern seen in the orbital angular momentum and magnetization (z axis orbital component) of electrons in the shells of atoms, as represented by the quantum numbers l and $m\left(=l_z\right)$. And also strikingly, the azimuth angle φ about which this topological winding occurs is the same azimuth in physical three-dimensional space through which this angular momentum is specified.

Clearly then, OET provides the basis for asserting that vectors with orientations in the set $\varphi + 4\pi n$ are not trivially-identical once topological OET has been considered, and that the differences between these inequivalent states map directly to the orbital and magnetic quantum structures of atoms and the nuclei and the orbital angular momentum quantum numbers which force exclusion. We also note in passing that the web animation at [31] which follows the same winding procedure we have used here albeit displayed from a bottom perspective view, is one example of how OE discussions often overlook Twist: This animation performs the disentangling operation $\{S, S\}$, so while it does indeed disentangle the threads, it still leaves the routinely-overlooked twist which in this case is the $\left|l' = +1, m' = -1\right\rangle$ state of Figure 5.

This leads us to three questions: First, are these $l' = 0, 1, 2, 3, ...$ and $-l' \leq m' \leq +l'$ concurrences merely coincidental, or can OET be used to provide a fully-topological understanding of electronic structure quantization (and by extension nuclear structure which is subject to similar quantized exclusion principles for proton and neutrons)? Second, how does this all relate (if at all) to the DWY monopoles which motivated this discussion in the first place because of the need to physically-distinguish rotational states with the same OE, i.e., states differing by a 2π or 4π rotation in $\nu = \mathcal{A} / \varphi$ in (3.8)? Third, because with the exception of the even denominator 2 which still needs discussion, (3.8) is an empirically-correct description of the observed odd-fractional FQHE denominators which apparently are observably topologically distinct and so can possibly be physical observables, how does this relate (if at all) to FQHE? And these three questions taken together lead to a fourth question: does the high-symmetry environment of FQHE where electrons are restrained to two dimensions at ultra-low temperatures approaching absolute zero reveal some type of genuine physical convergence of topology and atomic structure and the electric-magnetic symmetry of DWY magnetic monopoles, all emanating from $U(1)_{em}$ gauge theory?

We shall address all of these questions more fully starting in section 7. But first, we need to see if this topology can be further developed to map one other indispensable aspect of any conversation about electrons: their intrinsic spin ½.

6. Twist Part 2: Topological Modelling of Fermions and Conservation of Orbital Plus Spin Angular Momentum

We found in the last section that when rotating a vector (meaning, the arrow in the N→S bar) through $l' = 0,1,2,3,...$ double windings ($+4\pi$ rotations), the number of double twists (4π twists) that remain following disentangling using one of $2l'+1$ available disentangling operations is $-l' \le m' \le +l'$ with the positive and negative signs representing helicity handedness in relation to the +z axis. And this made us attentive to a *possible* connection with the analogous $l = 0,1,2,3,...$ and $-l \le m \le +l$ in one of $2l+1$ states from atomic structure. However, even if this possible connection is established to be a *true* connection, the finding that angles differing by 4π are topologically distinct still does not entirely solve the problem of physical observability because the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ to which we seek to solidify a connection is not by itself a physical observable. Observables must commute with the Hamiltonian, and when commuted with the Dirac Hamiltonian, $\left[H, \mathbf{L}\right] = -i\left(\boldsymbol{\alpha} \times \mathbf{P}\right)$. To fashion an *observable* angular momentum analogy, we need to also include the spin operator for which $\text{diag}\left(\boldsymbol{\Sigma}\right) = \left(\boldsymbol{\sigma}, \boldsymbol{\sigma}\right)$ with $\mathbf{S} \equiv \frac{1}{2}\boldsymbol{\Sigma}$, and for which $\left[H, \mathbf{S}\right] = +2i\left(\boldsymbol{\alpha} \times \mathbf{P}\right)$. Then, forming the total angular momentum $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma} = \mathbf{L} + \mathbf{S}$, we obtain $\left[H, \mathbf{J}\right] = 0$, and find that it is the total angular momentum $\mathbf{J}$ which is the conserved observable. It is from the Casimir operator $s = \frac{1}{2}$ in $\mathbf{S}^2 \left|\xi\right\rangle = s\left(s+1\right)\left|\xi\right\rangle$ and from the eigenvalues $\pm\frac{1}{2}$ in $S_z \left|\xi\right\rangle = s_z \left|\xi\right\rangle$, that the Dirac fermions acquire their intrinsic spin ½. Thus, if our goal is to develop a topological understanding of *observable* physics – as it must be – then we must advance the results of section 5 to provide a topological understanding of the intrinsic spin ½ of a fermion, and of the interplay between intrinsic spin and orbital angular momentum to conserve and render observable the total angular momentum. That will be the main objective of this section. So let us begin with the first question: topologically, in terms of OET, using the bar and ribbon apparatus of Figure 1(c), how do we represent intrinsic spin ½?

We saw in the last section that each unit of a quantum number in $l' = 0,1,2,3,...$ which is the *l* analogy represented one *double* rotation of $+4\pi$, and that each unit of a quantum number in $-l' \le m' \le +l'$ which is the $-l \le m \le +l$ analogy represented one *double* twist of 4π with associated handedness. So each quantized unit corresponded with two rotations and / or twists. Thus, each *single* rotation or twist with magnitude 2π should be represented by a half unit of a quantum number, i.e. by the quantum number ½ which is in fact the intrinsic spin of the electron. So our first conclusion is that an electron will be represented by in some way by a single rotation and / or twist and that the associated topological quantum number will be a half-integer.

Moreover, this first conclusion is supported by our finding following (3.7) that *if* the DWY monopoles exists, *then* the special case in which $\varphi = 2\pi$ and $\Lambda = 2\pi$ has an electric charge equal to that of a single electron. So there is an association already established between the DWY electron and $\varphi = 2\pi$ which represents a single rotation. Specifically, throughout the last section, we made use of the rotation $\varphi \to \varphi + 2l'$ which is to say we took the apparatus of Figure 3(b) in some unspecified azimuth orientation φ and then added a $+4\pi l'$ rotation to bring

the azimuth to $\varphi + 4\pi l'$. But we never really stated what the orientation of the original φ might be. However, if it is a single DWY electron that we are rotating and then disentangling, then what we found following (3.7) tells us that the initial state for this electron needs to be regarded as $\varphi = 2\pi$. Thus, for example, if a DWY electron with $\varphi = 2\pi$ is taken to be the initial state, then the rotation $\varphi \to \varphi + 4\pi l'$ is really $\varphi = 2\pi \to \varphi = 2\pi + 4\pi l' = 4\pi\left(l' + \frac{1}{2}\right)$.

Now, the fact that $l' + \frac{1}{2}$ naturally appears in this expression $\varphi = 2\pi \to \varphi = 4\pi\left(l' + \frac{1}{2}\right)$ a.k.a. $\not{\varphi} = 1 \to \not{\varphi} = 2\left(l' + \frac{1}{2}\right)$ means that the whole integer $l' = 0, 1, 2, 3, ...$ naturally gets added to the half integer ½ when the initial topological state is taken to be the $\varphi = 2\pi$ DWY electron. So keeping in mind that spin $s = \frac{1}{2}$ comes from the Casimir operator in $\mathbf{S}^2\left|\xi\right\rangle = s\left(s+1\right)\left|\xi\right\rangle$, we posit a topological spin analog $s' = \frac{1}{2}$, and we rewrite the foregoing rotation as $\not{\varphi} = 1 \to \not{\varphi} = 2\left(l' + s'\right)$. But of course the total angular momentum $j = l + s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}...$ comes from the Casimir operator in $\mathbf{J}^2\left|\xi\right\rangle = j\left(j+1\right)\left|\xi\right\rangle$ which combines $\mathbf{L}^2\left|\xi\right\rangle = l\left(l+1\right)\left|\xi\right\rangle$ with $\mathbf{S}^2\left|\xi\right\rangle = s\left(s+1\right)\left|\xi\right\rangle$ via $\mathbf{J} = \mathbf{L} + \mathbf{S}$ for which $\left[H, \mathbf{J}\right] = 0$. So the topological analog to the total angular momentum will need to be $j' = l' + s' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}....$ That is, if the electron begins in the topological state with $l' = 0$ and $j' = s' = \frac{1}{2}$, and then we apply $l' = 1, 2, 3...$ double rotations before disentangling the ribbons, the state we end up with will need to be characterized by $j' = l' + s' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}...$ in order to establish a suitable analog to the observable *j*. This means that the rotation $\not{\varphi} = 1 \to \not{\varphi} = 2\left(l' + s'\right)$ may be further rewritten as $\not{\varphi} = 1 \to \not{\varphi} = 2j' = 1, 3, 5, 7....$ But this is just the odd-integer denominator in (3.8), and that denominator originates in the exact same reduced azimuth $\not{\varphi} = 2j' = 1, 3, 5, 7....$ So if we go back and use this in (3.8), we may write:

$$\nu = \frac{\not{A}}{\not{\varphi}} = \frac{\not{A}}{2j'}; \quad \not{A} = 0, \pm 1, \pm 2, \pm 3...; \quad j' = l' + s' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}...; \quad l = 0, 1, 2, 3, 4...; \quad s' = \frac{1}{2}. \qquad (6.1)$$

Now, the denominator in the fill factor is expressed directly in terms of a quantum number $\not{\varphi} = 2j' = 1, 3, 5, 7...$ which is an odd integer, which has a topological interpretation by which each of these 4π-differing orientations is topologically distinct, which does correspond to what is observed in the FQHE, which uses the exact same reduced azimuth $\not{\varphi}$, and which analogizes to *j* which is a physical observable because $\left[H, \mathbf{J}\right] = 0$. But now, how, exactly, do we capture all of this in the bar and ribbon apparatus of Figure 1(c)? More precisely: how to we now modify the bar and ribbon apparatus of Figure 1(c) so that it properly represents all of this and provides an exact topological mapping to all of the angular momentum quantum numbers l, m, s, j, s_z, j_z?

The final clue involves the z-component of these angular momenta. The spin angular momentum has the eigenvalues $s_z = \pm\frac{1}{2}$ emerging from $S_z\left|\xi\right\rangle = s_z\left|\xi\right\rangle$ while the magnetization quantum number $-l \le m \le +l$ to which we analogize the double twist number $-l' \le m' \le +l'$ is the eigenvalue of the z-component of the orbital angular momentum $m = l_z$ as obtained from

$L_z|\xi\rangle = m|\xi\rangle$. The z-component of the conserved and observable total angular momentum is then obtained from $J_z|\xi\rangle = j_z|\xi\rangle$ with $J_z = L_z + S_z$ and $j_z = m + s_z$, with the integer plus ½ $-j \le j_z \le +j$ in one of $2j+1$ total angular momentum states. Therefore, the topological analog to the z-axis total angular momentum will need to be $j_z' = m' + s_z'$ with $-j' \le j_z' \le +j'$ in $2j'+1$ possible twist states, where j_z' is always an integer plus ½ and therefore $2j_z'$ is always an odd integer just like the FQHE denominator, aside from the denominator of 2.

So to pin this all down, because the question before us is what we need to do to the apparatus of Figure 1(c) to model intrinsic spin, with $j_z' = m' + s_z'$, let us first set $s_z' = 0$ thus $j_z' = m'$ which is not a physical electron state, but which is the model we developed in section 5 from Figure 1(c) before starting in this section to consider intrinsic spin $s_z = \pm\frac{1}{2}$. This quantum number m' in section 5 directly represented the number and handedness of the topological double twists following disentangling of the ribbons. This means that if $j_z' = m'$ as it was in section 5, then j_z' must similarly represent double twist number and handedness subsequent to disentangling. And this should not change once we introduce $s_z = \pm\frac{1}{2}$ to replace the $s_z' = 0$ which was implicit in section 5.

So let us now consider $j_z' = m' + s_z'$ in the situation where $m' = 0$ but $s_z = \pm\frac{1}{2}$, for example, for the states $|l' = +1, m' = 0\rangle$ in (5.1) or $\left(|l' = +2, m' = 0\rangle\right)$ in (5.2). In these states, once intrinsic spin is included we will have $j_z' = s_z' = \pm\frac{1}{2}$. Topologically, this represents half of a right- or left-handed double twist, i.e. one right- or left-handed single twist. So now we have our answer: for the topological analogy of Figure 1(c) to be able to account for intrinsic spin we must model the spin ½ DWY electron in its initial unrotated state with a single twist, i.e. a half double twist. Because there is both spin up and spin down along the z axis, this means that we will need two topological models, one in which the N ribbon (which by convention represents the state of the Figure 1(c) apparatus given twist conservation) has a single right-handed twist and the other in which it has a single left-handed twist.

Based on all of the above, we proceed in Figure 6 below to represent a fermion, using the "bar and ribbon" topological device of Figure 1(c), by introducing a single 2π twist into each ribbon, but with opposite helicity as between these two ribbons so as to conserve twist. Because we have adopted a convention utilizing twist conservation whereby the north ribbon specifies the OET handedness in relation to the +z axis, we will need two such bar and ribbon systems: one in which the north ribbon has one right-handed helicity twist to represent spin up, and the other in which it has one left-handed helicity twist to represent spin down. We denote the right-helicity ribbon with one half of a right-handed double twist using the quantum number $s_z' = +\frac{1}{2}$ and the left-helicity ribbon likewise as $s_z' = -\frac{1}{2}$, and we continue the convention whereby the "north" ribbon represents the overall topological system. Because we have not yet performed any $+4\pi$ rotations about the +z axis and then disentangled anything, these are configurations in the state $|l' = 0, m' = 0\rangle$. We have also included a second set of north (N) and south (S) references in

these Figures, with N placed on the OET "environment" bar and S placed in the middle of the OET "object" bar. The original N→S vector maintains the reference to the original homotopic deformations in Figure 1 here, of MTW's "object" and "threads" and "environment" illustration in Figure 31.6 of [18], into the bars and ribbons we are using here. Meanwhile, the supplementary N and S provide a way of referring to the relative relationships between the OET "environment" bar in the north and the "object" bar in the south. Which N and S we are talking about in any particular situation from here should be discernable by context.

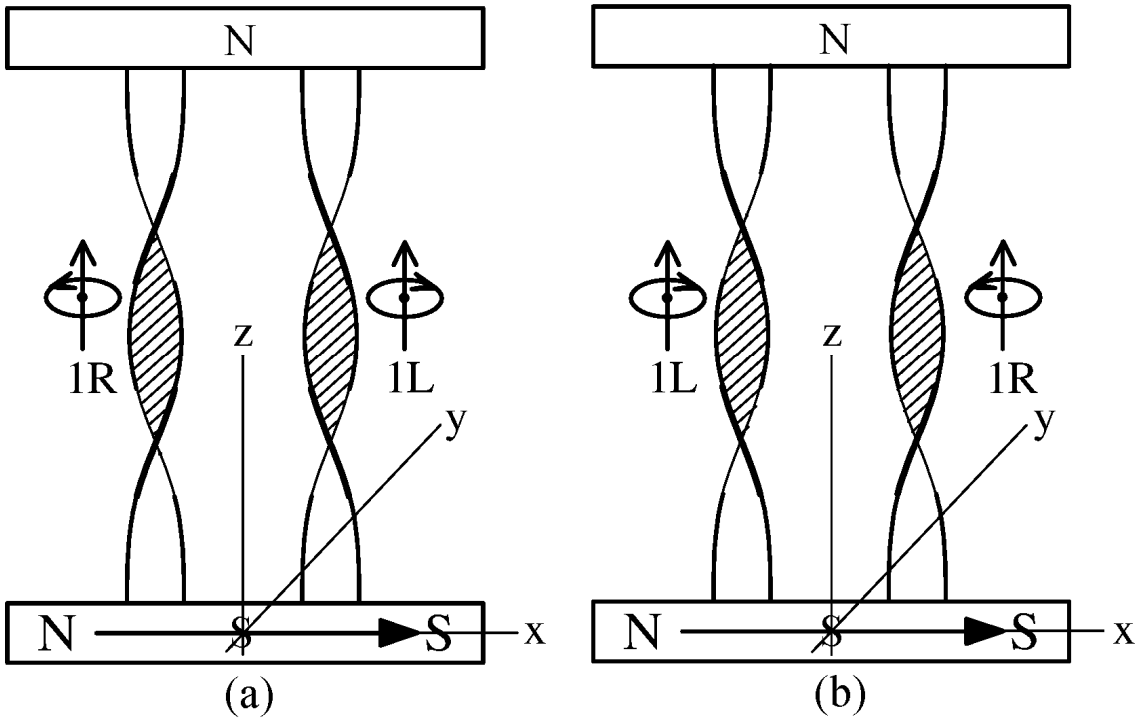

Figure 6: Topological Representation of spin *s*=½ DWY Electrons: (a) $\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle$ and (b) $\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle$

Now, given that we intend to use the configurations of Figure 6 above to represent spin up and spin down fermions $s_z=+\tfrac{1}{2}$ and $s_z=-\tfrac{1}{2}$, a practical question arises which may point to a deep physical result: Let us suppose that the reader has built the apparatus in Figure 2(a) a.k.a. 1(c) and now wishes produce the apparatuses of Figures 6(a) and 6(b). Is the reader required to unstaple or unglue (detach) the ribbons from Figure 2(a), given them each a twist, and then reattach them back? Or, can the reader merely perform some set of operations of rotation and disentangling to get from Figure 2(a) to Figures 6? This is a practical question for somebody who lacks a staple remover, but given the topological connections we are finding between twist and spin it is also a deep theoretical question, namely: *is there some way in which Figure 2(a) which represents no intrinsic spin can be topologically deformed into Figures 6 which represent a half unit of intrinsic spin?* That is to say, *are these representations of spin 0 and spin ½ part of the same homotopy group and thus topologically equivalent, or, are they topologically different?* The reason this is important is that if are eventually intending to have the number of double twists be a topological representation of total angular momentum about the z-axis j_z, then Figure 2 will eventually represent spin 0 and Figures 6 will eventually represent spin ½, and so

the topological deformation of Figure 2 into Figures 6 would suggest that one can use topological deformations to go from spin 0 scalars to a spin ½ fermions and by induction to higher spins. So the deep question this raises is this: *can topological deformations in the three-dimensional physical space be used to connect particles of different spins all within the same homotopy group, or do such different-spin particles belong to different homotopy groups whereby they are topologically distinct?*

As it turns out – which the reader who has built Figure 2(a) can easily confirm – there are no OET operations about either the z or y axes that will deform Figure 2(a) into Figures 6. The z-rotations have already been explored in the last section, and it turns out that rotation about the y axis followed by disentanglement produces results equivalent to the z rotations followed by disentanglement. However, it is possible to deform Figure 2(a) into Figures 6 by rotating the bottom N→S bar *about the x axis*, and then disentangling the ribbons. Specifically, using +x to define the axis of rotation, if one does a *right-handed* rotation of the N→S bar about the x axis through 2π (one rotation not two) letting the ribbons wind once about the bar, and then disentangles the north ribbon over the north pole and the south ribbon over the south pole, the resulting configuration is $s_z' = +\frac{1}{2}$ in Figure 6(a). With a *left-handed* rotation of the N→S bar about the x axis through 2π followed by the same disentangling, the result is Figure 6(b). So our representation of spin 0 in Figure 2(a) can be topologically deformed into our representation of spin ½ in Figures 6, but *only if we are permitted to rotate the N→S bar about the x axis*. If we are prevented from this rotation, and are only permitted rotations about y and z, then the different spins cannot be deformed into one another. It is also interesting to note, although these $\pm 2\pi$ spin-changing rotations about the x axis are *not* rotations about the azimuth φ defined around the z axis, that after we have disentangled the ribbons, the resulting twists do become transferred such that their $\pm 2\pi$ handedness and magnitude is now oriented about the z axis. Apparently, topologically, this $\pm 2\pi$ rotation about the x axis followed by disentangling the ribbons transmutes into the $\varphi = 2\pi$ azimuth which, as found after (3.7) and reviewed a few moments ago, is the ground state of the DWY electron.

Although tangential to the main development here, we take a moment to point out how all of this raises the physical question whether there are natural situations in three-dimensional space where rotations can occur about two of the three axes but not about the third. And in thinking about this question, one is drawn to the dynamical property of physical space whereby if one has an object with differing length > width > depth, say a length of 12 inches, a width of 6 inches and a depth of 1 inch, and then throws that object into the air while imparting rotation, a stable rotation can be achieved about the length and depth axes, *but not about the width axis*. So there is a physical precedent for "excluding" rotations about one of the three space axes. While we shall not pursue this collateral question here, it is perhaps worth musing whether this dynamical property of three-dimensional space can help in trying to understand spin supersymmetry, or its apparent observed absence.

Returning now from this brief digression, let us now rotate the azimuth φ of the N→S bar of each of Figures 6(a) and (b), which is inherently $\varphi = 2\pi$, through one $+4\pi$ double winding, then disentangle the ribbons using the $\{N,N\}$, $\{N,S\}$ and $\{S,S\}$ operations developed and discussed in section 5. We can draw some more Figures for this, but that is now

unnecessary because we have developed some shorthand for representing the results, and presumably the reader has built this apparatus and so can observe these results directly.

So, starting with Figure 6(a) which is in the state $\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle$ with one-half of a right-handed double twist in the north ribbon, we rotate $\varphi=2\pi\rightarrow\varphi=2\pi+4\pi=6\pi$, which brings about the $\left|l'=1\right\rangle$ state representing one double rotation. We learned at (5.1) and (5.2) that a disentangling which equally balances *N* and *S* restores the original number of and handedness of double twists, i.e., leaves m' unchanged from its original state. If one carries out these above operations it will become clear that $j'_z=m'+s'_z$ tells us the number of double twists in each final state. Thereafter, for each of the three combinations of disentangling, we arrive at:

$$\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle\rightarrow l'=+1\rightarrow\begin{cases}\{N,N\}\rightarrow\left|l'=+1,m'=+1,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle\\ \{N,S\}\rightarrow\left|l'=+1,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle\\ \{S,S\}\rightarrow\left|l'=+1,m'=-1,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle\end{cases}\quad .\quad (6.2)$$

Likewise, if we start with Figure 6(b) which is in the state $\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle$ and again rotate $\varphi=2\pi\rightarrow\varphi=2\pi+4\pi=6\pi$ which is represented by $\left|l'=1\right\rangle$, each of the disentangling options produces:

$$\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle\rightarrow l'=+1\rightarrow\begin{cases}\{N,N\}\rightarrow\left|l'=+1,m'=+1,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle\\ \{N,S\}\rightarrow\left|l'=+1,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle\\ \{S,S\}\rightarrow\left|l'=+1,m'=-1,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle\end{cases}\quad .\quad (6.3)$$

<u>*All*</u> of the foregoing states have $j'=\tfrac{3}{2}=l'+\tfrac{1}{2}$ which is the maximum number of double twists that can remain following a maximally-stretched all-N or all-S disentangling, i.e., following $\{N,N\}$ or $\{S,S\}$ in which m' and s'_z are both like-handed. And there are a total of six (6) distinct such $j'=\tfrac{3}{2}$ states, which is equal to the two original states of Figures 6 times the three distinct disentangling operations $\{N,N\}$, $\{N,S\}$ and $\{S,S\}$ that can be done following $2l'=+2$ single rotations from which there need to be two disentangling operations to disentangle the ribbons from one another. (Keep in mind, again, that the final result is invariant under the two temporal permutations (N,S) and (S,N) of the operation set $\{N,S\}$.)

Let's now start with each of Figures 6 which have $\varphi=2\pi$ and rotate four times to $\varphi=10\pi$ which brings them into the $l'=2$ state of two double azimuth rotations. To save on drawings, let us simply represent the results in the same way. Now from Figure 6(a) we have:

$$\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \rightarrow l'=+2 \rightarrow \begin{cases} \{N,N,N,N\} \rightarrow \left|l'=+2,m'=+2,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{5}{2}\right\rangle \\ \{N,N,N,S\} \rightarrow \left|l'=+2,m'=+1,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle \\ \{N,N,S,S\} \rightarrow \left|l'=+2,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{N,S,S,S\} \rightarrow \left|l'=+2,m'=-1,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \\ \{S,S,S,S\} \rightarrow \left|l'=+2,m'=-2,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle \end{cases} \quad .(6.4)$$

Likewise, if we start with Figure 6(b) which is in the state $\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle$ and again rotate to $l'=2$ which is $\varphi=10\pi$, each of the disentangling options produces:

$$\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \rightarrow l'=+2 \rightarrow \begin{cases} \{N,N,N,N\} \rightarrow \left|l'=+2,m'=+2,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle \\ \{N,N,N,S\} \rightarrow \left|l'=+2,m'=+1,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{N,N,S,S\} \rightarrow \left|l'=+2,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \\ \{N,S,S,S\} \rightarrow \left|l'=+2,m'=-1,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle \\ \{S,S,S,S\} \rightarrow \left|l'=+2,m'=-2,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{5}{2}\right\rangle \end{cases} \quad .(6.5)$$

All of the foregoing states have $j'=\tfrac{5}{2}=l'+\tfrac{1}{2}$ which is the maximum number of double twists that can remain following a maximally-stretched all-N or all-S disentangling, i.e., following $\{N,N,N,N\}$ or $\{S,S,S,S\}$ in which m' and s'_z are both like-handed. And there are a total of ten (10) distinct such $j'=\tfrac{5}{2}$ states. This is two original states times the five distinct disentangling operations $\{N,N,N,N\}$, $\{N,N,N,S\}$, $\{N,N,S,S\}$, $\{N,S,S,S\}$ and $\{S,S,S,S\}$ that can be done following $2l'=+4$ rotations from which there need to be four disentangling operations (for which the permutations of N and S ordering do not change the final result) to disentangle the ribbons from one another.

We can next go to $\varphi=14\pi$ which is $l'=3$ double rotations over and above the $\varphi=2\pi$ initial state and will find a total of fourteen (14) distinct $j'=\tfrac{7}{2}$ states, and the pattern will continue. Indeed, in general, for any given number of double rotations $l'=0,1,2,3...$ starting with the two $s'=\tfrac{1}{2}$ states shown in Figures 6(a) and (b), we will have $j'=l'+s'=l'+\tfrac{1}{2}=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{7}{2}...$ with $-j'\le j'_z\le+j'$, as well as $-l'\le m'\le+l'$. Making use of $j'=l'+\tfrac{1}{2}$ there will be a total of $4j'=4\left(l'+\tfrac{1}{2}\right)=2\left(2l'+1\right)$ distinct states for any given j'. So for respective $j'=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{7}{2}...$ there will be a total of $2,6,10,14...$ distinct states following the distinct sets of disentangling operations that can be performed, and as we would hope, with the mere removal of the "primes" from all of the above, this precisely maps into the angular momentum states observed in the electronic shells of an atom. In other words, this pattern of OET states – *derived wholly from the topology of OET in three space dimensions* – will continue to perfectly match the observed atomic structure for angular momentum, all the way through the entire periodic table.

These results deepen our attention to a possible connection between OET topology and electronic atomic structure, especially now that we have a j_z' quantum number representing the total number of double twists which are *observable* after all disentangling has been performed, which analogizes to the *observable* j_z eigenvalues in $J_z|\xi\rangle = j_z|\xi\rangle$ for which $\mathbf{J}=\mathbf{L}+\mathbf{S}$ and $[H,\mathbf{J}]=0$ and for which the Casimir relationship is $\mathbf{J}^2|\xi\rangle = j(j+1)|\xi\rangle$. Now, by representing spin ½ topologically with the twists in Figures 6, the observable number of double twists in the topological pattern following OET, namely j_z', now matches up precisely with the empirically-observable z-component of the total angular momentum, namely j_z. And in general the permitted states and numbers of these states also match perfectly with all of the angular momentum quantum numbers l, m, s, j, s_z, j_z. This renews our attention to the question whether OET be used to provide a fully-topological understanding of electronic structure quantization (and by extension nuclear structure which has a similar shell patter for each of protons and neutrons), and how this relates to the DWY monopoles and the odd-numbered FQHE denominators.

7. Review of Evidence that the Quantized Fractionalized DWY Electric Charges may be Synonymous with FQHE Quasiparticle Charges, and a Proposed Experimental Test

The whole original purpose of the last two sections (which yielded some unexpected analogies to atomic structure) was to show that although the gauge angle Λ and the azimuth angle φ appearing in (7.1) involve sets of angles differing by 2π which are geometrically indistinct, these angles are nonetheless *topologically-distinct* under OET and so can be physically observable, at least in terms of having this topological distinctness. This is a *necessary* (albeit not sufficient) condition for these angles to have physically-observable consequences, for example, as the numerator and denominator in the fill factor $\nu = \Lambda/\varphi$ of observed quantized and fractionalized DWY electric and magnetic charges. In this section we shall review facts and evidence widely known or thought to be true, and evidence which has been demonstrated thus far in this paper to be true. This evidentiary review will then lead us to propose an experiment to test some inferences from this evidence which inferences will be detailed in the next section.

To the best of our present knowledge, the Dirac-Wu-Yang (DWY) magnetic monopoles have never been observed. So if they do exist, they would exist under some highly specialized set of physical circumstances which certainly do not include ordinary observations of electrons either as free fermions or bound within the electronic shells of atomic orbitals at temperatures not near absolute zero. Because the DWY monopoles obey and indeed are defined by the Dirac Quantization and Fractionalization Condition (DQFC) which can be restated from (3.2) or (3.4) and (3.5) before we began to consider OET, as:

$$\frac{\mu e}{2\pi} = \nu = \frac{n}{l} = \frac{\cancel{\Lambda}}{\cancel{\varphi}} = \frac{\Lambda}{\varphi}; \quad \cancel{\Lambda} = 0, \pm 1, \pm 2, \pm 3...; \quad \cancel{\varphi} = 1, 2, 3, 4, 5, 6..., \tag{7.1}$$

and because this DQFC is symmetric under the interchange $\mu \leftrightarrow e$ of electric and magnetic charges, this means that the specialized set of conditions under which these monopoles do exist – if they exist – is one of perfect electric-magnetic symmetry in which Maxwell's classical equations are $J^{\mu} = \partial_{\sigma} F^{\sigma\mu}$ and $P^{\mu} = \partial_{\sigma} {}^{*}F^{\sigma\mu}$ where $F^{\sigma\mu}$ is the electromagnetic field strength tensor, ${}^{*}F^{\sigma\mu} = \frac{1}{2!}\varepsilon^{\alpha\beta\sigma\mu}F_{\alpha\beta}$ is the dual field strength tensor in flat spacetime, J^{μ} is the electric current four-density and P^{μ} the magnetic current four-density. So the DWY monopoles – if they exist – exist under natural circumstances which exhibit a high degree of symmetry, indeed, a high enough degree of symmetry to restore electric-magnetic duality to Maxwell's equations. So one question now becomes, what are the candidate physical conditions under which such a high degree of symmetry might arise and become observable?

To the best of our knowledge, there are two natural candidate circumstances under which nature displays a very high degree of symmetry. The first is at GUT energies; the second is at temperatures near absolute zero [27]. As discussed at the end of section 4, we expect that at GUT energies, and most certainly at ultra-GUT energies near the Planck scale defined by the Planck mass $M_P \cong 1.22\times 10^{19}$ GeV in $GM_P{}^2 \equiv \hbar c$ where G is the Newton gravitational constant, there will be many symmetries that we do not observe at laboratory energies. But $U(1)_{em}$ is not a high energy symmetry group; indeed, it arises when the electroweak $SU(2)_W \times U(1)_Y$ is broken down to $U(1)_{em}$ at energy scales established by the Fermi constant G_F and its associated relatively-low vev $v \cong 246\,\text{GeV}$, Further, the electric charge generator $Q = Y/2 + I_3$ sits across the generators Y and I_3 of the hypercharge and weak interactions. So GUTs are not the place to be looking for electric-magnetic symmetry, because $U(1)_{em}$ is far from having yet been "born" at GUT energies. The other candidate circumstance to look for high degrees of symmetry, namely temperatures near 0K, on the other hand, may be an ideal place to observe a $U(1)_{em}$ electric-magnetic symmetry. Other than the Fermi energies associated with the highest occupied states, virtually all other energy has been entirely drained out of the electrons, and under such ultra-low energy conditions $U(1)_{em}$ is certainly very firmly established. So if the DWY monopoles and the electric-magnetic symmetry they imply do exist under some specialized set of physical conditions, ultra-low temperatures appear to be the clearest and best candidate for being and providing that special set of conditions.

In this paper thus far, we have demonstrated that DWY analysis of $U(1)_{em}$ gauge theory does lead to charge quantization and fractionalization as in (7.1), and in the last two sections, we have demonstrated that these quantized numerators and fractionalized denominators are topologically distinct and thus are eligible be physically observed, at least on a topological basis. Further, we demonstrated in (3.8) that if the DWY monopoles exist *and if the only observed charges are those which can exist in a disentangled state*, then $\nu = \not{\Lambda}/\not{\varphi}$ in (7.1) will become restricted to only odd denominators $\not{\varphi} = 1 + 2l = 1,3,5,7,9\ldots$. Setting aside the observed even denominator $\not{\varphi} = 2$ which we shall separately consider in Section 10, it is an empirical fact that in FQHE, the observed states have precisely the same denominators $\not{\varphi} = 1 + 2l = 1,3,5,7,9\ldots$ as are shown in (3.8), and indeed, also looking to the reduced gauge angle $\not{\Lambda} = \Lambda/2\pi = n = 1,2,3\ldots$, have precisely the same filling factors $\nu = \not{\Lambda}/\not{\varphi}$ as are shown in (3.8). So *if* the DWY monopoles exist and *if* the only charges observed are ones which can be disentangled without

counter-rotation, *then* electric-magnetic duality which likely can only be restored and observed near the high-symmetry environment of 0K will exhibit a quantization and fractionalization which corresponds perfectly with the FQHE which also appears only near 0K, aside from the denominator $\varphi = 2$.

The next piece of evidence utilizes the topological findings of the last two sections and specifically (6.1). In order to establish that φ is a topological observable which is a necessary (but not sufficient) condition for it to be a physical observable, we ended up showing that $\varphi = 2j' = 1,3,5,7...$ which is equal to the maximum number of twists that can be observed once an unwound DWY electron state $\varphi = 1$ with $l' = 0$ hence $j' = s' = \frac{1}{2}$ has been disentangled using only all-North or all-South disentangling operations. So these DWY denominators $\varphi = 2j' = 1,3,5,7...$ do have a definitive connection to the states of topological twist under OET. Additionally, these j' and s' and the related topological quantum numbers l', m', s'_z, j'_z all map on a one-to-one basis, precisely with the angular momentum quantum numbers of the same labels simply without the "prime" designations. So *if* it can be demonstrated by theoretical argument or by experimental observation or by both that the angular momentum states of electrons in atomic shells *is topologically grounded* in this mapping of l', m', s'_z, j'_z to their unprimed cousins l, m, s_z, j_z from atomic theory, *then* these denominators would be given by $\varphi = 2j = 1,3,5,7...$ with the "prime" removed, and would be equal to twice the total angular momentum of the DWY electron. This would relate the DWY denominators to the total orbital angular momentum quantum numbers of the DWY electrons.

Additionally, because *j* is obtained from the Casimir operator $\mathbf{J}^2$ via $\mathbf{J}^2 \left| \xi \right\rangle = j\left(j+1 \right) \left| \xi \right\rangle$, which operator by definition commutes with each angular momentum generator via $\left[\mathbf{J}^2, \mathbf{J} \right] = 0$, and because $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is a conserved observable because $\left[H, \mathbf{J} \right] = 0$, this would connect the azimuth via $\varphi = 2j = 1,3,5,7...$ to a total angular momentum quantum number which is not only a topological observable, but is also a physical observable in atomic theory because $\left[H, \mathbf{J} \right] = 0$. This is an additional necessary (but still not sufficient) condition for φ to be a physical observable. Further, *if* it can be theoretically or experimentally demonstrated that the angular momentum l, m, s_z, j_z are topologically-grounded in their l', m', s'_z, j'_z cousins of OET, *then* this would mean the DWY denominators are intimately related to the quantized states of electrons in atomic shells.

Finally, if we are able to relate the DWY denominators $\varphi = 2j' = 1,3,5,7...$ to the total orbital angular momentum quantum numbers *j* of the DWY electrons as just laid out in the last paragraph, and *if* these DWY denominators can also be connected to the FQHE denominators, then the FQHE denominators would become connected to total orbital angular momentum, and one could start to look for *empirical* correlations between the two. Thus, via $\varphi = 2j = 1,3,5,7...$, the FQHE states with $\nu = n/3$ would all be states with $j = \frac{3}{2}$, the FQHE states with $\nu = n/5$ would all be states with $j = \frac{5}{2}$, the FQHE states with $\nu = n/7$ would all be states with $j = \frac{7}{2}$, and more generally, the FQHE states with $\nu = n/2j$ would all be states with total angular

momentum *j*. Thus, by testing for *j* in each of the FQHE fill factors $\nu = n/2j$, and more generally, by testing for l, m, s_z, j_z which are related to *j* to see if there is an observed correlation between the fill factor denominator and the hypothesized-to-be-related angular momentum quantum numbers, we would have an experimental route for testing whether all of these connections among topology and atomic structure and DWY monopoles are or are not physically supported by the natural world. *This final observation leads us to propose an experiment.*

If the one-to-one mapping developed here between the topological l', m', s_z', j_z' and the atomic angular momentum quantum numbers l, m, s_z, j_z is a true physical connection, which is to say if the pattern of angular momentum quantum numbers is rooted in the topology of OET, then we would be able to remove the primes from j', l', s' in (6.1), and so rewrite (6.1) as:

$$\nu = \frac{\not{n}}{\not{\varphi}} = \frac{\not{n}}{2j} = \frac{\not{n}}{2\left(l+s\right)} = \frac{\not{n}}{2\left(l+\frac{1}{2}\right)}; \qquad (7.2)$$
$$\not{n} = n = 0, \pm 1, \pm 2, \pm 3...; \quad j = l + s = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}...; \quad l = 0, 1, 2, 3...; \quad s = \tfrac{1}{2}$$

So now, the odd-integer DWY denominator $\not{\varphi} = 2j = 2\left(l+\frac{1}{2}\right) = 1, 3, 5, 7...$ would be characterized completely in terms of the observable total angular momentum quantum number *j* which is obtained from the Casimir relationship $\mathbf{J}^2\left|\xi\right\rangle = j\left(j+1\right)\left|\xi\right\rangle$ for a total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ for which $\left[\mathbf{J}^2, \mathbf{J}\right] = 0$ and $\left[H, \mathbf{J}\right] = 0$ and $\mathbf{J}_z\left|\xi\right\rangle = j_z\left|\xi\right\rangle$. But because $j = l + \frac{1}{2}$ with $l = 0, 1, 2, 3...$ we can just as readily characterize the DWY denominator in terms of *l* which has the Casimir relationship $\mathbf{L}^2\left|\xi\right\rangle = l\left(l+1\right)\left|\xi\right\rangle$. This $l = 0, 1, 2, 3...$, however, is the exact same *l* which is used to characterize atomic orbital shells with the designations s, p, d, f, etc., respectively named "sharp," principal," "diffuse," "fundamental," etc. Specifically, using $\left|l\right\rangle$ to denote "an electron in a particular *l* state," these designations of the shell in which a particular electron resides are defined as $s \equiv \left|l = 0\right\rangle$, $p \equiv \left|l = 1\right\rangle$, $d \equiv \left|l = 2\right\rangle$, $f \equiv \left|l = 3\right\rangle$, etc. But because $\not{\varphi} = 2\left(l+\frac{1}{2}\right)$, we can also write these in terms of the DWY denominators as $s \equiv \left|\not{\varphi} = 1\right\rangle$, $p \equiv \left|\not{\varphi} = 3\right\rangle$, $d \equiv \left|\not{\varphi} = 5\right\rangle$, $f \equiv \left|\not{\varphi} = 7\right\rangle$, etc.

This means that *if* the DWY monopoles are observed anywhere in the natural world, *then* the whole number charges $\not{\varphi} = 1$ with $\nu = n$ should exhibit properties reminiscent of *s* shell electrons; the 1/3 unit charges with $\not{\varphi} = 3$ thus $\nu = n/3$ should exhibit properties reminiscent of *p* shell electrons; the 1/5 unit charges with $\not{\varphi} = 5$ thus $\nu = n/5$ should exhibit properties reminiscent of *d* shell electrons; and the 1/7 unit charges with $\not{\varphi} = 7$ thus $\nu = n/7$ should exhibit properties reminiscent of *f* shell electrons; and so on.

Consequently, the proposed experiment, in principal is rather simple: We already know a great deal about the behaviors of electrons in atomic shells. For example, [32] and similar references which are available illustrate the real wavefunctions of various s, p, d, f electronic

states. Additionally, we know that there are two (2) distinct electronic states in the s shell with $\varphi = 1$, namely $s \equiv \left| l = 0; m = 0; s_z = \pm\tfrac{1}{2} \right\rangle$. There are six (6) distinct electronic states in the p shell with $\varphi = 3$, namely $p \equiv \left| l = +1; -1 \le m \le +1; s_z = \pm\tfrac{1}{2} \right\rangle$, see also their topological cousins with $l' = +1$ in (6.2) and (6.3). There are ten (10) distinct electronic states in the d shell with $\varphi = 5$, namely $d \equiv \left| l = +2; -2 \le m \le +2; s_z = \pm\tfrac{1}{2} \right\rangle$, see also their topological cousins with $l' = +2$ in (6.4) and (6.5). And there are fourteen (14) distinct electronic states in the f shell with $\varphi = 7$, namely $f \equiv \left| l = +3; -3 \le m \le +3; s_z = \pm\tfrac{1}{2} \right\rangle$. In general, $2\varphi = 4j = 4\left(l + \tfrac{1}{2}\right)$ is the number of distinct states for any given *l*. So if the various fill factors $\nu = n / 2\left(l + \tfrac{1}{2}\right)$ of the DWY monopole were to be observed somewhere in the natural world, and if the person observing these fill factors was to give the name "quasiparticles" to the charges that exhibit these quantized fractional fill factors, then the quasiparticles with the fill factor $\nu = n / 2\left(l + \tfrac{1}{2}\right)$ for any given *l* would be expected to come in $2\varphi = 4\left(l + \tfrac{1}{2}\right)$ "varieties," i.e., a number of varieties which is equal to twice the denominator, where by "varieties," we mean distinct states according to fermion Exclusion Principles.

So now the question becomes under what conditions in the natural world, one might be able to observe charges with the fill factors $\nu = n / 2\left(l + \tfrac{1}{2}\right)$. We know that *if* the atomic quantum numbers l, m, s_z, j_z are topologically rooted in OET, then $\nu = n / 2\left(l + \tfrac{1}{2}\right)$ are the fill factors of the DWY monopoles. But we also know that with the exception of the odd denominator 2, this is precisely the set of fill factors that is observed in the FQHE, where these fractionalized charges are in fact referred to as quasiparticles. So the question is whether these DWY monopole quantized fractionalized charges are in fact synonymous with the FQHE quantized fractionalized charges. Based on the foregoing evidence, one might well *suspect* that these are one and the same, but it is not possible at this juncture to do more than make the *inference* from all of the evidence discussed above that these *might* be one and the same. There are good *arguments* which can be made in support of this inference and these will be elaborated in the next section. And then in section 9 we shall endeavor to theoretically prove this inference by showing how a low-temperature duality symmetry can in fact be broken at higher temperatures consistently with the well-established non-observation of magnetic monopoles in daily experience.

But the most important direct *proof or disproof* of this inference, would be found not in in theoretical argumentation, but in the observation or non-observation of FQHE quasiparticles which for odd numbered denominators in the fill factor $\nu = n / 2\left(l + \tfrac{1}{2}\right)$ which exhibit properties reminiscent of electrons with orbital angular momentum *l*, including exhibiting $2\varphi = 4\left(l + \tfrac{1}{2}\right)$ distinct fermion exclusionary states for any given *l* hence fractional denominator $\varphi = 2\left(l + \tfrac{1}{2}\right)$. If such a correlation was to be observed between the FQHE denominators and the orbital quantum number *l* thus the atomic s, p, d, f etc. shell states, not only would this confirm that the odd-denominator DWY states are one and the same as the odd-denominator FQHE states and thus confirm the existence of DWY $U(1)_{em}$ magnetic monopoles near 0K in certain superconductors subjected to large perpendicular magnetic fields and so evidence – for the first

time since Maxwell's day – the existence of $U(1)_{em}$ magnetic monopoles and electric / magnetic duality in the physical environment of temperatures near 0K. Because such an observed correlation would have also been based upon having connected the topological OET quantum numbers l', m', s_z', j_z' to the atomic angular momentum quantum numbers l, m, s_z, j_z, this would confirm that atomic structure, and by implication nuclear structure, can be explained entirely on the basis of the topological OET characteristics of the three-dimensional physical space of four-dimensional spacetime. This in turn would affirmatively answer Misner, Thorne and Wheeler [18] by showing that orientation-entanglement analysis does have a very "detectable difference in the physics" that shows up at the root of atomic and nuclear structure, and it would compellingly validate Ross' belief [28] that the OE relationships have a tremendously important "topological role in physics." And at the most fundamental level of theoretical physics, it would extend the reach of geometrodynamic principles into the atoms and nuclei at the heart of the material universe and advance the view that one day it will become possible to fully explain the entirety of the natural world on the basis of no more and no less than spacetime geometry itself, and the topological OET features which are inherent to this geometry.

These are the reasons why it would be a valuable and worthwhile to conduct experiments to closely observe the odd-denominator FQHE quasiparticles to seek correlations to atomic angular momentum electronic shell structure. Now, let us examine more closely, some of the theoretical inferences which make it very plausible that such an experiment would yield positive results to confirm all of these connections.

8. Three Theoretical Inferences from the Evidence: Correlations between Topology and Atomic Structure, Correlations between Topological Freedom and Temperature, and the Identification of DWY Monopoles with FQHE; and Principles of Topological Least Action

As developed in the previous two sections the topological quantum numbers l', m', s_z', j_z' which summarize what happens under OET and the disentangling of various entangled states map directly on a one-to-one basis to the angular momentum quantum numbers l, m, s_z, j_z seen in the electronic shell structure of atoms. Also, the azimuth φ used to define the former topological numbers has exactly the same physical meaning as the azimuth φ about which the latter angular momentum is defined. Therefore, the first theoretical inference we shall make, or the first evidence-informed hypothesis if one prefers, is the following:

> Inference 1: Orientation-Entanglement-Twist (OET) has direct, physical relevance to and indeed is the topological basis for why the angular momentum quantum numbers l, m, s_z, j_z have the values and interrelationships that they do have. In other words, the one-to-one mapping between the topologically-evidenced l', m', s_z', j_z' and the experimentally-evidenced l, m, s_z, j_z is not just a happenstance concurrence. Rather, the experimental latter is a direct and immediate consequence of the topological former.

As we have already done in (7.2) in the last section, once it has been made, this inference / hypothesis allows us to remove the "primes" from all topological quantum numbers and now regard these as the angular momentum quantum numbers of electrons in atomic shells and of protons and neutrons in nuclear shells.

Although this mapping between l', m', s_z', j_z' and l, m, s_z, j_z was found because of the need to topologically-distinguish angular orientations differing from one another by 2π sans entanglement and 4π accounting for entanglement to provide a necessary foundation for the fractional numerators $\not{A} = n$ and denominators $\not{\varphi}$ in the DWY monopoles to be possible physical observables, *it is critical to understand that this topological-to-angular momentum mapping is totally and entirely distinct from the DWY monopoles*. It would have been entirely possible, without even a whisper about DWY monopoles, to have simply taken up Ross' call in [28] to "further work on the OE relations themselves and their topological role in physics," gone back to Figure 1(a) which is Figure 31.6 of MTW's [18], show that Figure 1(c) topologically deforms from and so belongs to the same homotopy group as Figure 1(a), and then proceed to analyze OET as we did in sections 5 and 6. Then, having each quantized integer represent one *double* winding rotation or *double* twist, and using single offsetting twists (net helicity zero) as in Figures 6 to represent an electron, we would have assigned the electron a reduced azimuth $\not{\varphi} = 1$ to represent the topological dimensional transmutation which occurs after a $\pm 2\pi$ rotation about the x axis followed by disentangling the ribbons transfers the *single rotation about x* into *a single twist along z*, as reviewed after Figures 6. Then, we would have rotated the N→S vector by $l' = 1, 2, 3, 4...$ double rotations, and after any specific double winding, disentangled the N and S ribbons using various combinations of the N and S disentangling operations. Doing so, we would have found that the total number of double twists $j_z = m + s_z$ and the combinations of operations which led to those double twists could be characterized for $l' = 0$ double rotations by $s' \equiv \left| l' = 0; m' = 0; s_z' = \pm\tfrac{1}{2} \right\rangle$; for $l' = +1$ double rotations by $p' \equiv \left| l' = +1; -1 \le m' \le +1; s_z' = \pm\tfrac{1}{2} \right\rangle$; for $l' = +2$ double rotations by $d' \equiv \left| l' = +2; -2 \le m' \le +2; s_z' = \pm\tfrac{1}{2} \right\rangle$, and for $l' = +3$ by $f' \equiv \left| l' = +3; -3 \le m' \le +3; s_z' = \pm\tfrac{1}{2} \right\rangle$, etc. And we then would have seen that these l', m', s_z', j_z' topological quantum numbers map precisely with the atomic l, m, s_z, j_z angular momentum quantum numbers and would have been asking as we are presently whether this is a coincidence or a real physical connection.

So because this mapping between l', m', s_z', j_z' and l, m, s_z, j_z is entirely independent and distinct from anything having to do with DWY monopoles or FQHE, and is simply the consequence of "closely studying the OE relations themselves and their topological role in physics" per [28], we can evaluate this first theoretical inference that l', m', s_z', j_z' and l, m, s_z, j_z are genuinely physically connected without any thought or reference whatsoever to DWY monopoles or FQHE. So, setting aside any thought of DWY monopoles or FQHE, and thinking solely about atomic (and if we wish, nuclear) structure, is this a plausible inference?

At present, the l, m, s_z, j_z angular momentum quantum numbers (taken also with the principal number *n* not yet examined, which will make its topological appearance in section 9)

with $l = 0,1,2,3...$ and $-l \le m \le +l$ and $s_z = \pm\frac{1}{2}$ and $j_z = m + s_z$ (and $l < n$) are simply a set of rules that are required to enforce Exclusion of electronic states in atomic shells. But these rules are heavily validated empirically by the structure of the Periodic Table of the Elements itself. If, as a theoretical physicist one adopts the creditable view which has been taken by the likes of Einstein [21] and Weyl [22], [23], [24] and Wheeler [19], [20] that geometry and by extension topology should eventually be understood to furnish the basis for explaining all of physical reality, then on broad, fundamental geometrodynamic and topological principles, *there would have to be some geometric or topological explanation* for the precise l, m, s_z, j_z observed in atomic structure, even if that explanation is not presently known. If one further credits the study of orientation-entanglement in [18] as being a primary approach to understand the topology of the physical three-dimensional space of spacetime, and then finds as we have done here that this topology produces a set of l', m', s_z', j_z' which map precisely to l, m, s_z, j_z, the inference that these are connected to one another appears almost inescapable. For, if this mapping between l', m', s_z', j_z' of OET and l, m, s_z, j_z of angular momentum each using the same azimuth φ is not the reason why the l, m, s_z, j_z are empirically observed to be they are, then what is the reason?

So if this inference is correct and the l', m', s_z', j_z' of OET are in fact the foundation for the l, m, s_z, j_z observed in atoms, what things can we learn from the topology which might better inform our understanding of electrons in atomic shells, and thereafter allow us to account for DWY monopoles and FQHE? To consider his, let us return to Figures 6, and more carefully walk through the process already somewhat reviewed in section 5, of rotating the N→S vector through $l' = +1$ double winding about the z axis. It is advisable for the reader to have constructed this bar and ribbon apparatus and use it to confirm what is about to be discussed.

The first thing to notice is that this operation makes use of all three of the x, y, z spatial dimensions. The ribbons are aligned parallel with the z axis, and the rotation $\varphi \to \varphi + 4\pi$ takes place through the x-y plane. Now as we started to discuss following Figure 2, if we think of Figures 6 as if the N→S bar (OET "object") was the seat of a child's swing and the ribbons were a pair of "chains" (OET "threads") which have offsetting twists (total twist conservation) and which hang from the top bar which is the "fulcrum" of the swing (OET "environment"), then after we have done this rotation, if we now "let go" and give the swing seat freedom to rotate in the x-y plane, then under the force of "gravity," the swing seat will start to rotate about the z axis oppositely to how it was originally wound, it will go somewhat past its original configuration due to its rotational inertia, and it will thereafter slow to a stop. Then it will reverse direction, and do this through a few pendulous cycles until the damping effect of friction has drawn off all energy and the swing seat has returned to its initial state of Figures 6, which initial state we may think of energetically as a "ground state."

But as before, let us say we do *not* let the N→S vector rotate any more. Instead, let us remove the degree of freedom along the y axis and lock everything into the two-dimensional x-z plane, not allowing the N→S vector to rotate back. That is, let us now "freeze," pun very intended, the OET "object" in relation to its "environment." What will happen then? As discussed following Figures 3, if we do permit a small $y = \pm\varepsilon$ ribbon incursion into the y axis simply to allow the ribbons to be moved around the N→S bar which is now immobilized from

rotating relative to the environment bar, then – by virtue of having done a $+4\pi$ rather than a $+2\pi$ rotation – we can still disentangle the ribbons from one another and end up back at a disentangled state. But, depending upon what ribbon operations we use to disentangle, if we do not balance the number of operations using the N ribbon with the number of operations using the S ribbon, then we will introduce some additional double twists which were not there at the start. And if we regard a less-twisted ribbon as representing a lower energy state than a more-twisted ribbon, then we will not have returned fully to the ground state. For, rather than getting back to $s' \equiv \left| l' = 0; m' = 0; s'_z = \pm\tfrac{1}{2} \right\rangle$ where each ribbon had no twists other than the single twist used to represent spin ½, we will end up with an $m' = \pm 1$ double twist. And if the original twist $s'_z = \pm\tfrac{1}{2}$ aligns with the $m' = \pm 1$, then we will have $j'_z = m' + s'_z = \pm\tfrac{3}{2}$ totaling three single twists, rather than $j'_z = \pm\tfrac{1}{2}$ and one single twist that we started out with. And this state with $p' \equiv \left| l' = +1; -1 \le m' \le +1; s'_z = \pm\tfrac{1}{2} \right\rangle$, when $j'_z = \pm\tfrac{3}{2}$ with a pair of three offsetting twists, will be energetically elevated from the ground state $s' \equiv \left| l' = 0; m' = 0; s'_z = \pm\tfrac{1}{2} \right\rangle$ with its pair of single offsetting twists.

What we are able to deduce from all of this, are an apparent set of topological "least action" or "least energy" or "topological geodesic" dynamical principles which it is helpful to keep in mind as we think about how to relate the OET topology to temperature which is a driving variable in FQHE, and thus think about temperature topologically:

> Topological Least Action Dynamical Principles: If an N→S bar has been rotated from its ground state through an even number of rotations in the x-y plane and had its ribbons become entangled with one another as a result, and if nothing prevents the N→S bar from rotating back, then this bar will naturally rotate back to its ground state with no entanglement. However, if there is a force which prevents the N→S bar from disentangling by a counter-rotation through the x-y plane, then that N→S bar will still naturally evolve to become disentangled by a passing the ribbons around the N→S bar with a small but finite $y = \pm\varepsilon$ incursion into the y axis, so long as there is nothing also preventing this small $y = \pm\varepsilon$ incursion. Further, if this small $y = \pm\varepsilon$ incursion is permitted, this disentangling through ribbon passage around the N→S bar will occur even if the net result of this disentangling is that more twists are created. Therefore, disentangled ribbons generally define a lower energy state than entangled ribbons, irrespective of twists. Finally, if $\varepsilon = 0$ and no incursion is permitted into the y axis whatsoever, then the bars and ribbons will become completely frozen as is, with no entangling or disentangling occurring at all.

How do we know this is true? *If* our first theoretical inference is correct that l, m, s_z, j_z emanate from the topological l', m', s'_z, j'_z, *then* we know that this is true from the l, m, s_z, j_z structure of the atoms themselves in the periodic table. Why? If the electrons in the atoms preferred to remain in states of topological entanglement, i.e., if states of greater entanglement had lower energy than states with additional twists, then we would not observe such things as p

shell electrons with the states $p \equiv \left| l = +1; -1 \le m \le +1; s_z = \pm\tfrac{1}{2} \right\rangle$ or d shell electrons with $d \equiv \left| l = +2; -2 \le m \le +2; s_z = \pm\tfrac{1}{2} \right\rangle$ with $j_z = m + s_z$, etc., given that when the primes are returned to these quantum numbers, we have $j_z' = m' + s_z'$ representing the number of offsetting double twists at the end of the disentangling process. The fact that we *do observe* $j_z = \pm\tfrac{3}{2}$ corresponding to 3 twists in some of the p electrons, and *do observe* $j_z = \pm\tfrac{3}{2}, \pm\tfrac{5}{2}$ corresponding to 3 or 5 twists in some of the d electrons, and *do observe* $j_z = \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \pm\tfrac{7}{2}$ corresponding to 3, 5 or 7 twists in some of the f electrons, all of which are more twists than the *single twist* $j_z = s_z = \pm\tfrac{1}{2}$ of the $s \equiv \left| l = 0; m = 0; s_z = \pm\tfrac{1}{2} \right\rangle$ individual Figure 6 electrons that we started with, tells us that to maintain exclusion, nature will disentangle the electrons and put them into states with extra twists instead of leaving them entangled, *if it can*, i.e., if it is allowed a small $y = \pm\varepsilon$ incursion into the y axis to disentangle ribbons. Because nature will always migrate to the lowest permitted energy state when it can, the fact that we do observe $j_z = \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \pm\tfrac{7}{2}$ tells us that disentangled states have a lower energy than states which remain entangled. States such as those with $j_z = \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \pm\tfrac{7}{2}$ are the elevated energy states that fermions are required to enter into when they assemble into atomic systems, in order to satisfy the requirement that they maintain Exclusion. And in all of this, physical principles of least action and lowest energy are seen to correspond to topological principles of least entanglement and least twist and a priority for least entanglement over least twist which means generally that less energy is needed to maintain twist than to maintain entanglement. (One can suppose that there are some cases where nature might prefer a small entanglement over a very large number of twists, but for the cases studied here that does not appear to have presented itself.)

With this introduction to topological dynamics, let us now talk about what happens when an electron is added to an orbital shell of an atom, or conversely, when it is removed from a shell. Energetically, of course, some amount of ionization energy needs to be provided for the electron to join the atom, and these required ionization energies have been well-catalogued empirically, see, e.g., the web references [33] in graphical form and [34] in tabular form. But our interest in this discussion is to understand the *topological* processes which occur when an electron joins or leaves an atom. And to use a concrete example, let us suppose that we are starting off with a fluorine F nucleus that has nine (9) protons, but the orbital structure only has eight (8) electrons, so this atom is positively ionized and needs to secure a ninth electron to achieve neutrality. The reason we choose F, is that this is the first element for which at least one of the electrons *must* have a total angular momentum $j_z = \pm\tfrac{3}{2}$, which corresponds to three-halves of a topological double twist $j_z' = \pm\tfrac{3}{2}$, which is a conserved observable, and which is larger than the $j_z = s_z = \pm\tfrac{1}{2}$ of a free electron and in particular has more one more double twist than the electrons in Figure 6.

So, introducing the principal quantum number n with $l < n$ as usual, the first two electrons which bring us through H and He have $1s \equiv \left| n = 1, l = 0, m = 0, s_z = \pm\tfrac{1}{2}, j_z = \pm\tfrac{1}{2} \right\rangle$. The next two electrons which pass Li and Be if they keep to the lowest energy states have $2s \equiv \left| n = 2, l = 0, m = 0, s_z = \pm\tfrac{1}{2}, j_z = \pm\tfrac{1}{2} \right\rangle$. The next two electrons for B and C with

$2p_0 \equiv \left| n=2, l=+1, m=0, s_z=\pm\frac{1}{2}, j_z=\pm\frac{1}{2} \right\rangle$ for the first time have a non-zero *l*, which means topologically, their N→S bar (whatever that corresponds to in physical reality) has been wound through $l'=+1$ double rotations. The 0 subscript in p_0 designates $m=0$. But then, once the two bars have been given a frozen alignment, the disentangling would take place using a balanced $\{N,S\}$ operation, yielding $m'=0$ new twists, so that the total number of double twists remains at $j_z'=\pm\frac{1}{2}$. For N and O we must have $2p_{1-} \equiv \left| n=2, l=+1, m=\pm 1, s_z=\mp\frac{1}{2}, j_z=\pm\frac{1}{2} \right\rangle$ which for the first time introduces a non-zero $m=\pm 1$, (the 1 subscript in $2p_{1-}$). But if the spin $s_z=\mp\frac{1}{2}$ is aligned opposite to *m* (the – subscript in $2p_{1-}$), we will still have $j_z=\pm\frac{1}{2}$, which still corresponds to $j_z'=\pm\frac{1}{2}$ of a double twist. It is only for F that for the first time, we must have at least one $2p_{1+} \equiv \left| n=2, l=+1, m=\pm 1, s_z=\pm\frac{1}{2}, j_z=\pm\frac{3}{2} \right\rangle$ electron, and this electron will have the topology of $j_z'=\pm\frac{3}{2}$ double twists, or three single twists.

So, when this $2p_{1+}$ electron with topological $j_z'=\pm\frac{3}{2}$ is added to create a neutral F atom, what happens, *topologically*? Energetically, we can refer to [34] and find out that 17.4228 eV of energy is provided to a free 1*s* electron topologically represented by Figures 6, which energy enables the electron to elevate into $2p_{1+}$ and join the F. If we had a Ne nucleus with ten (10) protons but only nine (9) electrons thus positively ionized, and wanted a tenth electron to join, then another 1*s* electron would need to be given an even larger 21.5645 eV of energy and turned into the second $2p_{1+}$ electron in Ne, also with the $j_z'=\pm\frac{3}{2}$ topology of three full twists. But the question we are driving at is topological: what do these ionization energies purchase, *topologically*? What they purchase is a $\varphi \to \varphi + 4\pi$ rotation of the free 1*s* electron through a $l'=+1$ double winding. Once this double winding has occurred, the electron is suited to join the atom because it has sufficient l'. (This is the case for the earlier $l'=+1$ electrons in Li, Be, B, C, N, O also, but these still could maintain $j_z'=\pm\frac{1}{2}$ and so did not need quite as much energy to add a full double twist. For example, the new electron in N needs 14.5341 eV and that in O needs 13.6181 eV.)

Now, however, for the electron to join the atom after the $\varphi \to \varphi + 4\pi$ rotation has been purchased with some ionization energy, any further rotation must cease, and the N→S "object" bar needs to become fixed relative to its "environment." After all, this electron is now joining an atom, and as such, it is no longer free, but will have to fall in line as part of a group of electrons in orbital shells and so will have its freedom to reorient with respect to its environment, i.e., the rest of the atom, wholly or partially removed. *This is a very important point, because this tells us for an electron to join an atom, it must surrender a degree of freedom, at least insofar as its N→S "vector bar" is no longer able to fully rotate through the x-y plane.* Finally, because entanglement is still an elevated energy state which will be lowered by topological least action if possible, so long as there remains some freedom for a small $y=\pm\varepsilon$ incursion along the y axis, the ribbons can still be disentangled, but with the result that there may be some additional twists that were not present when the electron was a free electron. So, some of this ionization energy also purchases the latitude for the $y=\pm\varepsilon$ spacing needed for disentangling. And in the end, after everything is disentangled, some of this energy will be stored in the extra topological twists

that were not present when the $2p_{1-} \equiv \left| n=2, l=+1, m=\pm 1, s_z=\mp\tfrac{1}{2}, j_z=\pm\tfrac{1}{2} \right\rangle$ electrons were free $\left| l=0, m=0, s_z=\pm\tfrac{1}{2}, j_z=\pm\tfrac{1}{2} \right\rangle$ electrons.

With this background, let us now start to think again about low-temperature FQHE where the electrons are tightly constrained to two dimensions. As a result of the discussion just concluded, we understand that in *any* temperature environment, *simply by being in the shells of atoms*, electrons are already somewhat topologically constrained to two dimensions, because their N→S vector bars are not permitted to make full rotations in the x-y plane. But, in this still-unfrozen (i.e., not yet near 0K state), there is still enough freedom in the y dimension schematically represented as $y=\pm\varepsilon$ for the ribbons to disentangle themselves from the higher energy state in which they are entangled with one another, into the lower energy states where they are no longer entangled but may gain some extra topological twists which they did not have when the electron was free. It may well be and very likely is the case even when the electrons *are* in the shells of atoms, that *some limited rotation* is still permitted through the x-y axis. But this permitted albeit limited rotation – to whatever degree it is permitted – is insufficient for the electron to disentangle by undergoing a $\varphi \to \varphi - 4\pi$ rotation to undo the original $\varphi \to \varphi + 4\pi$ rotation like the child's swing will do if one simply lets go. So having the ribbons / threads disentangle is the next best choice, and this puts the electron into its observed exclusionary quantum states as has been reviewed.

So now, let's discuss temperature and heat generally, with an eye firmly pealed on FQHE. Undoubtedly, the removal of heat from any object removes energy and removes some freedom to move about in the physical three-dimensional space of spacetime. Through the Boltzmann relationship $E = kT$ one can obtain the energy equivalent *E* of a given temperature *T*. A gas which is cooled has less energy and reduces the physical agitation of its atoms and molecules in the x-y-z physical space. When the gas is cooled enough to phase transition to liquid, there is a further reduction of freedom in the physical space. When the liquid turns solid, various crystalline or lattice or similar structures are formed and there is even less spatial freedom. And when all heat is removed and the temperature approaches absolute zero, the spatial freedom is at an absolute minimum and the only energy left in an electron is the Fermi energy of the state it occupies (and of course its rest mass). And for a superconductor exhibiting FQHE, in this state near 0K with large perpendicular magnetic fields applied, the electrons or quasiparticles are understood to be tightly restricted to two dimensions. But what is really happening here, topologically? More directly to the point, we have articulated the theoretical geometrodynamic view that all of nature ought to be understood based on geometry and the topological properties of geometry. So: *what is the correct topological understanding of heat and of temperature which measures heat?*

Clearly, the very act of joining an atomic shell removes from an electron as represented in Figures 6, *some* topological freedom to move about in the y-dimension, but leaves *enough residual freedom* in the y-dimension at least for the ribbons or "threads" which connect the electron to its environment to become disentangled, and presumably for some limited rotation in the x-y plane, but just not enough for a full rotation which would allow a $\varphi \to \varphi + 4\pi l'$ rotation to simply reverse itself via $\varphi \to \varphi - 4\pi l'$. We have spoken abstractly of this residual freedom in terms of a $y=\pm\varepsilon$ incursion into the y-axis, which is certainly needed for ribbons to disentangle.

But now let's progress this discussion further. Is this length ε which we are schematically using to represent freedom in the y dimension a fixed number, or can it vary? And if it can vary, in relation to what *physical observables* might ε vary? Given all that we have just reviewed, heat and temperature would certainly be good candidates. So let us now introduce our second inference or informed hypothesis:

> Inference 2: The magnitude of topological freedom which an electron has to operate in the y dimension i.e., in all three space dimensions, correlates not only with whether it is free or in an atomic shell, but also with heat and / or temperature.

How might this work? Let's start with very high temperatures, where many elements are in plasma phase, and where the defining feature of the plasma is rampant ionization. What is plasma? It is a phase of matter in which electrons are freed from their nuclei and float around in something of a soup and are exceptionally responsive to electromagnetic forces. And, topologically, what is ionization? As just discussed, when we add an electron to an atom which changes the atom's ionization, we supply enough ionization energy to rotate through $\varphi \to \varphi + 4\pi l'$, hitch the electron up to the rest of the atom by removing enough of its x-y rotational freedom to stop it from unwinding via $\varphi \to \varphi - 4\pi l'$, and then disentangle the ribbons instead. So reversing course, how might we now *remove* an electron from a neutral atom and create an ionized state, and so do with enough atoms to create a plasma with a high degree of ionization? Simply, provide enough heat to restore the freedom of many electrons to rotate through the x-y plane fully without restriction so they can uncouple from the atoms and become the freely-flowing electrons of a plasma. So in the highest temperature extreme of a plasma, the schematic ε in $y = \pm\varepsilon$ is not a small length, it is a large length. Specifically, it is large enough so that there is complete x-y-z freedom and the electrons can topologically rotate in the x-y plane at will and so ionize in and out of atoms equally at will.

At the other extreme is 0K. Between plasma phase and solids near 0K, the schematic ε in $y = \pm\varepsilon$ goes from offering no y-axis restriction, i.e. complete 3-dimensional freedom to electrons in the plasma phase, to some y-axis restriction sufficient to keep the electrons bound to the atoms in a gas or liquid or solid state with ε growing smaller and smaller as the temperature decreases, to no y-axis freedom at all at 0K. So stopping short of trying to quantify this for the moment, we certainly expect qualitatively that a decrease in temperature and / or heat will correlate with a decrease in the $y = \pm\varepsilon$ freedom afforded to an electron and that the transition from ultra-high to ultra-low temperatures is topologically understood as a transition from electrons with unrestricted x-y-z freedom to some restriction along the y axis to rigid, frozen two-dimensional constraint that entirely removes all access to the y-dimension. So what happens at absolute zero? Now $\varepsilon = 0$. There is no freedom whatsoever for *anything* topological to utilize the y dimension. There is no permitted rotation whatsoever of the N→S vector in relation to the environment i.e. relative to the rest of the atom and its nuclei. And very importantly, there is no movement whatsoever of the ribbons / threads through the y-dimension. If something is entangled it stays entangled. If something is not tangled but is twisted, it stays not tangled but twisted. Whatever it is, it continues to be. Everything is – to use a very apt descriptor – entirely frozen at 0K. To the question of why there is an absolute zero of

temperature in the first place, the topological OET answer is this: absolute zero is reached when an OET object is given no latitude at all to rotate into the y dimension, and the threads which connect that object to its environment also have no freedom to move at all into the y dimension. It is a complete and total freezing of an electron which has some temperature-correlated three-dimensional freedom at higher temperatures, to an electron entirely topologically restrained to two dimensions at 0K. So throughout the entire transition from high to low temperature during which the y-freedom goes from being unrestricted to being more and more restricted, the final transition is to $\varepsilon = 0$ at 0K. Now the ribbons can no longer move at all into the y dimension. Now, there is a *qualitative* change that has occurred from any other state where $\varepsilon > 0$ because even the ribbons / threads cannot move. Now the entire OET for each electron is frozen into the two dimensional x-z plane.

If we think of this total removal of freedom in the y-dimension as a very tight "tuning" of the electron's topological properties which removes extraneous topological activity in the y dimension – sort of like tightly tuning a bobsled to the tracks on which it slides to minimize extraneous jostling movement normal to the tracks and ensure a smoother ride with less resistance – then we can think about the superconductivity observed in certain metals at low temperature as the result of fine tuning the electron topology tightly into two dimensions: By fine tuning the electron topology in relation to its environment by removing the y-dimension freedom, so too are removed the frictional forces and therefore resistance that is observed absent this fine tuning. If the electrons are the bobsled "object" and the superconducting metal is the tracks "environment" and the resistance arises from the OET relationship between the object and its environment, then near 0K the electron objects are tightly tuned to the their environmental tracks by the complete removal of the y freedom and thus the resistance of jostling between object and environment is also removed.

All of this provides a qualitative, topological understand of superconductivity. So now, what about DWY monopoles and FQHE? And how do we make this all quantitative? We have demonstrated that *if* DWY monopoles exist and their implied electric / magnetic symmetry exists under some physical conditions, *then* their electric and magnetic charges will be quantized and fractionalized. We have also demonstrated that *if* these DWY charges are all to be in disentangled states, *then* the fractional denominators for these charges will be odd integers, see, e.g., (7.2). In sum: *if* the DWY monopoles exist near 0K in disentangled states only, then quantized fractionalized charges with only odd integer denominators will be observed near 0K. And, if DWY monopoles exist and the only permitted *entangled* state is $\varphi = 2$ (which is entangled because this state is a $\varphi = 2\pi \rightarrow \varphi = 4\pi$ rotation of the electron through 2π which state cannot be OE disentangled), *then* the quantized fractionalized charges will have only odd denominators, with the exception of the even denominator 2. Finally, it is empirically-established that FQHE is a near-0K quantized, fractionalized charge phenomenon which does exhibit only odd integer denominators, with the exception of the exclusive even denominator 2.

All of this evidence leads us to make a third evidence-based inference which we shall pursue quantitatively in the next section:

> Inference 3: DWY monopoles and electric / magnetic symmetry *do exist in nature* under some physical conditions, and the physical conditions under which they

> exist is in certain materials near 0K. Thus, near 0K, one of the symmetries restored is the electric-magnetic symmetry which gives rise to the formation of DWY monopoles with quantized fractionalized odd-denominator charges as found in (7.2). The electrons underlying these DWY monopoles must be topologically disentangled, with the exception that the only permitted entangled state is $\varphi = 2$, and therefore, the observed charges will have the only odd denominators, with the exception of the even denominator 2. Because this precisely maps to both the conditions under which FQHE is observed and to the quantitative features of FQHE, we infer that these are not separate and distinctive phenomena: *rather, they are one and the same phenomenon*.

As a consequence of this inference, what we ordinarily think of as fractionally-charged quasiparticles, are instead seen to be ordinary electrons with a quantized fractional charge which is $\nu = n/2\left(l+\frac{1}{2}\right) = n/2j$ with the orbital quantum number $l = 0,1,2,3...$ based on (7.2). In other words, $\nu = n/3$ quasiparticles are electrons in the $l = 1$ orbital angular momentum state which were topologically rotated through $l' = 1$ double windings prior to joining an atomic shell, which are now frozen into their orientation and entanglement with all y-freedom removed, $\varepsilon = 0$. And $\nu = n/5$ quasiparticles are electrons in the $l = 2$ orbital state following $l' = 2$ double windings, frozen to $\varepsilon = 0$, etc.

The challenge which we now take up in the next section, is to understand how it is that we can pass between a state near 0K where DWY monopoles do exist and so the gauge theory inexorably leads to quantized fractionalized charges, to the observed states at all other temperatures where there have never been monopoles observed and charges are quantized but not fractionalized (other than fractionally-charged quarks, of course, which come from $Q = Y/2 + I_3$ and not FQHE). One might picture that topologically, the "pinching" of the ribbons when $\varepsilon = 0$ at 0K with all topological freedom in the y dimension completely removed somehow causes the electric and magnetic fields to start to do strange things so that the magnetic fields suddenly exhibit net non-zero surface fluxes $\oiint F = \mu$ and the charges suddenly become quantized independently of $Q = Y/2 + I_3$ and also become fractionalized. And the OET topology provides us a backdrop, because we know we are looking at a situation in which the "N→S object" bars and ribbons at the bottom (south side) of Figures 6 are completely fixed and permitted no independent freedom in relation to the "environment" bars at the top (north side) of Figures 6. But to fully work this through, the question of how a transition might occur between no monopoles and no fractionalization in most physical situations, to there being monopoles and FQHE fractionalization in certain materials near 0K, is best approached by returning to $U(1)_{em}$ gauge theory, and the DWY monopoles already studied in sections 1 through 3.

9. How DWY Monopoles and Charge Quantization and the FQHE near 0K, Symmetry Break into a Thermal Residue and Energy Quantization and Atomic Shell Structure at Larger Temperatures

Let us now return to section 1 and retrace our development very carefully to see if we can garner a better topological understanding of how the DWY monopoles might come into being

including a restoration of electric / magnetic symmetry near 0K. Topologically, we now understand that near 0K the electrons become topologically confined to the x-z plane without any freedom of incursion whatsoever into the y dimension and the south "object" bars in Figures 6 become completely immobilized in relation to the north "environment" bars.

We start with equation (1.3), $e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi$, because *if* equation (1.3) is true, *then* restating (3.2), it is also true that the electric charge *e* and the magnetic charge μ will be related to one another by $e\mu=2\pi(n/l)$ where *n* and *l* are both integers. This would mean three things all at once: if (1.3) is true, then 1) there is an electric / magnetic duality symmetry because $e\mu=2\pi(n/l)$ is invariant under $e\leftrightarrow\mu$ interchange; 2) these charges are quantized because *n* is an integer; and 3) these charges are fractionalized because *l* is also an integer. This of course is the Dirac Quantization *and Fractionalization* Condition (DQFC). In section 1 we segregated the study of the DQC and so developed $e\mu=2\pi n$ in (1.6) without the fractionalization to keep section 1 limited only to what is already well known and widely accepted. But in reality, beyond this pedagogically-motivated segregation of the DQC from the full DQFC, there is nothing that prevents us from asserting that *if* (1.3) is true, *then* $e\mu=2\pi(n/l)$ is true. Equation (1.3) is solved by (1.4), and (1.4) is in turn fully solved for *all* states in (3.1) by $e\mu=2\pi(n/l)$. Logically, the electric / magnetic symmetry and the quantization and fractionalization all emerge together. The quantization can only be separated from the fractionalization by what is effectively restricting consideration to $l=1$. Later, we connected these quantum numbers $n/l=\Lambda/\varphi$ to the topology of physical x-y-z space and showed for example that if we restrict ourselves to only disentangled electronic states there would only be odd denominators. But it is the DQFC $e\mu=2\pi(n/l)$ of (3.2) which is the raw logical conclusion flowing from (1.3).

The reason this is important is because in everyday experience, we do *not* observe electric magnetic symmetry, and we do *not* observe charge fractionalization (pace quark charges). We only observe charge quantization and that is for a different reason, namely because of the electric charge generator $Q=Y/2+I_3$ emerging from electroweak symmetry breaking. While $e\mu=2\pi(n/l)$ may be true under some specialized set of physical conditions, and while we have inferred and are presently working to develop the view that this is true near 0K and is observed as the FQHE, we can state with certainty that $e\mu=2\pi(n/l)$ is *not true in general*, because it is not observed in general. If we look only at the logical implication relations $A\rightarrow B$, then what we found earlier is $\left[e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi\right]\rightarrow\left[e\mu=2\pi(n/l)\right]$. The contrapositive $\sim B\rightarrow\sim A$ of this is $\sim\left[e\mu=2\pi(n/l)\right]\rightarrow\sim\left[e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi\right]$. In all known observations to date, certainly in observations away from 0K, we have found that $e\mu=2\pi(n/l)$ is *not observed*, i.e. $\sim\left[e\mu=2\pi(n/l)\right]$, because electric / magnetic duality is and charge fractionalization is not generally observed. So the logical conclusion from contrapositive logic is that generally, $\sim\left[e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi\right]$. That is, equation (1.3) is *not true* as a general rule because there is neither observed electric / magnetic duality nor any fractionalization

as a general rule. And if the inference is correct that the DQFC is only manifest near 0K, then $e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi$ can also only be true near 0K. So there must be something that is routinely being overlooked not only by this author but by everyone else in the DWY monopole derivation, before we even get to (1.3). Thus, we need to return to section 1 and scour everything that gets us to (1.3) in light of all we have learned here since (1.3), to find out what is being missed.

Specifically, if the third inference in the last section is correct that the DWY monopoles are physically real near 0K and responsible for FQHE, then $e^{-i\Lambda}de^{i\Lambda}/ie=(\mu/2\pi)d\varphi$ is also physically true, *but only near 0K*. So as we carefully turn over how we got to (1.3), we need to look for anything that is assumed to be generally true, when in fact it is only true near 0K. So let's start to scour.

We know that $F=dA$ is a generally true relationship, because its consequences are observed throughout electrodynamics and in quantum physics, and that dd=0 is a mathematical identity of differential forms geometry which is also true in general. We also know that $A\rightarrow A'=A+e^{-i\Lambda}de^{i\Lambda}/ie$ in (1.1) is a correct and generally-true statement of how a gauge field transforms, due to the gauge symmetry which is manifest throughout nature generally. We know that if a magnetic charge exists it will be defined by $\mu=\oiint F$ and that using $F=(\mu/4\pi)d\cos\theta d\varphi$ in this surface integral properly reproduces $\mu=\oiint F$ because $\oiint(\mu/4\pi)d\cos\theta d\varphi=(\mu/4\pi)\int_0^{\pi}d\cos\theta\int_0^{2\pi}d\varphi$ which evaluates upon definite integration to $(\mu/4\pi)\cos\theta\big|_0^{\pi}\varphi\big|_0^{2\pi}=\mu$. Further, because $F=dA$ and dd=0 we know that $F=dA=(\mu/4\pi)d\cos\theta d\varphi=(\mu/4\pi)d(\cos\theta-K)d\varphi$ will be correctly reproduced with any constant K in $A=(\mu/4\pi)(\cos\theta-K)d\varphi$. Of course, these relationships containing μ presuppose a magnetic charge μ, but that is the proposition being tested, not an oversight in logic. Finally, because general coordinate invariance allows us any choice of coordinates, we can choose $A_N=(\mu/4\pi)(\cos\theta-1)d\varphi$ and $A_S=(\mu/4\pi)(\cos\theta+1)d\varphi$ to avoid any indeterminacy at the north and south poles, and this too is not limited to any special circumstance such as 0K. And therefore, $A_S-A_N=(\mu/2\pi)d\varphi$ is a proper statement of a perfectly general relationship between these two coordinate patches in a generally valid and fully determinate system of coordinates. So with all of these ingredients being correct and generally true, what are we missing?

Starting with $A_S-A_N=(\mu/2\pi)d\varphi$, we can easily rewrite this as $A_S=A_N+(\mu/2\pi)d\varphi$ as in (1.2), and we are still on *terra firma*. But now, when we take the next step and regard $A_S\equiv A'_N$ as simply a gauge-transformed state A'_N of A_N, and proceed to write $A'_N=A_N+(\mu/2\pi)d\varphi$, the problem begins. For as soon as we write $A_S=A_N+(\mu/2\pi)d\varphi$ in the form of the gauge transformation $A'_N=A_N+(\mu/2\pi)d\varphi$, then the combination with the certainly-true $A\rightarrow A'=A+e^{-i\Lambda}de^{i\Lambda}/ie$ in (1.1) leads us to (1.3), and (1.3) in turn inexorably

leads us to electric magnetic duality and fractionalized charges that we do not generally observe. So what is wrong here? Why is it a misstep to regard A_S as a gauge-transformed A_N, i.e., to assume that $A_S \equiv A'_N$, *at least in general*? What if, as a general rule, *in the physical world*, there actually *is no gauge transformation by which the north patch can be transformed into the south patch?* What if these north and south patches of the gauge field $A = A_\mu dx^\mu$ are observably-distinct in general and their observable distinctness is only removed near 0K? Or, going back to topology, what if the north environment bars and the south object bars in Figures 6 have some freedom relative to one another in general, and only have their mutually-separate freedoms removed near 0K?

In other words, might it be that A_N can only be gauge transformed into A_S in the limited environment of temperatures near 0K, and might it be that otherwise, *in the physical world not at 0K*, there is an *observable distinctness* between the A_N and A_S? And might it be that this is topologically represented by the immobilization between the OET "object" and the OET "environment" near 0K juxtaposed against the relative freedom between the object and its environment in the physical world not at 0K? Let us study this possibility more closely.

When we assumed to arrive at (1.3) that $A_S \equiv A'_N$, we were assuming that these two gauge field patches over the closed surface $\oiint F = \mu$ surrounding the magnetic charge μ differ from one another by nothing more than a gauge transformation. Written using the vector potentials as $A_{S\mu} = A'_{N\ \mu} = A_{N\mu} + \partial_\mu \Lambda$, this $A_S \equiv A'_N$ is an assumption that the gauge potentials on the two patches about what then turns out to be an electric and magnetic charge, differ by nothing more than the gradient $\partial_\mu \Lambda$ of a local *unobservable* phase $\Lambda(x^\mu)$. And because $\Lambda(x^\mu)$ is not observable, we are assuming that there is no observable distinction between the gauge fields on the two patches. But suppose these north and south gauge field patches *are observably distinct*, in physical reality. Suppose that in physical reality $A_S \neq A'_N$ as a general rule, i.e., that there is no gauge transformation that can get us from A_N to A_S or vice versa. Suppose instead that the assumption $A_S \equiv A'_N$ is only true and thus permitted close to 0K, and that other than at 0K there is no transformation whatsoever by which the north coordinate patch can be gauge transformed into the south patch.

Quantitatively, suppose instead that $A'_{N\ \mu} = A_{S\mu} + \varepsilon_\mu$ for the gauge field of each charge, where ε_μ is some four-vector which *cannot* be expressed merely as the gradient $\partial_\mu \Lambda$ of an unobservable local phase $\Lambda(x^\mu)$ and so cannot simply be gauged away into $\partial_\mu \Lambda$. Using the one-form $\varepsilon = \varepsilon_\mu dx^\mu$, this may be written as $A'_N = A_S + \varepsilon$, and the deliberate analogy is to the $y = \pm\varepsilon$ spatial freedom except when $\varepsilon = 0$ near 0K which we used during our topological discussions. The gauge transformation (1.1) is still $A'_N = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$ for the north patch, and the north and south patches are still related by $A_S = A_N + (\mu / 2\pi) d\varphi$. But now, rather that

assuming as we did at (1.3) that $A_S \equiv A'_N$, we are assuming that in general, $A'_N = A_S + \varepsilon$, where $\varepsilon = \varepsilon_\mu dx^\mu \neq 0$ in general and *is a physical observable*.

Because the gauge field A_μ has physical dimensions of energy, this new vector ε_μ will likewise have physical dimensions of energy. The *form of energy* represented by A_μ is *electrical potential energy*, so if $A_{N\mu}$ and $A_{S\mu}$ differ by more than just a gauge transformation, this tells us that the form of energy represented by ε_μ must be a *different form of energy*. And given the context of this discussion, the suspicion which we now need to confirm is that ε_μ represents heat energy. If ε_μ does represent heat energy then we would have $\varepsilon_\mu = 0$ at 0K, and $A'_{N\,\mu} = A_{S\mu} + \varepsilon_\mu$ would become $A'_{N\,\mu} = A_{S\mu}$ at 0K, and the DQFC would come to life at 0K, and we could understand the FQHE using the DWY monopoles at 0K. And then we would be saying that the north and south gauge patches A_N and A_S differ from one another by merely a gauge transformation *in the absence of all heat*, but differ from one another by more than a gauge transformation in the presence of heat energy represented by ε_μ. And this would in some way help to unify $U(1)_{em}$ electromagnetic gauge theory with thermodynamics. So let's now develop this formally.

First we may write $A_S = A'_N - \varepsilon = A_N + (\mu / 2\pi)\, d\varphi$ by combining the two expressions containing A_S. Then, writing this as $A'_N = A_N + (\mu / 2\pi)\, d\varphi + \varepsilon = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$ which also employs the gauge transformation $A'_N = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$, and eliminating A_N yields:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi + \varepsilon \,. \tag{9.1}$$

This is the generalization of $e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi)\, d\varphi$ in (1.3), to the circumstance where there is an observable distinctness between the north and south gauge patches which cannot be gauged away by a gauge transformation. Based on all we have discussed, we expect that ε will be temperature dependent, i.e. that $\varepsilon = \varepsilon(T)$, and that at absolute zero, $\varepsilon(0\text{K}) = 0$. Thus, very close to 0K, (9.1) will become $e^{-i\Lambda} de^{i\Lambda} / ie = (\mu / 2\pi)\, d\varphi$, and based on what was discussed at the start of this section this will mean that $e\mu = 2\pi(n / l)$ near 0K. This in turn will mean that there will be an electric / magnetic symmetry near 0K with quantized fractionalized charges. And as also discussed, if the only permitted states are disentangled states, this will yield only odd denominators just like in the FQHE. And if we also permit a single entangled state $\varphi = 2$ (which we shall study in the next section), then this would entirely reproduce the observed FQHE in all respects, with only the denominators $\varphi = 1, 2, 3, 5, 7, 9 \ldots$.

With (9.1) however, we now have a way to study what happens away from 0K, in the physical domains where we do not observe magnetic monopoles, or fractionalized charges other than those of the quarks. So let us find the solutions to (9.1). First, related in some to-be-

determined way to $\varepsilon(T)$, let us posit a "thermal function" $\tau(T)$ which is also a function of temperature. Then, working from the earlier solution (1.4), and using the reduced azimuth $\varphi = \varphi / 2\pi$ which we have previously related to the orbital and angular momentum quantum numbers via $\varphi = 2j = 2\left(l + \frac{1}{2}\right)$, see (7.2), let us write:

$$\exp(i\Lambda) = \exp\left(ie\mu \frac{\varphi}{2\pi} + ie\tau \right) = \exp\left(ie\mu\varphi + ie\tau \right) = \exp\left(ie\mu\varphi \right) \exp\left(ie\tau \right) \tag{9.2}$$

If we then place this into the left hand side of (9.1), we obtain:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi + d\tau = \frac{\mu}{2\pi} d\varphi + \varepsilon . \tag{9.3}$$

Therefore, (9.2) is the solution to (9.1) on the condition that:

$$\varepsilon = d\tau = \varepsilon_\mu dx^\mu = \partial_\mu \tau dx^\mu , \tag{9.4}$$

or in tensor language:

$$\varepsilon_\mu = \partial_\mu \tau . \tag{9.5}$$

So the vector $\varepsilon_\mu(T)$ is the spacetime gradient of a thermal scalar $\tau(T)$, i.e., it is some sort of thermal function gradient. The relationship $A'_N = A_S + \varepsilon$ between the north and south gauge field patches is equivalent to $A'_{N\mu} = A_{S\mu} + \varepsilon_\mu = A_{S\mu} + \partial_\mu \tau(T)$. But the scalar $\tau(T)$ is an observable function of temperature, unlike the unobservable phase $\Lambda(x^\mu)$ in the gauge / phase transformation $A'_\mu = A_\mu + \partial_\mu \Lambda(x^\mu)$.

Now, let's get to work on the solution (9.2). If we first approach this purely mathematically, and if we consider only orientation whereby $\exp(ie\mu\varphi)$ for any of $\varphi = 0,1,2,3...$ is regarded to be equal to $\exp(ie\mu\varphi)$ for any other $\varphi = 0,1,2,3...$, then:

$$1 \cdot \exp\left(ie\tau \right) = \exp\left(ie\mu \cdot 1 \right) \exp\left(ie\tau \right) = \exp\left(ie\mu \cdot 2 \right) \exp\left(ie\tau \right) = \exp\left(ie\mu \cdot 3 \right) \exp\left(ie\tau \right) .... \tag{9.6}$$

Were we to then divide out the new term $\exp(ie\tau)$ from each of the above, we would arrive at exactly what we found in (3.1), which yields the DQFC of (3.2), namely $e\mu = 2\pi(n/l)$. But the thermal function $\tau(T)$ was supposed to get rid of the fractionalized charge and break the electric magnetic symmetry at higher temperatures, because that is what we observe at higher temperatures. So what have we missed now?

We regarded $\tau(T)$ as a function of temperature, but we also now know that the azimuth is in general related to electron angular momentum by $\varphi = 2j = 2\left(l+\frac{1}{2}\right)$, see (7.2). Keeping in mind that $\tau(T)$ is a function used to characterize the thermal character *of individual charges*, which means that it specifies the degree to which the A_N and A_S for an individual DWY electron cannot be gauge-transformed into one another, and because individual electrons are distinguished from one another by their orbital and principal quantum numbers, let us now regard $\tau(T)$ to be a function not only of temperature, but also of the $\varphi = 2j = 2\left(l+\frac{1}{2}\right)$ of that DWY electron. That is, we now regard this thermal scalar to be a $\tau\left(T,\varphi\right)$, so that its gradient $\varepsilon_\mu\left(T,\varphi\right) = \partial_\mu \tau\left(T,\varphi\right)$ is similarly a function of $\varphi = 2j = 2\left(l+\frac{1}{2}\right)$. To make this explicit, let us now write (9.2) as:

$$\exp\left(i\Lambda\right) = \exp\left(ie\mu\varphi + ie\tau(T,\varphi)\right) \tag{9.7}$$

Next, recall that back at (3.7) we determined that topologically, $\varphi = 1$ represents the electron bar and ribbon systems of Figures 6, and that $\varphi = 2j = 2\left(l+\frac{1}{2}\right)$ with $s = \frac{1}{2}$ in general. The states which are not entangled are $l = 0,1,2,3...$ i.e., $\varphi = 1,3,5,7....$ We also know that $\varphi = 0$ represents $s = 0$, which is not an electron at all, but is the spinless (twistless) bar and ribbon apparatus of Figure 2(a). So because $\varphi = 0$ is no electron charge at all, we regard $\tau(T,\varphi)$ to be always equal to zero for $\varphi = 0$, i.e., $\tau(T,\varphi = 0) = 0$. Thus, regarding any $\exp\left(ie\mu\varphi\right)\exp\left(ie\tau(T,\varphi)\right)$ for a disentangled electron $\varphi = 1,3,5,7...$ to be equal to $\exp\left(ie\mu\varphi\right)\exp\left(ie\tau(T,\varphi)\right)$ for any other disentangled electron, and also equating this to the $\varphi = 0$ function $\exp\left(ie\mu\cdot 0\right)\exp\left(ie\tau(T,\varphi = 0)\right) = 1 = \exp\left(i2\pi n\right)$, we now write (9.7), contra (9.6), in a way that the thermal scalar τ does not factor out, as:

$$\begin{aligned}\exp\left(i\Lambda\right) = \exp\left(i2\pi n\right) = 1 &= \exp\left(ie\mu\cdot 1 + ie\tau(T,\varphi = 1)\right) = \exp\left(ie\mu\cdot 3 + ie\tau(T,\varphi = 3)\right)\\ &= \exp\left(ie\mu\cdot 5 + ie\tau(T,\varphi = 5)\right) = \exp\left(ie\mu\cdot 7 + ie\tau(T,\varphi = 7)\right)...\end{aligned} \tag{9.8}$$

We may then extract the equivalent, disentangled states:

$$\Lambda = 2\pi n = e\mu\cdot\varphi + e\tau(T,\varphi) \tag{9.9}$$

where $\varphi = 2j = 2\left(l+\frac{1}{2}\right) = 1,3,5,7....$ In terms of magnetic charge, and mindful that the reduced gauge angle $\mathcal{A} = \Lambda/2\pi = n$, we then restructure (9.9) into:

$$\mu = \frac{\mathcal{A}}{\varphi}\frac{2\pi}{e} - \frac{\tau(T,\varphi)}{\varphi}. \tag{9.10}$$

Now let us consider two specializations of (9.10). By design, we expect the thermal function $\tau(T,\varphi)$ to be zero at T=0K, i.e., $\tau(0\text{K},\varphi) = 0$, which means that over a spacetime

region where the temperature is 0K everywhere at all times, $\varepsilon_\mu = \partial_\mu \tau\left(0\text{K}, \not{\varphi}\right) = 0$ and $\varepsilon = \varepsilon_\mu dx^\mu = 0$. This corresponds to the topological $\varepsilon = 0$ in $y = \pm\varepsilon$. So at absolute zero, where $\tau(0\text{K}, \not{\varphi}) = 0$, (9.10) becomes:

$$\mu = \frac{\not{\text{A}}}{\not{\varphi}} \frac{2\pi}{e} = \nu u_{\text{u}}, \tag{9.11}$$

or, in terms of the electric charge,

$$e = \frac{\not{\text{A}}}{\not{\varphi}} \frac{2\pi}{\mu} = \nu e_{\text{u}}, \tag{9.12}$$

with $\nu = \not{\text{A}} / \not{\varphi}$ and $u_{\text{u}} = 2\pi / e$ and $e_{\text{u}} = 2\pi / \mu$, just as before. Because $\not{\text{A}} = 0, 1, 2, 3...$ and the disentangled states are $\not{\varphi} = 1, 3, 5, 7...$, and because these states are all topologically observably distinct based on OET, (9.12), which is a near-0K specialization of (9.10), describes exactly what is observed in the FQHE, if we take the single additional step of permitting $\not{\varphi} = 2$ as the only allowed *entangled* electron state. Again, we shall look more closely at $\not{\varphi} = 2$ in the next section. In this state, $\mu e = 2\pi\left(\not{\text{A}} / \not{\varphi}\right)$, there is quantization, there is fractionalization, there is an electric / magnetic symmetry under $\mu \leftrightarrow e$ interchange, and the untangled charge states have fill factors with the odd denominators found in the FQHE.

The other specialization of (9.10) is at warmer temperatures. The magnetic charge μ in (9.11) is based supposing that $\mu = \oiint F$, and then asking via the use of gauge theory, what would happen *if* these magnetic charges were to exist under some set of physical circumstances? We see that what would happen, is that we would observe a charge fractionalization which looks very much like the FQHE near 0K. But because we do not ever observe a magnetic charge at warmer temperatures, this means that $\mu = \oiint F = 0$ except in the FQHE environment. So, let's simply go to (9.10) and set $\mu = 0$, which we regard as an act of "breaking the electric / magnetic symmetry" and entering a thermal environment with $\tau(T, \not{\varphi}) \neq 0$, and find the result. With $\mu = 0$ but $\tau(T, \not{\varphi}) \neq 0$, (9.10) now becomes

$$e = \not{\text{A}} \frac{2\pi}{\tau(T, \not{\varphi})} = n \frac{2\pi}{\tau(T, \not{\varphi})} = n \frac{2\pi}{\tau(T, 2j)} = n \frac{2\pi}{\tau(T, 2l+1)}, \tag{9.13}$$

where we have also used $\not{\varphi} = 2j = 2l + 1$ to introduce the angular momentum Casimir quantum numbers *j* and *l*. Isolating the thermal scalar instead, and also using the DQC magnetic charge unit $u_{\text{u}} = 2\pi / e$, this becomes:

$$\tau(T, n, \not{\varphi}) = \tau(T, n, 2j) = \tau(T, n, 2l+1) = n \frac{2\pi}{e} = n\mu_{\text{u}}, \tag{9.14}$$

where we have now written this as $\tau(T,n,\not{\varphi})$ in recognition that this thermal function is also a function of the quantum number $\not{\text{A}}=n$ as well as $\not{\varphi}=2j=2l+1$.

Now, there are some notable similarities between the low-temperature (9.11) and (9.12) which are responsible for the FQHE and the higher-temperature (9.13) and (9.14). These parallels are highlighted if we write the near-0K (9.11) and (9.12) to show the FQHE and its electric / magnetic duality including DWY electric magnetic charges with $\not{\text{A}}=1,2,3,4...$ and $\not{\varphi}=1,2,3,5,7,9...$ (all disentangled odd denominators except $\not{\varphi}=2$ to be reviewed in the next section) as:

$$e\mu=2\pi\frac{\not{\text{A}}}{\not{\varphi}}, \tag{9.15}$$

and then write (9.14) in parallel form including $\not{\text{A}}=n$ as:

$$e\frac{\tau(T,n,\not{\varphi})}{\not{\varphi}}=2\pi\frac{\not{\text{A}}}{\not{\varphi}}. \tag{9.16}$$

We see that when the low temperature symmetry breaks including breaking the duality symmetry, the magnetic charge μ is replaced by $\tau(T,n,\not{\varphi})/\not{\varphi}$ which has the dimensionality and character of a magnetic charge but is instead a *thermal scalar charge*. So in an extremely fundamental way, the magnetic monopole charges μ that appear right at 0K and motivate FQHE, turn into a thermal charge $\tau(T,n,\not{\varphi})/\not{\varphi}$ at higher energies. Given that $\mu\rightarrow\tau(T,n,\not{\varphi})/\not{\varphi}$ once we rise from 0K and break the low temperature duality symmetry, and given that $\tau(T,n,\not{\varphi})$ is fundamentally a thermal scalar, we may unify Maxwell's electromagnetic theory with thermodynamic theory in a very deep way at the microscopic level of individual electrons by understanding that *heat itself* is what replaces the magnetic monopoles of 0K once the temperature rises. *Thermal energy, which of course pervades natural experience, is the form in which we observe the broken residue of the DYW magnetic monopoles that give rise to FQHE very close to absolute zero.*

Now we return to (9.5) where we found that the thermal scalar $\tau(T,n,\not{\varphi})$ is related by $\varepsilon_\mu=\partial_\mu\tau$ to the ε_μ which via $A'_{N\,\mu}=A_{S\,\mu}+\varepsilon_\mu$ specifies the extent to which the north gauge patch cannot be gauge transformed into the south patch and vice versa. This $\varepsilon=\varepsilon_\mu dx^\mu$ was developed to formally represent the topological freedom $y=\pm\varepsilon$ which the "ribbons and bars" of Figures 6 are given for OET rotation and disentangling. Now, in (9.14), we have found that $\tau(T,n,\not{\varphi})=2\pi n/e$. Se we can directly determine from this, that:

$$\varepsilon_\mu(T,n,\not{\varphi})=\partial_\mu\tau(T,n,\not{\varphi})=2\pi n\partial_\mu\left(\frac{1}{e}\right)=-\frac{2\pi}{e^2}n\partial_\mu e=-\frac{1}{4\pi}\frac{2\pi}{\alpha}n\partial_\mu e=-\frac{1}{4\pi}\frac{2}{g-2}n\partial_\mu e, \tag{9.17}$$

where we have also included the running electromagnetic coupling $4\pi\alpha = e^2$, and in the final expression included the one-loop g-factor "anomaly" for an individual electron $g/2 = 1+\alpha/2\pi$ first found by Schwinger in [35]. The above is in natural units $\hbar = c = 1$, but with these two fundamental constants restored $e \to e/\sqrt{\hbar c}$ is a dimensionless number. So as already discussed, ε_μ has a mass dimension of +1, i.e., it has dimensions of energy. But because $\not{\varphi} = 2j = 2l+1$, aside from being a function of the quantum numbers *n* and j, $\varepsilon_\mu(T,n,\not{\varphi})$ is also a function of temperature. Thus, as an energy-dimensioned four vector with thermal character, we anticipate that ε_μ will bear some discernable relation to the energy equivalent of any given temperature, $E = kT$, and as we anticipated, that $\varepsilon_\mu(T,n,\not{\varphi})$ is a form of heat energy which separates the north and south gauge patches by more than a gauge transformation. While $2/(g-2) \cong 861.023$ at $\alpha \cong 1/137.036$ for low probe energy is a large number, $\partial_\mu e \cong 0$ is a very small number at low energy, i.e., the running electromagnetic charge / coupling is very flat at low energy and does not start a discernable ascent until energies start to reach the GeV domain. So at 0K where $E = kT = 0$, we also expect that $\varepsilon_\mu = 0$, consistent with how this is ε is understood to analogize to the topological $y = \pm\varepsilon$.

Additionally, because the energy dimensioned $4\pi\varepsilon_\mu = -2n\partial_\mu e/(g-2) \propto n$ is quantized in proportion to $n = \not{A}$, it is clear that *n* defines a *quantized energy level*. Because $\varepsilon_\mu(T,n,\not{\varphi})$ is also a function of the angular momentum Casimir quantum numbers $\not{\varphi} = 2j = 2l+1$, and because the principal quantum number *n* also specifies quantized energy levels and is the only atomic number that we have not yet attempted to topologically explain, and because $n \propto \varepsilon_\mu$ and ε_μ is an energy distinct from the electromagnetic potential and appears to be a thermal energy at least in part, we shall now add a fourth evidence-informed inference to the three inferences in section 8:

> Inference 4: At finite, non-zero temperatures, after the electric / magnetic duality is broken and the magnetic charge μ has been replaced by its "thermal residue" $\tau(T,n,\not{\varphi})/\not{\varphi}$, the reduced gauge angle $n = \not{A}$ is one and the same as the principal quantum number seen in atomic structure.

With this final inference our topological understanding of atomic structure is complete. All of the angular momentum quantum numbers *l*, *s*, *j*, *m*, s_z and j_z have already been understood in terms of OET rotations and "thread" twists of an OET "object" relative to its "environment." But the gauge angle $n = \not{A}$ has also been present all along as an integer quantum number which brings about the *quantization of charge* in DWY monopole theory. Now, when we break the low temperature symmetry and the magnetic monopole μ becomes a thermal residue $\tau(T,n,\not{\varphi})/\not{\varphi}$, the thermal residue gradient $\varepsilon_\mu(T,n,\not{\varphi}) = \partial_\mu\tau(T,n,\not{\varphi}) \propto n$, which is an energy, becomes quantized in proportion to this very same $n = \not{A}$, so that this same *n* now brings about the *quantization of energy* in the electron shells of atoms.

So what we now see is that at ultra-low temperatures, the reduced gauge angle $n = \mathcal{A}$ is responsible for quantization of charge in the DWY monopoles and supplies *the numerator in the FQHE*. But at higher temperatures after the electromagnetic duality is broken, the reduced gauge angle $n = \mathcal{A}$ now becomes responsible for quantization of energy in atomic shells, and becomes *the principal quantum number in atomic structure*. And this energy quantization driven by the principal quantum number, is also related via the temperature *T* dependence of $\varepsilon_\mu(T, n, \varphi)$, to the "freedom" which the electron has to operate topologically spatially perpendicularly to its two-dimensional restraint at 0K. Therefore, $\varepsilon_\mu(T, n, \varphi)$ which should be further studied, appears to be a measure of the intrinsic heat content of that electron, and therefore a microscopic thermodynamic variable. Whether ε_μ contains energy content from a form other than heat we leave as a question for separate consideration. But what we do know for certain about ε_μ is that it does include heat energy, and it does not include electromagnetic potential energy.

To the extent that this thermal magnetic charge residue $\tau(T, n, \varphi)$ and its energy-dimensioned gradient $\varepsilon_\mu(T, n, \varphi) = \partial_\mu \tau(T, n, \varphi) \propto n$, as it may be further developed, proves capable of explaining the existence of heat at the microscopic level of individual electrons, and to the extent that this microscopic understanding of heat for individual electrons can be related to the usual statistical understanding of heat based on the movement of collective systems, one may be able to entertain the prospect that *the very existence of heat energy in the universe* is the observed residue of the $U(1)_{em}$ magnetic charges that exist near 0K but quickly disappear to be replaced by heat energy once the low-temperature duality symmetry is broken. It certainly appears to be true that *at least some* of the heat energy in the universe is from the magnetic monopole residue $\mu \to \tau(T, n, \varphi)/\varphi$. The question being posed here, is whether in some fashion, *all* of the heat energy in the universe can be traced back to this magnetic monopole residue. If so, it would be rather ironic to find that the long-pursued magnetic monopole makes its presence observably detectable, as a thermal energy residue that animates all of nature.

With the connection between 0K and other higher temperatures developed in this section, we now understand that at 0K there is indeed a duality symmetry between electric and magnetic charges, that DWY $U(1)_{em}$ magnetic monopoles do exist, and that these electric and magnetic charges are observed via the FQHE fill factor $\nu = \mathcal{A}/\varphi$ with integer numerators $\mathcal{A} = n = 0, 1, 2, 3...$ and with odd-integer denominators $\varphi = 2j = 2\left(l + \frac{1}{2}\right) = 1, 3, 5, 7...$ shown in (7.2), with the exception of the even denominator $\varphi = 2$ which we shall examine in detail next. We also now understand that once the temperature rises from 0K the electric-magnetic duality becomes broken and the magnetic monopole charge μ is replaced by a "thermal residue" charge $\tau(T, n, \varphi)/\varphi$ which is a function of temperature *T*, the same $\mathcal{A} = n$ which is the FQHE denominator, and the same $\varphi = 2j = 2\left(l + \frac{1}{2}\right) = 1, 3, 5, 7...$ which is the FQHE numerator. The gradient $\varepsilon_\mu = \partial_\mu \tau \propto \mathcal{A} = n$ of this thermal residue, which is an energy-dimensioned four-vector, has its energy quantized in proportion to *n*, and taken together with all the other topological connections to the angular momentum quantum numbers *l*, *s*, *j*, *m*, s_z and j_z, this appears to qualify $\mathcal{A} = n$ at higher temperatures as the principal quantum number in atomic shell structure. So just as the DWY monopoles become a thermal residue at higher temperatures, so too does the

DWY electric and magnetic charge quantum number $\mathbb{A}=n$ become the principal energy quantum number for atomic (and presumably nuclear) orbital shells.

If we do identify *n* from the DWY monopoles as the principal quantum number in this way, there is one question raised by the empirical data that still needs to be considered. In atomic systems, the principal quantum number constrains the orbital quantum number by the relationship $l<n$. But if we write the FQHE fill factor $\nu=\mathbb{A}/\varphi$ as $\nu=n/2\left(l+\frac{1}{2}\right)$ with $\mathbb{A}=n$ and $\varphi=2\left(l+\frac{1}{2}\right)$, then we see that there are some observed FQHE states which violate this constraint. For example, if $n=l=1$ we obtain $\nu=1/3$, which violates $l<n$ but is clearly observed. The same is true, for example, of $\nu=2/5$ with $n=l=2$ and of $\nu=3/7$ with $n=l=3$, which are also observed. If one were to observe $\nu=1/5$ with $n=1$ and $l=2$, or $\nu=1/7$ or $\nu=2/7$ with $n=1$ and $n=2$ respectively but $l=3$ in both cases, then we would actually be observing states for which $n<l$. So the question: how might the constraint $l<n$ at higher temperatures be removed near 0K?

There appear to be two interrelated explanations of how this could be the case. First, at low temperature, *n* measures *charge quantization*. It is only at higher temperatures after the duality symmetry has been broken that it starts to measure *energy quantization*. So even if $n<l$ is a constraint once *n* begins to measure energy quanta, that may not mandate that $n<l$ still has to apply when *n* is measuring charge not energy quantization. Second, and closely related, is the fact that even at low temperatures, fermions still must satisfy the Exclusion principle. And in atoms at low temperatures, electrons in the atomic shells of these atoms still must have quantum numbered states available into which they can elevate to satisfy exclusion. So if *n* is proportional to heat energy, and if there is no heat energy at 0K, and if *n* is removed as an energy quantum number and converted over to a charge quantum number at 0K, then the electrons still must be able to enter states of $l>0$ to satisfy Exclusion, even when there is no heat so *n* would become zero if it was still a measure of heat, but because there is no heat *n* has now converted over from a measure of heat to a measure of charge. Put differently, once *n* is no longer an exclusionary quantum number measuring heat energy but is simply a measure of charge quantization, fermions still need to enter elevated states to satisfy Exclusion, and so *l* will decouple from *n* and states for which $l=n$ and even $l>n$ will now be permitted.

The final piece of the puzzle which now needs to be explained, is the even-numbered FQHE denominator $\varphi=2$.

10. Paired, Entangled Electron States with Even-Integer FQHE Denominator 2, why Larger Even-Integer FQHE Denominators are not Observed, and some Additional Proposed Experimental Tests

Based on (9.11) and (9.12), which was the earlier (7.2), we now know that when the thermal scalar $\tau(T,n,\varphi)=0$, in other words, at or very near 0K, there is an electric / magnetic duality symmetry, $\mu e=2\pi\mathbb{A}/\varphi$ with a fill factor $\nu=\mathbb{A}/\varphi$, with $\mathbb{A}=0,1,2,3...$, and, if the DWY electrons are to exist in disentangled states only, $\varphi=1,3,5,7....$ All of this accords perfectly with the quantized and odd-fractionalized FQHE with one exception: the empirical

evidence that $\varphi = 2$ is also an observed state [6], [7], [8], [9], [10], and that this is the only even denominator state observed. So the empirical evidence is telling us that at 0K, the DWY electrons are indeed all in disentangled states, with the singular exception of $\varphi = 2$. Therefore, let is make use of this empirical observation to try to understand, theoretically, what motivates this pattern of observation for the even-denominator FQHE, as well as the observed absence of any other even-integer denominators.

We used the first of four evidence-based inferences made in sections 8 and 9 to introduce the view that the observed angular momentum quantum numbers l, m, s_z, j_z in atomic orbital shells is a consequence of precisely-analogous l', m', s_z', j_z' numbers used to summarize OET topology because of a precise one-to-one mapping between these. Once this connection was made, we came to understand that the azimuth angle $\varphi = 2j = 2l+1 = 2l+2s$ in terms of the Casimir numbers *j* and *l* where the Casimir $s = \frac{1}{2}$, because $\varphi = 2j' = 2l'+1 = 2l'+2s'$ based on the topology to which this maps. Therefore $\varphi = 1,3,5,7...$ corresponds to $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and $l = 0,1,2,3...$. So what does it means when we empirically observe $\varphi = 2$?

Topologically, although we are entangling the DWY electrons with $\varphi = 2$, they are still intrinsic spin ½, electrons, so their spin Casimir $s = \frac{1}{2}$ will not change. Any changes will be to the other angular momentum quantum numbers. Based on $\varphi = 2j = 2l + 2s$, this *topologically entangled* state $\varphi = 2$ has $j = 1$ and $l = s = \frac{1}{2}$. So this entangled state is a boson state with total observable angular momentum Casimir $j = 1$ in $\mathbf{J}^2 \left| \xi \right\rangle = j\left(j+1\right) \left| \xi \right\rangle$. We also know that $\mathbf{L}^2 \left| \xi \right\rangle = l\left(l+1\right) \left| \xi \right\rangle$ and $\mathbf{S}^2 \left| \xi \right\rangle = s\left(s+1\right) \left| \xi \right\rangle$. So along the z-axis, the observed eigenvalues will be $L_z \left| \xi \right\rangle = m \left| \xi \right\rangle = \pm \frac{1}{2} \left| \xi \right\rangle$ and $S_z \left| \xi \right\rangle = s_z \left| \xi \right\rangle = \pm \frac{1}{2} \left| \xi \right\rangle$. As a shorthand, let us now define $\left| \uparrow_m \right\rangle \equiv \left| m = +\frac{1}{2} \right\rangle$, $\left| \downarrow_m \right\rangle \equiv \left| m = -\frac{1}{2} \right\rangle$, $\left| \uparrow_s \right\rangle \equiv \left| s_z = +\frac{1}{2} \right\rangle$, and $\left| \downarrow_s \right\rangle \equiv \left| s_z = -\frac{1}{2} \right\rangle$. Therefore, the four possible state combinations are $\left| \uparrow_m, \uparrow_{s_z} \right\rangle$, $\left| \uparrow_m, \downarrow_{s_z} \right\rangle$, $\left| \downarrow_m, \uparrow_{s_z} \right\rangle$ and $\left| \downarrow_m, \downarrow_{s_z} \right\rangle$. Then, because $j_z = m + s_z$, we can group these into the familiar triplet and singlet states:

$$\begin{cases} \left| j = 1, j_z = +1 \right\rangle \equiv \left| \uparrow_m, \uparrow_{s_z} \right\rangle \\ \left| j = 1, j_z = 0 \right\rangle \equiv \left| \frac{1}{\sqrt{2}} \left(\uparrow_m, \downarrow_{s_z} + \downarrow_m, \uparrow_{s_z} \right) \right\rangle \\ \left| j = 1, j_z = -1 \right\rangle \equiv \left| \uparrow_m, \uparrow_{s_z} \right\rangle \end{cases} \qquad \left| j = 0, j_z = 0 \right\rangle \equiv \left| \frac{1}{\sqrt{2}} \left(\uparrow_m, \downarrow_{s_z} - \downarrow_m, \uparrow_{s_z} \right) \right\rangle \tag{10.1}$$

This pattern will be familiar because it is the pattern of spin ordinarily found in massive vector bosons, with a spin 1 triplet which includes two transverse and one longitudinal polarization, as well as a scalar $j = 0$ singlet.

So, the empirically-observed $\varphi = 2$ FQHE denominator is telling us that the even denominator $\varphi = 2$ is evidencing a boson spin state consisting of entangled electrons, with three spin 1 degrees of freedom and a single spin 0 degree of freedom. At the same time, all of the odd FQHE denominators are evidencing individual entanglement-free electron states of $\varphi = 1, 3, 5, 7...$ which we know respectively as s, p, d, f… electrons.

This allows us to expand the experiment proposed in section 7: It has already been suggested to closely study the observed FQHE charge states for correlations between the $\varphi = 1, 3, 5, 7$ FQHE denominators and the known observed characteristics of the s, p, d, f electrons, respectively. A further empirical prediction from (10.1) is that close experimental study of the $\varphi = 2$ even-denominator states should demonstrate that these are boson states in which two electrons have become entangled together, i.e., these are *electron pair* states, and that these pair states when studied closely experimentally should correlate with the boson spin pattern of (10.1). While the "Cooper pairs" model of electron pairing [36] may well come to mind, for the moment let us not be that specific. What does seem to be clear, is that there the $\varphi = 2$ states do involve boson spins in which two electrons become paired by entanglement, and that these should exhibit angular momentum characteristics which are decidedly-distinct from the angular momentum characteristics of all the odd-integer denominator states.

Further, now that section 9 has made clear how the FQHE can indeed be regarded as a consequence of the electric / magnetic duality of FQHE because we have shown how this duality can be broken at higher temperatures so that the monopoles are no longer observed and leave in their place a thermal reside observed as heat, although these monopole charges may be slight in relation to the applied perpendicular magnetic fields, it would be worthwhile to see if some clever experiment can be designed to filter out the noise of the experiment from the signal of the monopoles so that these monopoles might be directly detected.

Returning to the $\varphi = 1, 3, 5, 7$ FQHE denominators and their correlation with s, p, d, f electrons, let us predict one further correlation that should be experimentally observable. With all electrons in the lowest permitted Exclusionary states, the elements from ${}_{1}\mathrm{H}$ through ${}_{20}\mathrm{Ca}$ (with Z=1 and Z=20 respectively) will only contain s and p electrons for which $\varphi = 1, 3$. The elements from ${}_{21}\mathrm{Sc}$ through ${}_{56}\mathrm{Ba}$ must all naturally contain some p electrons for which $\varphi = 5$, even before any magnetic field is applied which could excite higher orbits. And once we enter the Lanthanides at ${}_{57}\mathrm{La}$, then there must also naturally be some f electrons for which $\varphi = 7$, before anything else happens. Now, the FQHE is not observed in free space; it is observed in certain superconducting metals, see for example, the partial listing of simple-structure superconductors at [37] which also lists highest transition temperatures and critical magnetic fields. So if the FQHE denominator is in fact tied to the s, p, d, f… states of electrons, then more electrons with larger l and therefore larger φ will naturally be made available by metals which contain heavier elements than those with lighter elements.

With this in mind, the final correlation proposed for experimental observation is to characterize the observed FQHE in relation to the elements in the particular superconducting metals used to discern this effect. The *prediction* is that metals which contain elements with higher Z on the periodic table, will all be metals in which larger FQHE denominators reveal

themselves more readily in response to smaller applied perpendicular magnetic fields. Specifically, one does not need to rely as much on the magnetic fields to raise the orbital levels into the fractional states, because there are already electrons at higher orbitals in the natural state of these metals. Thus, for example, the list at [37] shows that all of $_{57}$La, $_{70}$Yb, $_{72}$Hf, $_{73}$Ta $_{74}$W, $_{75}$Re, $_{76}$Os, $_{77}$Ir, $_{80}$Hg, $_{81}$Tl, $_{82}$Pb $_{90}$Th, $_{91}$Pa, $_{92}$U or compounds containing these are superconductors, and these will all naturally contain f electrons for which $l = 3$ and $\varphi = 7$. Because it takes a larger perpendicular magnetic field to generate the FQHE states which display smaller fractions, this correlation would suggest that the larger denominators can be brought into observation with smaller applied magnetic fields in high-*Z* superconductors such as $_{82}$Pb or $_{90}$Th with naturally-provided d-state and f-state electrons, than in low-*Z* superconductors with $_{13}$Al or $_{14}$Si or $_{22}$Ti or various carbon compound superconductors which have a natural surfeit of d and f electrons. Perhaps $_{82}$Pb (lead) with a fairly high critical temperature 7.19K and a fairly small critical magnetic field 0.08T is a good element to use to see how readily the large-denominators states can be produced with smaller magnetic fields.

The other factor that will correlate with generating larger φ with smaller magnetic fields will be how tightly bound the d and f electrons are to their atoms. The more the d or f electrons are accessible at the outer shells with smaller energies, the more readily the application of a magnetic field will be able to stimulate them to displaying their higher orbitals via the FQHE. In this regard, the Lanthanoid superconductors which have f-electron binding energies of 5.57 to 6.25 eV [33], [34] may prove to be the best candidates for generating high φ fractions with smaller applied magnetic fields.

The final question as regards the observed FQHE denominators, is why $\varphi = 2$ is the only observed denominator, and why we do not observe any other even denominators $\varphi = 4, 6, 8...$ which via $\varphi = 2j = 2l + 1 = 2l + 2s$ would correspond to $j = 2, 3, 4...$ and $l = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}....$ This is answered by reference to quantum statistics and the Exclusion Principle: For multiple electrons to exist as part of a single quantum system, each of the electrons must have a set of quantum numbers which differs from the quantum number set of any other electron in the same system, and this requirement remains in place even near 0K. This is why, for example, electrons still maintain a non-zero Fermi energy to be able to occupy exclusionary states, even at 0K. So in any given principal shell *n*, once there are two s electrons with $\varphi = 1$ and $l = 0$ and $j = \frac{1}{2}$, in order to add a third electron we must put that new electron into a p state with $l = 1$ and $j = \frac{3}{2}$, which corresponds with $\varphi = 3$ and $l = 1$ and $j = \frac{3}{2}$. In short, we must have higher-*j* states to satisfy Exclusion in larger systems of electrons. And near 0K, this raises the Fermi level. On the other hand for bosons there is no such requirement. One can have as many bosons as one would like with $\varphi = 2$ and the $j = 1$ spin characteristics of (10.1) all in the same system, without ever having to supply a $\varphi = 4$ and $j = 2$ or any higher spin state boson. Because nature always seeks the lowest energy state and because larger-*j* states require more energy than smaller-*j* states as evidenced by the ionization energies [33], [34], the higher-denominator $\varphi = 4, 6, 8...$ boson states are not required by Exclusion, and they are not favored energetically, so they are not observed in the FQHE.

11. Summary and Conclusion, and Consolidation of Experimental Predictions

A complete analysis of the gauge symmetries of Dirac Monopoles following the approach pioneered by Wu and Yang [15], [16] (DWY monopoles) results in an electric / magnetic duality symmetry with charges that are quantized *and fractionalized* according to the Dirac Quantization and Fractionalization Condition (DQFC) $e\mu = 2\pi\left(\not{\Lambda} / \not{\varphi}\right)$, where $\not{\Lambda} = \Lambda / 2\pi = 0,1,2,3...$ is a reduced gauge angle and $\not{\varphi} = \varphi / 2\pi = 1,2,3,4...$ is a reduced azimuth angle following the DWY analysis, see (3.4) and (3.5). When orientation alone is considered, these angles differing by 2π are geometrically indistinct from one another. However, when topological Orientation-Entanglement as taught by Misner, Thorne and Wheeler [18] is accounted for, and when we also account for the topological *twisting* of the threads following various disentangling operations, then all angles differing from one another by 2π and even by 4π are seen to be observably topologically distinct based on Orientation-Entanglement and Twist (OET). Moreover, when we characterize the various topological OET states based on rotations through $4\pi l'$ where $l' = 1,2,3...$, followed by disentangling of the rotated state, we unexpectedly discover that this OET topology can be characterized by a set of quantum numbers $l', m', s', s'_z, j', j'_z$ representing rotation and twist which map precisely, on a one-to-one basis, to the l, m, s, s_z, j, j_z observed in the orbital, spin and total angular momentum of electrons in atomic shells. Finding that the topology of a free electron is represented by $\not{\varphi} = 1$, and if the DQFC $e\mu = 2\pi\left(\not{\Lambda} / \not{\varphi}\right)$ is applied only to states which differ from this by integer multiples of $+4\pi$ and so have the same orientation-entanglement "version" as $\not{\varphi} = 1$, then the denominators $\not{\varphi} = 1,3,5,7...$ will be restricted to odd integers only, and this provides a topological explanation for the similar odd-integer denominators of FQHE. At the same time, the only observed even-FQHE denominator $\not{\varphi} = 2$ is seen to correspond to the spin 1 boson states (10.1), and the absence of larger even-integer FQHE denominators is understood on the basis that whereas fermions are subject to quantum Exclusion, bosons are not.

The central conceptual hurdle in this development is the fact that all of the foregoing is rooted in the DWY analysis which predicts the DQFC $e\mu = 2\pi\left(\not{\Lambda} / \not{\varphi}\right)$ and therefore a duality symmetry under $e \leftrightarrow \mu$ electric / magnetic charge interchange, and the fact that magnetic charges are definitively not observed, at least (if FQHE is connected to the DWY DQFC) in any environment other than near 0K. A related conceptual hurdle rests in fact that the DWY analysis involves three-dimensional systems of electrons whereas electrons in superconducting materials subjected to ultra-low temperature and large perpendicular magnetic fields in the FQHE environment are substantially restricted to two spatial dimensions. The OET topology does allow us to understand qualitatively, how a lowering of temperature correlates with a gradual restriction in, and eventual removal of, one space degree of freedom from these electrons. But to make the quantitative connection required to completely clear these hurdles, it is necessary to recognize that there is a hidden assumption that goes into the usual DWY analysis, namely, that the south patch of the gauge field A_S differs from the north patch A_N by nothing more than a gauge transformation, and so can be represented by $A_S = A'_N = A_N + e^{-i\Lambda} de^{i\Lambda} / ie$. Specifically,

this assumption leads by inexorable logic to the Dirac prediction of electric / magnetic duality, and because such a duality is not physically observed (at least at temperatures not near 0K), this means by contrapositive logic that $A_S = A'_N$ is a flawed assumption (again, at least at temperatures not near 0K). As a result, we are required *by the empirical evidence of no observed duality a.k.a. no observed magnetic monopoles* to modify this assumption into a new supposition that $A'_N = A_S + \varepsilon(T)$, where $\varepsilon(T) = \varepsilon_\mu dx^\mu$ is a physically-observable function of temperature which is equal to zero at 0K, but otherwise nonzero, and where $\varepsilon_\mu(T)$ has physical energy dimensionality but represents heat and possibly other forms of energy *other than* the electromagnetic gauge potential A_μ. Thus, at all temperatures but 0K, there is an observable physical difference between the north and south DWY gauge field patches. With this modification, we find in (9.9) that the DQFC generalizes to $\Lambda = 2\pi n = e\mu \cdot \varphi + e\tau(T, \varphi)$ where $\varepsilon_\mu = \partial_\mu \tau$ and where $\tau(T, \varphi)$ is a thermal scalar which is a function of temperature as well as of the topological winding reduced azimuth φ (and also of n). At 0K, $\tau(T, n, \varphi) = 0$ which recovers the DQFC $2\pi n = e\mu \cdot \varphi$ and leads to the FQHE as discussed in the previous paragraph. But when the low-temperature duality is broken and the magnetic monopoles $\oiint F = \mu = 0$, the DQFC is replaced by $2\pi n = e\tau(T, n, \varphi)$, which means that $\mu \to \tau(T, n, \varphi)/\varphi$ becomes a "thermal residue" of the magnetic charge μ once the temperature rises from 0K and the electric magnetic duality symmetry is broken.

The experimental tests proposed at the end of section 7 for the odd-FQHE denominators $\varphi = 1, 3, 5, 7...$ and also in section 10 for the even denominator $\varphi = 2$ and high-Z superconductors are intended to validate not only the overall inference that the DQFC $e\mu = 2\pi(\mathcal{A}/\varphi) = 2\pi\nu$ is responsible for the FQHE, but also, the first inference in section 8 that the angular momentum quantum numbers observed in the electronic structure of atomic shells (and by extension in nuclear structure) are a direct consequence of the OET topology. This is because, if the DQFC truly is responsible for the FQHE, and if the OET topology really is responsible for atomic structure, then because the reduced azimuth φ in the FQHE fill factor $\nu = \mathcal{A}/\varphi$ is thereby related by $\varphi = 2j = 2l + 1 = 2l + 2s$ to the Casimir quantum numbers for total, orbital and spin angular momentum, the odd-integer FQHE denominators $\varphi = 1, 3, 5, 7, 9, 11...$ will characterize electrons in the respective orbital angular momentum states $l = 0, 1, 2, 3, 4, 5...$ for which the respective shorthands s, p, d, f, g, h are customarily employed. And, this also means that the $\varphi = 2$ denominator will characterize a pair of entangled electrons forming boson states summarized by (10.1).

All of this is consolidated into Figure 7 below, which is reproduced from [38], [39] and adapted from Figure 3 in [40], and which contains added annotations showing how each fractional denominator $\varphi = 1, 3, 5, 7, 9, 11...$ is expected to correlate with the orbital angular momentum states of s, p, d, f, g, h electrons and how $\varphi = 2$ is expected to correlate to spin 1 boson polarization states. The experiments proposed here, are then easily summarized: each of the odd-denominator fractional states shown in Figure 7 is predicted to correlate to the angular momentum states signified by the s, p, d, f, g, h, annotations, and the single state with even

denominator $\varphi = 2$ is predicted to correlate to the angular momentum states of a spin 1 boson. The $\nu = 2/3 = 6/9$ state may be of special interest, because it can be generated by both p orbital and g orbital electrons, and so it can be predicted that both of these orbital characteristics may be correlated to $\nu = 2/3$. Experimental tests which observe these angular momentum correlations would support the theoretical results presented here; while a finding that these correlations do not exist would provide contradiction to these theoretical results. Additionally, because superconductors with large *Z* on the periodic table naturally contain more d and f shell electrons than smaller-*Z* elements before any magnetic field is ever applied, it is predicted that the larger denominator FQHE states such as $\varphi = 5, 7, 9, 11...$ can be stimulated with smaller applied magnetic fields using higher-*Z* versus lower-*Z* superconductors, which is illustrated by the horizontal arrows showing how large-*Z* materials should stretch the magnetic field axis to the right, i.e., move the observation of smaller fractional charges to the left along the magnetic field axis. Finally, if the "noise" of the large perpendicular magnetic fields introduced in the FQHE experiments can somehow be separated from the "signal" of the DWY magnetic monopoles $\oiint F = \mu$ which are predicted by these results to exist near 0K and be tied to the FQHE by $e\mu = 2\pi(\mathcal{A}/\varphi) = 2\pi\nu$ – which might be assisted by the use of high-*Z* materials, then these experiments should most certainly also seek to directly observe these magnetic monopoles, which have been the object of inquiry and pursuit since ever the time of Maxwell and Heaviside.

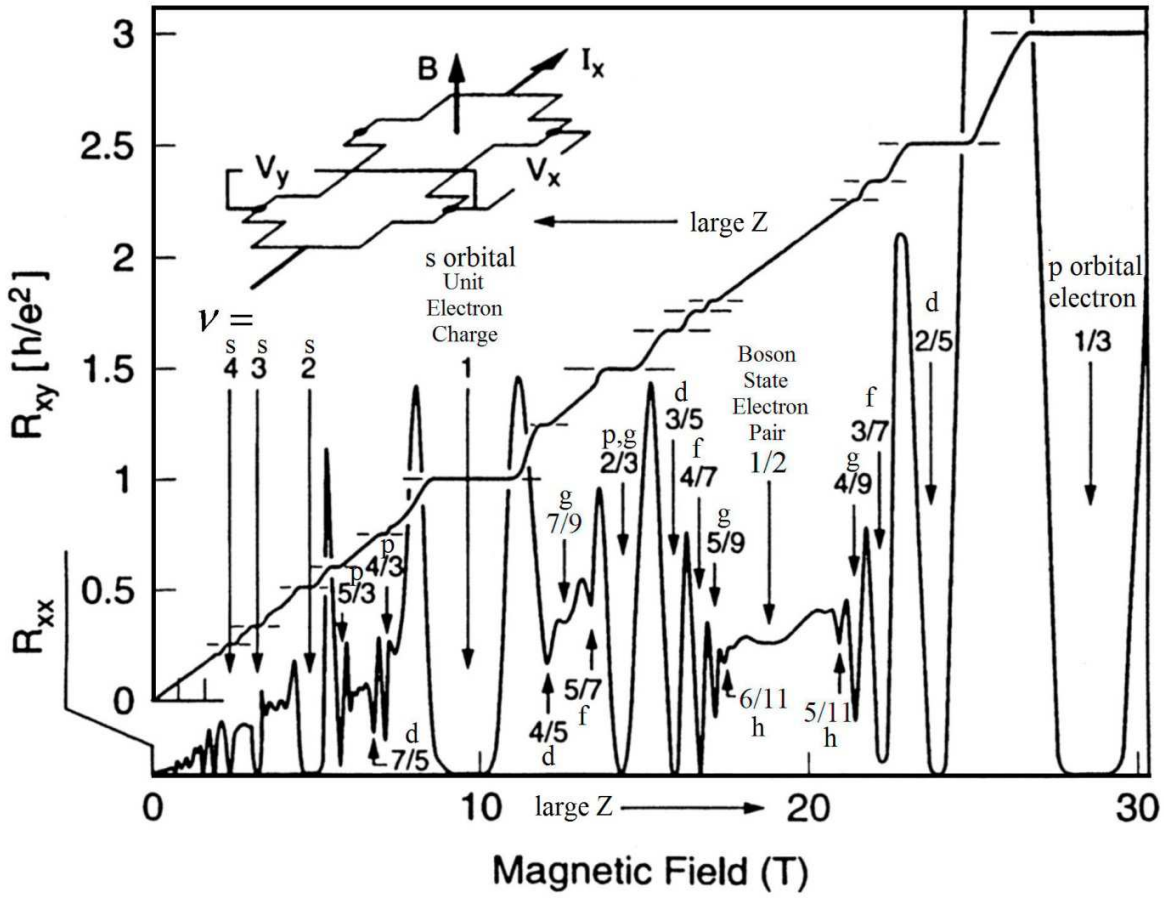

Figure 7: Figure 1: Fractional Quantum Hall Effect, reproduced from [38], [39], and adapted from [40], with added annotation

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