Quaternion Dynamics, Part 1 – Functions, Derivatives, and Integrals

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Summary

Definitions are presented for "quaternion functions" of a quaternion. Polynomial and exponential quaternion functions are presented. Derivatives and integrals of these quaternion functions are developed. It is shown that quaternion multiplication can be represented by matrix multiplication provided the matrix has a specific type of structure. It is also shown that differentiation and integration are similar to their non-quaternion applications.

Preface

Knowledge of quaternions and Linear Algebra is required.

Discussion

In Part II of "Elements of Quaternions", Hamilton develops the Calculus of quaternions. His thinking in that text seems to have been restricted to situations where the coefficients of a quaternion are functions of an independent parameter such as time. It does not appear that he considered the possibility of a quaternion being a function of another quaternion. It also does not appear that he considered the derivative of a quaternion taken with respect to a quaternion. The objective of this text is to extend Hamilton's work to include these possibilities.

Development

Background:

Let us begin by defining quaternions **X** and **Y** such that:

$$\mathbf{X} = x_0 + x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k} = x_0 + \mathbf{x}$$

and

$$\mathbf{Y} = y_0 + y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k} = y_0 + \mathbf{y}$$

Here, a quaternion is designated by a **bold-faced** CAPITAL letter. A vector is designated by a **bold-faced** lower-case letter. A scalar is designated by a lower-case letter in regular font.

Quaternions are used to describe physical, three-dimensional space. The unit vectors **i**, **j**, and **k** form a three-dimensional coordinate system. The six values x_i, x_j, x_k, y_i, y_j, and y_k each represent distances in the direction of the respective unit vectors. But what of the values x_0 and y_0? The author speculates that these values are starting points. Therefore, the physical position denoted by quaternion **X** is the result of beginning at point x_0 and moving distances x_i, x_j, and x_k in the direction of the respective unit vectors. A similar statement is true for quaternion **Y**. Thought of in this way, the values for x_0 and y_0 default to zero.

Now let us suppose that **X** and **Y** represent separate coordinate systems. Let us further suppose that the **i**-**j**-**k** orientations of the coordinate systems are equal. Essentially, the systems are the same except they have different starting points (i.e., the coordinate systems are translated). For such a system, the starting points x_0 and y_0 are related as follows:

$$y_0 = x_0 + \Delta\mathbf{x}_0 \; and \; x_0 = y_0 + \Delta\mathbf{y}_0$$

These relations allow either **X** or **Y** to be expressed with respect to the reference point of the other coordinate system. The vector difference associated with the translation is merged with the other vector terms of the respective quaternion.

Quaternions were developed by Hamilton as a method of expressing the ratio between two vectors. Therefore, let us now consider that application. Let the quaternion **Q** represent the ratio between vector **y** and vector **x**. The quaternion **Q** is therefore:

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k} = \frac{\mathbf{y}}{\mathbf{x}} = \frac{y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k}}{x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k}};\ x_0 = y_0 = 0$$

If both sides are multiplied (on the right-hand side of **Q**) by the denominator, the result is:

Equation 0:

$$\mathbf{Qx} = \mathbf{y}$$

$$\left(q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}\right)\left(x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k}\right) = y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k}$$

When this multiplication is performed, the result is the following system of equations:

$$-\left(q_i x_i + q_j x_j + q_k x_k\right) = 0$$

$$\left(q_0 x_i + q_j x_k - q_k x_j\right)\mathbf{i} = y_i\mathbf{i}$$

$$\left(q_0 x_j - q_i x_k + q_k x_i\right)\mathbf{j} = y_j\mathbf{j}$$

$$\left(q_0 x_k + q_i x_j - q_j x_i\right)\mathbf{k} = y_k\mathbf{k}$$

The objective is to solve for the four terms of **Q**. Therefore, this system is expressed as the following matrix multiplication:

Equation 0.1:

$$\begin{bmatrix} 0 & -x_i & -x_j & -x_k \\ +x_i & 0 & +x_k & -x_j \\ +x_j & -x_k & 0 & +x_i \\ +x_k & +x_j & -x_i & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

If the transpose of the coefficient matrix is pre-multiplied by the coefficient matrix, the result is a diagonal matrix as follows:

$$\begin{bmatrix} 0 & +x_i & +x_j & +x_k \\ -x_i & 0 & -x_k & +x_j \\ -x_j & +x_k & 0 & -x_i \\ -x_k & -x_j & +x_i & 0 \end{bmatrix}\begin{bmatrix} 0 & -x_i & -x_j & -x_k \\ +x_i & 0 & +x_k & -x_j \\ +x_j & -x_k & 0 & +x_i \\ +x_k & +x_j & -x_i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\left(x_i^2 + x_j^2 + x_k^2\right)$$

Therefore, the inverse of the coefficient matrix is simply the transpose matrix divided by the sum of the squares of x_i, x_j, and x_k. This allows the coefficients of quaternion **Q** to be determined as follows:

Equation 0.2:

$$\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \frac{1}{x_i^2 + x_j^2 + x_k^2} \begin{bmatrix} 0 & +x_i & +x_j & +x_k \\ -x_i & 0 & -x_k & +x_j \\ -x_j & +x_k & 0 & -x_i \\ -x_k & -x_j & +x_i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

Equation 0.2.1:

$$q_0 = \frac{x_i y_i + x_j y_j + x_k y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.2.2:

$$q_i = \frac{-x_k y_j + x_j y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.2.3

$$q_j = \frac{x_k y_i - x_i y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.2.4:

$$q_k = \frac{-x_j y_i + x_i y_j}{x_i^2 + x_j^2 + x_k^2}$$

When discussing the interpretation of the scalar term of **X**, the author speculated that the x_0 term is the starting point and the vector portion of **X** is added to that point to produce a final position. So then, what is the meaning of Equation 0.2.1 since it describes the scalar value q_0 produced by the ratio **y/x**? The units of measure (or lack thereof) associated with Equations 0.2.1 – 0.2.4 are a clue. The dimensions of the individual x terms and y terms are all length. Therefore, q_0, q_i, q_j, and q_k are each dimensionless since they have units of $length^2/length^2$. This should not be a surprise since **Q** is the ratio between two space vectors and the vectors' units of measure (length) should cancel when the ratio is taken. Therefore, the terms of **Q** do not have a *spatial* meaning. The meaning of **Q** is simply to convert between **x** and **y**. Pre-multiplication of vector **x** by quaternion **Q** maps vector **x** into vector **y**.

There is something very noteworthy regarding the terms of **Q**. The sum of the three xy terms in Equation 0.2.1 is the dot product of the vector **x** and the vector **y**. The other xy terms in Equations 0.2.2 - 0.2.4 form the cross product of the vector **x** and the vector **y**. The sum of the squares of the three terms of **x** is the square of the length of vector **x**. Therefore, quaternion **Q** can be expressed as follows:

Equation 0.2.5:

$$\mathbf{Q}_R = \frac{\mathbf{y}}{\mathbf{x}} = \frac{1}{||\mathbf{x}||^2}(\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \times \mathbf{y});\ q_0 = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||^2}\ and\ \mathbf{q} = \frac{\mathbf{x} \times \mathbf{y}}{||\mathbf{x}||^2}$$

The subscript R is being used to designate right-hand side multiplication. The author will note that both Newton's Law of Gravity and Coulomb's Law of Electrostatics are inversely proportional to the square of the distance. Equation 0.2.5 incorporates this feature with the added bonus of both scalar and vector forms. Therefore, Equation 0.2.5 could be used to describe - but not explain - both forces provided the vectors **x** and **y** are properly selected and provided consideration is given to the system units of measure and the direction of action. The author will also note that in electro-magnetism, the Lorentz Force is described as **F** = q(**E** + **v** x **B**) where **F** is the resulting force, q is the particle charge, **E** is the electric field, **v** is the particle velocity, and **B** is the magnetic field. It seems clear to the author that the quaternion mathematics of Hamilton is a tool that can *describe* many of the observed physical laws of nature.

Next let us consider the case where **Q** is the ratio between two quaternions.

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k} = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{y_0 + y_i\mathbf{i} + y_j\mathbf{j} + y_k\mathbf{k}}{x_0 + x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k}}$$

$$\mathbf{QX} = \mathbf{Y}$$

Repeating the exercise above produces the following system of equations:

Equation 0.3:

$$\begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & +x_k & -x_j \\ +x_j & -x_k & +x_0 & +x_i \\ +x_k & +x_j & -x_i & +x_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

If the transpose of the coefficient matrix is pre-multiplied by the coefficient matrix, the result is a diagonal matrix as follows:

$$\begin{bmatrix} +x_0 & +x_i & +x_j & +x_k \\ -x_i & +x_0 & -x_k & +x_j \\ -x_j & +x_k & +x_0 & -x_i \\ -x_k & -x_j & +x_i & +x_0 \end{bmatrix} \begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & +x_k & -x_j \\ +x_j & -x_k & +x_0 & +x_i \\ +x_k & +x_j & -x_i & +x_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(x_0^2 + x_i^2 + x_j^2 + x_k^2\right)$$

Therefore, the inverse of the coefficient matrix is simply the transpose matrix divided by the sum of the four squares. This allows the coefficients of quaternion **Q** to be determined as follows:

Equation 0.4:

$$\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \frac{1}{x_0^2 + x_i^2 + x_j^2 + x_k^2} \begin{bmatrix} +x_0 & +x_i & +x_j & +x_k \\ -x_i & +x_0 & -x_k & +x_j \\ -x_j & +x_k & +x_0 & -x_i \\ -x_k & -x_j & +x_i & +x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

Equation 0.4.1:

$$q_0 = \frac{x_0 y_0 + \left(x_i y_i + x_j y_j + x_k y_k\right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.4.2:

$$q_i = \frac{-x_i y_0 + x_0 y_i + \left(-x_k y_j + x_j y_k\right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.4.3

$$q_j = \frac{-x_j y_0 + x_0 y_j + (x_k y_i - x_i y_k)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.4.4:

$$q_k = \frac{-x_k y_0 + x_0 y_k + \left(-x_j y_i + x_i y_j\right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Therefore, the quaternion **Q** can be expressed as follows:

Equation 0.4.5:

$$\mathbf{Q}_R = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{1}{x_0^2 + x_i^2 + x_j^2 + x_k^2}(x_0 y_0 + x_0\mathbf{y} - y_0\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\times\mathbf{y})$$

A consequence of this analysis is that the units of measure associated with the scalar x_0 must be the same as the units of measure associated with the vector **x**. A similar requirement is placed upon the scalar y_0 and the vector **y**.

In Equation 0, it would have been equally valid to place the quaternion **X** to the left of **Q** instead of to the right of **Q**. Therefore, this exercise will be repeated for that form. Equation 0.1 becomes:

Equation 0.5:

$$\begin{bmatrix} 0 & -x_i & -x_j & -x_k \\ +x_i & 0 & -x_k & +x_j \\ +x_j & +x_k & 0 & -x_i \\ +x_k & -x_j & +x_i & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

Multiplication by the inverse matrix then produces:

Equation 0.6:

$$\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \frac{1}{x_i^2 + x_j^2 + x_k^2} \begin{bmatrix} 0 & +x_i & +x_j & +x_k \\ -x_i & 0 & +x_k & -x_j \\ -x_j & -x_k & 0 & +x_i \\ -x_k & +x_j & -x_i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

This then allows the values for the coefficients of **Q** to be determined as follows:

Equation 0.6.1:

$$q_0 = \frac{x_i y_i + x_j y_j + x_k y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.6.2:

$$q_i = \frac{x_k y_j - x_j y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.6.3:

$$q_j = \frac{-x_k y_i + x_i y_k}{x_i^2 + x_j^2 + x_k^2}$$

Equation 0.6.4:

$$q_k = \frac{x_j y_i - x_i y_j}{x_i^2 + x_j^2 + x_k^2}$$

A quick comparison between the right-hand **Q** and the left-hand **Q** indicates that they share the same scalar term but that they have opposite vector terms. Therefore,

Equation 0.6.5:

$$\mathbf{Q}_L = \frac{\mathbf{y}}{\mathbf{x}} = \frac{1}{||\mathbf{x}||^2}(\mathbf{x}\cdot\mathbf{y} - \mathbf{x}\times\mathbf{y});\ q_0 = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||^2}\ and\ \mathbf{q} = \frac{\mathbf{x}\times\mathbf{y}}{||\mathbf{x}||^2}$$

The subscript L is being used to designate left-hand side multiplication. The quaternions $\mathbf{Q}_R$ and $\mathbf{Q}_L$ are complex conjugates.

Continuing the left-hand side evaluation for the ratio between two quaternions produces the following:

Equation 0.7:

$$\begin{bmatrix} q_0 \\ q_i \\ q_j \\ q_k \end{bmatrix} = \frac{1}{x_0^2 + x_i^2 + x_j^2 + x_k^2} \begin{bmatrix} +x_0 & +x_i & +x_j & +x_k \\ -x_i & +x_0 & +x_k & -x_j \\ -x_j & -x_k & +x_0 & +x_i \\ -x_k & +x_j & -x_i & +x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix}$$

This then allows the values for the coefficients of **Q** to be determined as follows:

Equation 0.7.1:

$$q_0 = \frac{x_0 y_0 + \left(x_i y_i + x_j y_j + x_k y_k \right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.7.2:

$$q_i = \frac{-x_i y_0 + x_0 y_i + \left(x_k y_j - x_j y_k \right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.7.3:

$$q_j = \frac{-x_j y_0 + x_0 y_j + \left(-x_k y_i + x_i y_k \right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Equation 0.7.4:

$$q_k = \frac{-x_k y_0 + x_0 y_k + \left(x_j y_i - x_i y_j \right)}{x_0^2 + x_i^2 + x_j^2 + x_k^2}$$

Therefore, the quaternion **Q** can be expressed as follows:

Equation 0.7.5:

$$\mathbf{Q}_L = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{1}{x_0^2 + x_i^2 + x_j^2 + x_k^2} \left(x_0 y_0 - x_0 \mathbf{y} + y_0 \mathbf{x} + \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \times \mathbf{y} \right)$$

Next, let us define a generic quaternion function **F**(**X**) such that **Y** = **F**(**X**). In principle, **F**(**X**) could be anything although the author is primarily considering functions such as polynomials and exponentials. The author will not attempt to develop the sine and cosine functions. Matrices will be used extensively. The utility of the matrix formulation will become apparent during the discussion of differentiation and integration.

Polynomials:

In algebra, one of the simplest functions is the line. This is usually presented as y = mx + b. The quaternion equivalent of this is:

$$\mathbf{Y} = \mathbf{MX} + \mathbf{B}$$

The multiplication **MX** is as follows:

$$\mathbf{MX} = (m_0 + m_i\mathbf{i} + m_j\mathbf{j} + m_k\mathbf{k})(x_0 + x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k})$$

The detailed steps are as follows:

$$(m_0x_0 - m_ix_i - m_jx_j - m_kx_k) +$$

$$(m_ix_0 + m_0x_i - m_kx_j + m_jx_k)\mathbf{i} +$$

$$(m_jx_0 + m_kx_i + m_0x_j - m_ix_k)\mathbf{j} +$$

$$(m_kx_0 - m_jx_i + m_ix_j + m_0x_k)\mathbf{k}$$

The next concept is absolutely crucial to understanding all of the material presented here. The quaternion multiplication **MX** can be represented as a matrix multiplication as follows:

$$\mathbf{MX} = \begin{bmatrix} +m_0 & -m_i & -m_j & -m_k \\ +m_i & +m_0 & -m_k & +m_j \\ +m_j & +m_k & +m_0 & -m_i \\ +m_k & -m_j & +m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix} = [m][x]$$

The quaternion characteristics (i.e., the values 1, **i**, **j**, and **k**) are contained in the column matrix [x]. The square matrix [m] contains only scalar values. Formulating the problem is this manner allows the matrix methods of linear algebra to be combined with quaternions. Now, the quaternion "line" can be expressed as follows:

Equation 1:

$$\mathbf{Y} = \mathbf{MX} + \mathbf{B} = \begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix} = \begin{bmatrix} +m_0 & -m_i & -m_j & -m_k \\ +m_i & +m_0 & -m_k & +m_j \\ +m_j & +m_k & +m_0 & -m_i \\ +m_k & -m_j & +m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix} + \begin{bmatrix} b_0 \\ b_i \\ b_j \\ b_k \end{bmatrix} = [y] = [m][x] + [b]$$

The author requests that the reader carefully examine the 4x4 matrix [m] above. It should be noted that the matrix [m] consists of four 2x2 matrices as follows:

Equation 1.1:

$$[m] = \begin{bmatrix} +[a] & -[b] \\ +[b] & +[a] \end{bmatrix}$$

Where

Equation 1.1.1:

$$[a] = \begin{bmatrix} +m_0 & -m_i \\ +m_i & +m_0 \end{bmatrix}; and\ [b] = \begin{bmatrix} +m_j & +m_k \\ +m_k & -m_j \end{bmatrix}$$

This is a special type of symmetry that appears in Linear Algebra when dealing with matrix inversion.

Equation 1 can be solved for **X** as follows:

$$[m]^{-1}([y] - [b]) = [x]$$

where [y], [m], [x], and [b] are as presented in Equation 1. This requires the determination of the inverse of matrix [m]. At first glance, this would appear to be difficult, since it requires solving for the 16 terms of the inverse matrix. Fortunately, the symmetry of the problem (as presented in Equation 1.1) simplifies this greatly. Multiplication of the coefficient matrix by its transpose produces a diagonal matrix. For ease of reading, the author will substitute the letters A, B, C, and D for the terms of the coefficient matrix.

$$[m]^T[m] = [?]$$

$$\begin{bmatrix} +A & +B & +C & +D \\ -B & +A & +D & -C \\ -C & -D & +A & +B \\ -D & +C & -B & +A \end{bmatrix} \begin{bmatrix} +A & -B & -C & -D \\ +B & +A & -D & +C \\ +C & +D & +A & -B \\ +D & -C & +B & +A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (A^2 + B^2 + C^2 + D^2)$$

Therefore, the inverse matrix is simply the transpose matrix divided by the sum of the four squares.

Equation 1.2:

$$[m]^{-1} = \frac{1}{m_0^2 + m_i^2 + m_j^2 + m_k^2}[m]^T = \frac{1}{m_0^2 + m_i^2 + m_j^2 + m_k^2} \begin{bmatrix} +m_0 & +m_i & +m_j & +m_k \\ -m_i & +m_0 & +m_k & -m_j \\ -m_j & -m_k & +m_0 & +m_i \\ -m_k & +m_j & -m_i & +m_0 \end{bmatrix}$$

Now let us very briefly consider the complex conjugate of **M**. The conjugate is usually denoted using the asterisk symbol (*). The complex conjugate of **M** is therefore denoted as **M***. Conjugates are defined as follows:

Equation 1.2.1:

$$\mathbf{M} = m_0 + \mathbf{m};\ \mathbf{M}^* = m_0 - \mathbf{m}$$

The term **m** represents the vector portion of **M**. The conjugate is noteworthy because $[m]^T$ = [m*]. For the special case where the transpose matrix is also the inverse matrix, the sum of the four squares must be equal to one. This follows directly from Equation 1.2.

For quaternions, the order of multiplication is important. Therefore, the reader might reasonably ask what would be the result if the order of multiplication in Equation 1 were to be changed from **MX** to **XM** instead? The multiplication **XM** is as follows:

$$\mathbf{XM} = (x_0 + x_i\mathbf{i} + x_j\mathbf{j} + x_k\mathbf{k})(m_0 + m_i\mathbf{i} + m_j\mathbf{j} + m_k\mathbf{k})$$

The detailed steps are as follows:

$$(x_0m_0 - x_im_i - x_jm_j - x_km_k) +$$

$$(x_im_0 + x_0m_i - x_km_j + x_jm_k)\mathbf{i} +$$

$$(x_jm_0 + x_km_i + x_0m_j - x_im_k)\mathbf{j} +$$

$$(x_km_0 - x_jm_i + x_im_j + x_0m_k)\mathbf{k}$$

If the terms are rearranged so as to preserve the **MX** matrix form of Equation 1, these detailed steps become:

$$(x_0m_0 - x_im_i - x_jm_j - x_km_k) +$$

$$(x_0m_i + x_im_0 + x_jm_k - x_km_j)\mathbf{i} +$$

$$(x_0m_j - x_im_k + x_jm_0 + x_km_i)\mathbf{j} +$$

$$(x_0m_k + x_im_j - x_jm_i + x_km_0)\mathbf{k}$$

This multiplication is then expressed as a matrix as follows:

$$\begin{bmatrix} +m_0 & -m_i & -m_j & -m_k \\ +m_i & +m_0 & +m_k & -m_j \\ +m_j & -m_k & +m_0 & +m_i \\ +m_k & +m_j & -m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

This coefficient matrix does not match either the coefficient matrix from Equation 1 or the transpose of that matrix. Instead, this matrix is a curious blend of the two. Column 1 and row 1 are from the original coefficient matrix. The remainder is from the transpose.

This makes the author curious regarding the possible forms that the **MX** multiplication of Equation 1 can take. These are **MX**, **M*X**, **XM**, and **XM***. They can be represented by the following matrix multiplications:

$$\mathbf{MX} = \begin{bmatrix} +m_0 & -m_i & -m_j & -m_k \\ +m_i & +m_0 & -m_k & +m_j \\ +m_j & +m_k & +m_0 & -m_i \\ +m_k & -m_j & +m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

$$\mathbf{M}^*\mathbf{X} = \begin{bmatrix} +m_0 & +m_i & +m_j & +m_k \\ -m_i & +m_0 & +m_k & -m_j \\ -m_j & -m_k & +m_0 & +m_i \\ -m_k & +m_j & -m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

$$\mathbf{XM} = \begin{bmatrix} +m_0 & -m_i & -m_j & -m_k \\ +m_i & +m_0 & +m_k & -m_j \\ +m_j & -m_k & +m_0 & +m_i \\ +m_k & +m_j & -m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

$$\mathbf{XM}^* = \begin{bmatrix} +m_0 & +m_i & +m_j & +m_k \\ -m_i & +m_0 & -m_k & +m_j \\ -m_j & +m_k & +m_0 & -m_i \\ -m_k & -m_j & +m_i & +m_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

The author anticipates that these forms will be useful in the future. If the four coefficient matrices are added together, the result is a diagonal matrix with the value $4m_0$ along the diagonal.

Next, let us consider a simple quadratic such as **Y** = $\mathbf{X}^2$. This can be thought of as being **Y** = **XX**. Therefore, Equation 1 can be used with **M** = **X** and **B** = **0**.

$$\mathbf{Y} = \mathbf{X}^2 = \begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & -x_k & +x_j \\ +x_j & +x_k & +x_0 & -x_i \\ +x_k & -x_j & +x_i & +x_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

Now suppose that **Y** = $\mathbf{X}^3$. This can be written as **Y** = $\mathbf{XX}^2$. This can then be combined with the relation for $\mathbf{X}^2$ to produce:

$$\mathbf{Y} = \mathbf{X}^3 = \begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & -x_k & +x_j \\ +x_j & +x_k & +x_0 & -x_i \\ +x_k & -x_j & +x_i & +x_0 \end{bmatrix}^2 \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

By logical extension, it follows that:

Equation 1.3:

$$\mathbf{Y} = \mathbf{X}^N = \begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & -x_k & +x_j \\ +x_j & +x_k & +x_0 & -x_i \\ +x_k & -x_j & +x_i & +x_0 \end{bmatrix}^{N-1} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix}$$

For N = 1, the coefficient matrix becomes the Identity Matrix. For N = 0, the coefficient matrix becomes the Inverse Matrix. The value of N is restricted to integer values since the (N - 1) term that is used as an exponent on the coefficient matrix must be an integer.

The above relations make it possible to produce a generic quaternion-based polynomial as follows:

Equation 1.4:

$$\mathbf{Y} = \mathbf{A}_N\mathbf{X}^N + \mathbf{A}_{N-1}\mathbf{X}^{N-1} + \cdots + \mathbf{A}_1\mathbf{X} + \mathbf{A}_0$$

where the various terms can be inferred from Equation 1 and Equation 1.3. As an example, consider the following:

$$\mathbf{Y} = \mathbf{A}\mathbf{X}^2 + \mathbf{B}\mathbf{X} + \mathbf{C}$$

$$\begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix} = \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k \\ +a_i & +a_0 & -a_k & +a_j \\ +a_j & +a_k & +a_0 & -a_i \\ +a_k & -a_j & +a_i & +a_0 \end{bmatrix} \begin{bmatrix} +x_0 & -x_i & -x_j & -x_k \\ +x_i & +x_0 & -x_k & +x_j \\ +x_j & +x_k & +x_0 & -x_i \\ +x_k & -x_j & +x_i & +x_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix} + \begin{bmatrix} +b_0 & -b_i & -b_j & -b_k \\ +b_i & +b_0 & -b_k & +b_j \\ +b_j & +b_k & +b_0 & -b_i \\ +b_k & -b_j & +b_i & +b_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_i \\ x_j \\ x_k \end{bmatrix} + \begin{bmatrix} c_0 \\ c_i \\ c_j \\ c_k \end{bmatrix}$$

Multiplication is associative for matrices and quaternions.

Exponentials:

Now let us consider exponentials of a quaternion. We will begin with Euler's Equation.

$$e^{ix} = \cos(x) + i\sin(x)$$

Since cos(-x) = cos(x) and sin(-x) = -sin(x), Euler's Equation can also be written as:

$$e^{-ix} = \cos(x) - i\sin(x)$$

The *i* (*italicized i*) term used in this equation is normally thought to be the complex *i* with it being defined by $i^2 = -1$. Part of Hamilton's definitions for the unit vectors **i**, **j**, and **k** is that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk}$. Given the similarity of these definitions, would it be correct to substitute one of Hamilton's unit vectors for the complex *i* in Euler's Equation? The author will present several pieces of evidence that support the affirmative. The essence of the author's argument is that the *behavior* of Euler's Equation remains true, but the complex *i* remains distinct from the unit vectors **i**, **j**, and **k**. The following few paragraphs will support this thinking.

Since $i^2 = \mathbf{i}^2$, the most that could be stated would be that $i = \pm\mathbf{i}$. If **i** is substituted into Euler's Equation for *i*, the ± behavior of the sine and cosine terms considers this, and Euler's Equation remains valid. The same reasoning applies for the unit vectors **j** and **k**.

For the unit vectors, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ (= i^2). Despite this definition, **i** ≠ **j** ≠ **k** since the unit vectors are in different directions. The unit vectors are distinct from each other despite being defined by the same equality. There is no reason to think that the complex *i* is not distinct also, despite being defined by the same equality.

Quaternions are composed of four terms with one term being scalar and the other three terms being vectors. It seems reasonable to think that the complex *i* is associated with the scalar term. This would provide some "symmetry" between the scalar and vector portions.

Consider the following:

$$\mathbf{j}e^{\mathbf{i}x} = \mathbf{j}[\cos(x) + \mathbf{i}\sin(x)] = \mathbf{j}\cos(x) - \mathbf{k}\sin(x)$$

Multiplication of Euler's Equation by one of the unit vectors can produce a vector as shown immediately above.

The dot product of two vectors is equal to the product of their lengths multiplied by the cosine of the angle between them. The cross product of two vectors is equal to the product of their lengths multiplied by the sine of the angle between them multiplied by a unit vector perpendicular to the vectors. Therefore, it is possible to construct Euler's Equation using the dot product and cross product of two arbitrary vectors as follows:

$$e^{\mathbf{i}x} = \cos(x) + \mathbf{i}\sin(x) = \frac{1}{\|\mathbf{a}\|\|\mathbf{b}\|}[\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b}]\,;\; a_i = 0, b_i = 0$$

In each example presented here, Euler's Equation works equally well whether the complex *i* is used or a unit vector is used. Therefore, the author is convinced that Euler's Equation can be considered to be a quaternion with a single vector term.

The next step is to extend Euler's Equation to include the unit vectors **j** and **k**. Ideally, the author hopes for a relationship similar to the following:

Equation 2:

$$e^{\mathbf{q}} = \cos(\theta_0) + \mathbf{i}\sin(\theta_i) + \mathbf{j}\sin(\theta_j) + \mathbf{k}\sin(\theta_k)$$

Where:

Equation 2.1:

$$\mathbf{q} = q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$$

Since Euler's Equation does not include a scalar term as part of the exponential, the author will exclude it for now. A scalar term can easily be incorporated in a later step. If Equation 2 and Equation 2.1 are combined, the result is:

Equation 2.2:

$$e^{\mathbf{q}} = e^{(q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k})} = e^{\mathbf{i}q_i}e^{\mathbf{j}q_j}e^{\mathbf{k}q_k} = \cos(\theta_0) + \mathbf{i}\sin(\theta_i) + \mathbf{j}\sin(\theta_j) + \mathbf{k}\sin(\theta_k)$$

Euler's Equation can be applied individually to the unit vectors as follows:

Equation 2.3:

$$e^{\mathbf{i}q_i}e^{\mathbf{j}q_j}e^{\mathbf{k}q_k} = [\cos(q_i) + \mathbf{i}\sin(q_i)][\cos(q_j) + \mathbf{j}\sin(q_j)][\cos(q_k) + \mathbf{k}\sin(q_k)]$$

The terms on the right-hand side of Equation 2 can now be determined by performing the multiplication presented in Equation 2.3. The results of this multiplication are as follows:

Equation 2.4.1:

$$\cos(\theta_0) = \cos(q_i)\cos(q_j)\cos(q_k) - \sin(q_i)\sin(q_j)\sin(q_k)$$

Equation 2.4.2:

$$\sin(\theta_i) = \sin(q_i)\cos(q_j)\cos(q_k) + \cos(q_i)\sin(q_j)\sin(q_k)$$

Equation 2.4.3:

$$\sin(\theta_j) = \cos(q_i)\sin(q_j)\cos(q_k) - \sin(q_i)\cos(q_j)\sin(q_k)$$

Equation 2.4.4:

$$\sin(\theta_k) = \sin(q_i)\sin(q_j)\cos(q_k) + \cos(q_i)\cos(q_j)\sin(q_k)$$

This system of equations can also be represented by the following matrix multiplication (see **Polynomials** above):

Equation 2.5:

$$e^{\mathbf{q}} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_i \\ \sin\theta_j \\ \sin\theta_k \end{bmatrix} = \begin{bmatrix} \cos q_i & -\sin q_i & 0 & 0 \\ \sin q_i & \cos q_i & 0 & 0 \\ 0 & 0 & \cos q_i & -\sin q_i \\ 0 & 0 & \sin q_i & \cos q_i \end{bmatrix} \begin{bmatrix} \cos q_j & 0 & -\sin q_j & 0 \\ 0 & \cos q_j & 0 & \sin q_j \\ \sin q_j & 0 & \cos q_j & 0 \\ 0 & -\sin q_j & 0 & \cos q_j \end{bmatrix} \begin{bmatrix} \cos q_k \\ 0 \\ 0 \\ \sin q_k \end{bmatrix}$$

A generic quaternion exponential is then produced as follows:

Equation 2.6:

$$e^{\mathbf{Q}} = e^{q_0+\mathbf{q}} = e^{q_0}e^{\mathbf{q}}$$

where:

Equation 2.6.1:

$$\mathbf{Q} = q_0 + \mathbf{q}$$

Therefore, the exponential of q_0 becomes a pre-multiplier for Equation 2.5 as follows:

Equation 2.7:

$$e^{\mathbf{Q}} = e^{q_0}\begin{bmatrix} \cos\theta_0 \\ \sin\theta_i \\ \sin\theta_j \\ \sin\theta_k \end{bmatrix} = e^{q_0}\begin{bmatrix} \cos q_i & -\sin q_i & 0 & 0 \\ \sin q_i & \cos q_i & 0 & 0 \\ 0 & 0 & \cos q_i & -\sin q_i \\ 0 & 0 & \sin q_i & \cos q_i \end{bmatrix} \begin{bmatrix} \cos q_j & 0 & -\sin q_j & 0 \\ 0 & \cos q_j & 0 & \sin q_j \\ \sin q_j & 0 & \cos q_j & 0 \\ 0 & -\sin q_j & 0 & \cos q_j \end{bmatrix} \begin{bmatrix} \cos q_k \\ 0 \\ 0 \\ \sin q_k \end{bmatrix}$$

In the discussion of the complex *i*, the author speculated that the complex *i* was associated with the scalar term. Equation 2.7 supports this. The complex *i* could operate upon the q_0 term.

Differentiation:

In this section the author will discuss and develop a method of differentiation as it applies to the quaternion functions presented above. The author begins with a classical definition followed by development of the differential operator. The author then presents the derivatives of the functions presented above.

The first objective is to determine a meaning for the 1st derivative d**Y**/d**X**. The definition of the 1st derivative is:

Equation 3:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\Delta\mathbf{X}\to 0}\left[\frac{\mathbf{Y}(\mathbf{X} + \Delta\mathbf{X}) - \mathbf{Y}(\mathbf{X})}{\Delta\mathbf{X}}\right]$$

The difficulty of this equation is that both **X** and **Y** contain four terms. The author will present two methods of determining the derivative.

Method 1: Since both the numerator and the denominator of the right-hand side of Equation 3 are quaternions, and since the ratio between two quaternions is also a quaternion, the derivative on the left-hand side must be a quaternion. Therefore, let us make the following definitions:

Equation 3.1:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \mathbf{F}'(\mathbf{X}) = f'_0 + f'_i\mathbf{i} + f'_j\mathbf{j} + f'_k\mathbf{k} = \frac{d}{d\mathbf{X}}\begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix} = \begin{bmatrix} f'_0 \\ f'_i \\ f'_j \\ f'_k \end{bmatrix}$$

and

Equation 3.2:

$$d\mathbf{X} = dx_0 + dx_i\mathbf{i} + dx_j\mathbf{j} + dx_k\mathbf{k}$$

Multiplying Equation 3.1 by this definition for d**X** produces:

Equation 3.3:

$$d\mathbf{Y} = \mathbf{F}'(\mathbf{X})d\mathbf{X} = \left(f'_0 + f'_i\mathbf{i} + f'_j\mathbf{j} + f'_k\mathbf{k}\right)\left(dx_0 + dx_i\mathbf{i} + dx_j\mathbf{j} + dx_k\mathbf{k}\right)$$

There is some ambiguity here since the order of multiplication is important for the unit vectors and the author has *chosen* to place the d**X** to the right of **F**'(**X**).

Now let us define d**Y** as follows:

Equation 3.4:

$$d\mathbf{Y} = dy_0 + dy_i\mathbf{i} + dy_j\mathbf{j} + dy_k\mathbf{k}$$

Substituting this into Equation 3.3 produces:

Equation 3.5:

$$dy_0 + dy_i\mathbf{i} + dy_j\mathbf{j} + dy_k\mathbf{k} = \left(f'_0 + f'_i\mathbf{i} + f'_j\mathbf{j} + f'_k\mathbf{k}\right)\left(dx_0 + dx_i\mathbf{i} + dx_j\mathbf{j} + dx_k\mathbf{k}\right)$$

When the multiplication on the right-hand side is performed, the result is a system of four simultaneous equations that can be represented in matrix form as follows:

Equation 3.6:

$$\begin{bmatrix} +f'_0 & -f'_i & -f'_j & -f'_k \\ +f'_i & +f'_0 & -f'_k & +f'_j \\ +f'_j & +f'_k & +f'_0 & -f'_i \\ +f'_k & -f'_j & +f'_i & +f'_0 \end{bmatrix} \begin{bmatrix} dx_0 \\ dx_i \\ dx_j \\ dx_k \end{bmatrix} = \begin{bmatrix} dy_0 \\ dy_i \\ dy_j \\ dy_k \end{bmatrix}$$

or as

Equation 3.7:

$$[f']d\mathbf{X} = d\mathbf{Y};\ [f'] = \frac{d\mathbf{Y}}{d\mathbf{X}}$$

Equation 3.7 is a compact way of expressing Equation 3.6. The elements of [f'] are all scalar values. The quaternion attributes are contained in d**X** and d**Y**. The derivative of **Y** with respect to **X** may be thought of as a quaternion or as a matrix. The matrix *structure* ensures that this is true. **Please note the following:** The coefficient matrix in Equation 3.6 has the same matrix structure as the coefficient matrix in Equation 1 in the discussion of **Polynomials**.

In multi-variable Calculus, there is a distinction between the total derivative and the various partial derivatives. The total derivative is defined as follows:

Equation 3.8:

$$dy = \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \cdots + \frac{\partial y}{\partial x_n}dx_n$$

Therefore, the coefficient matrix presented in Equation 3.6 can be represented by a system of partial derivatives as follows:

Equation 3.8.1:

$$\begin{bmatrix} +f'_0 & -f'_i & -f'_j & -f'_k \\ +f'_i & +f'_0 & -f'_k & +f'_j \\ +f'_j & +f'_k & +f'_0 & -f'_i \\ +f'_k & -f'_j & +f'_i & +f'_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial y_0}{\partial x_0} & \frac{\partial y_0}{\partial x_i} & \frac{\partial y_0}{\partial x_j} & \frac{\partial y_0}{\partial x_k} \\ \frac{\partial y_i}{\partial x_0} & \frac{\partial y_i}{\partial x_i} & \frac{\partial y_i}{\partial x_j} & \frac{\partial y_i}{\partial x_k} \\ \frac{\partial y_j}{\partial x_0} & \frac{\partial y_j}{\partial x_i} & \frac{\partial y_j}{\partial x_j} & \frac{\partial y_j}{\partial x_k} \\ \frac{\partial y_k}{\partial x_0} & \frac{\partial y_k}{\partial x_i} & \frac{\partial y_k}{\partial x_j} & \frac{\partial y_k}{\partial x_k} \end{bmatrix}$$

Now let us consider again the definition for the derivative presented by Equation 1.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\Delta\mathbf{X}\to 0}\left[\frac{\mathbf{Y}(\mathbf{X} + \Delta\mathbf{X}) - \mathbf{Y}(\mathbf{X})}{\Delta\mathbf{X}}\right]$$

It seems clear that this derivative can be determined by individually incrementing each of the terms of **Y** by each of the individual terms of **X**. The resulting matrix is a Jacobian Matrix. It also seems clear that the differential operator d/d**X** must operate as follows:

Equation 3.8.2:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \frac{d}{d\mathbf{X}}\begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix} = \begin{bmatrix} \frac{\partial y_0}{\partial x_0} & \frac{\partial y_0}{\partial x_i} & \frac{\partial y_0}{\partial x_j} & \frac{\partial y_0}{\partial x_k} \\ \frac{\partial y_i}{\partial x_0} & \frac{\partial y_i}{\partial x_i} & \frac{\partial y_i}{\partial x_j} & \frac{\partial y_i}{\partial x_k} \\ \frac{\partial y_j}{\partial x_0} & \frac{\partial y_j}{\partial x_i} & \frac{\partial y_j}{\partial x_j} & \frac{\partial y_j}{\partial x_k} \\ \frac{\partial y_k}{\partial x_0} & \frac{\partial y_k}{\partial x_i} & \frac{\partial y_k}{\partial x_j} & \frac{\partial y_k}{\partial x_k} \end{bmatrix} = \begin{bmatrix} f'_0 \\ f'_i \\ f'_j \\ f'_k \end{bmatrix}$$

or alternately

Equation 3.8.2.1:

$$d\mathbf{Y} = d\begin{bmatrix} y_0 \\ y_i \\ y_j \\ y_k \end{bmatrix} = \begin{bmatrix} \frac{\partial y_0}{\partial x_0} & \frac{\partial y_0}{\partial x_i} & \frac{\partial y_0}{\partial x_j} & \frac{\partial y_0}{\partial x_k} \\ \frac{\partial y_i}{\partial x_0} & \frac{\partial y_i}{\partial x_i} & \frac{\partial y_i}{\partial x_j} & \frac{\partial y_i}{\partial x_k} \\ \frac{\partial y_j}{\partial x_0} & \frac{\partial y_j}{\partial x_i} & \frac{\partial y_j}{\partial x_j} & \frac{\partial y_j}{\partial x_k} \\ \frac{\partial y_k}{\partial x_0} & \frac{\partial y_k}{\partial x_i} & \frac{\partial y_k}{\partial x_j} & \frac{\partial y_k}{\partial x_k} \end{bmatrix} d\mathbf{X} = \mathbf{F}' d\mathbf{X}$$

Higher order derivatives are determined by successive substitutions. For example, the 2nd derivative is determined as follows:

$$\frac{d^2\mathbf{Y}}{d\mathbf{X}^2} = \frac{d}{d\mathbf{X}}\begin{bmatrix} f'_0 \\ f'_i \\ f'_j \\ f'_k \end{bmatrix} = F''(\mathbf{X}) = \begin{bmatrix} f''_0 \\ f''_i \\ f''_j \\ f''_k \end{bmatrix}$$

and

$$\begin{bmatrix} +f''_0 & -f''_i & -f''_j & -f''_k \\ +f''_i & +f''_0 & -f''_k & +f''_j \\ +f''_j & +f''_k & +f''_0 & -f''_i \\ +f''_k & -f''_j & +f''_i & +f''_0 \end{bmatrix}\begin{bmatrix} dx_0 \\ dx_i \\ dx_j \\ dx_k \end{bmatrix} = \begin{bmatrix} df'_0 \\ df'_i \\ df'_j \\ df'_k \end{bmatrix}$$

Method 2: This method is based upon the definition presented by Equation 3.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\Delta\mathbf{X}\to 0}\left[\frac{\mathbf{Y}(\mathbf{X} + \Delta\mathbf{X}) - \mathbf{Y}(\mathbf{X})}{\Delta\mathbf{X}}\right]$$

Rather than use the Δ as a difference operator and then take the limit as it reduces to zero, the author proposes to multiply the quaternion **X** by a scalar value λ and then take the limit as λ goes to zero. This will assure that all of the coefficients of **X** go to zero together. Equation 3 is therefore re-written as follows:

Equation 3.9:

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda\to 0}\left[\frac{\mathbf{Y}(\mathbf{X} + \lambda\mathbf{X}) - \mathbf{Y}(\mathbf{X})}{\lambda\mathbf{X}}\right];\ \lambda \in R$$

Now let us begin the discussion regarding the differentials of quaternion polynomials and quaternion exponentials as described in the sections **Polynomials** and **Exponentials**. It will quickly be seen that the Equation 3.9 form of the definition is very simple to use.

We will begin with the constant function **Y** = **A**.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda\to 0}\left[\frac{\mathbf{A} - \mathbf{A}}{\lambda\mathbf{X}}\right] = \lim_{\lambda\to 0}\left[\frac{\mathbf{0}}{\lambda\mathbf{X}}\right] = \mathbf{0}$$

This is the zero quaternion.

The next function to consider is **Y** = **AX**.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda\to 0}\left[\frac{\mathbf{A}(\mathbf{X} + \lambda\mathbf{X}) - \mathbf{A}\mathbf{X}}{\lambda\mathbf{X}}\right] = \lim_{\lambda\to 0}\left[\frac{\lambda\mathbf{A}\mathbf{X}}{\lambda\mathbf{X}}\right] = \mathbf{A}$$

The next polynomial function is $\mathbf{Y} = \mathbf{AX}^2$.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda \to 0}\left[\frac{\mathbf{A}(\mathbf{X} + \lambda\mathbf{X})^2 - \mathbf{AX}^2}{\lambda\mathbf{X}}\right] = \lim_{\lambda \to 0}\left[\frac{(1+\lambda)^2\mathbf{AX}^2 - \mathbf{AX}^2}{\lambda\mathbf{X}}\right] = \lim_{\lambda \to 0}\left[\frac{(2\lambda + \lambda^2)\mathbf{AX}^2}{\lambda\mathbf{X}}\right] = 2\mathbf{AX}$$

The next function to consider is $\mathbf{Y} = \mathbf{AX}^3$.

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda \to 0}\left[\frac{\mathbf{A}(\mathbf{X} + \lambda\mathbf{X})^3 - \mathbf{AX}^3}{\lambda\mathbf{X}}\right] = \lim_{\lambda \to 0}\left[\frac{(1+\lambda)^3\mathbf{AX}^3 - \mathbf{AX}^3}{\lambda\mathbf{X}}\right] = \lim_{\lambda \to 0}\left[\frac{(3\lambda + 3\lambda^2 + \lambda^3)\mathbf{AX}^3}{\lambda\mathbf{X}}\right] = 3\mathbf{AX}^2$$

At this point, it is clear that quaternion differentials of quaternion polynomials behave exactly the same as their non-quaternion counterparts.

The next functions to consider are the exponentials. Let $\mathbf{Y} = e^{\mathbf{X}}$.

$$\mathbf{Y} = e^{\mathbf{X}}$$

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \lim_{\lambda \to 0}\frac{e^{\mathbf{X}+\lambda\mathbf{X}} - e^{\mathbf{X}}}{\lambda\mathbf{X}} = \lim_{\lambda \to 0}\frac{e^{\mathbf{X}}(e^{\lambda\mathbf{X}} - 1)}{\lambda\mathbf{X}} = e^{\mathbf{X}}\lim_{\lambda \to 0}\frac{(e^{\lambda\mathbf{X}} - 1)}{\lambda\mathbf{X}} = e^{\mathbf{X}}\lim_{\lambda \to 0}\frac{\mathbf{X}e^{\lambda\mathbf{X}}}{\mathbf{X}}$$

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = e^{\mathbf{X}}$$

The author used L' Hôpital's Rule in the above.

It seems that differentiation of a quaternion exponential is also identical to the non-quaternion counterpart.

Integration:

Given the similar behavior of differentiation for quaternion and non-quaternion functions, integration of a quaternion function is expected to behave like its non-quaternion counterpart. The only variations that the author expects concern the constant of integration and the limits of integration.

When indefinite integration is used, a constant of integration must be added to the result. For quaternion integration, this constant of integration must be a quaternion. It is of course possible that this constant quaternion has zero for its scalar and/or vector portions.

When definite integration is used, the limits of integration must satisfy the quaternion function. For example, consider the following:

$$\int_{\mathbf{T}_0}^{\mathbf{T}_1} \mathbf{V}\, d\mathbf{T} = \int_{\mathbf{X}_0}^{\mathbf{X}_1} d\mathbf{X}$$

This can easily be solved to give $\mathbf{V}(\mathbf{T}_1 - \mathbf{T}_0) = (\mathbf{X}_1 - \mathbf{X}_0)$. The puzzling aspect of this is that there are so very many values for **T** and **X** that cannot satisfy this equation. This will be more fully developed in future work.

The next observation concerns the order of multiplication of the integrand. The usual convention in Calculus would be to write the integrand as **V** d**T** rather than d**T** **V**. As was seen in the section on **Polynomials**, reversing the order of multiplication will produce the complex conjugate. Therefore, the complex conjugate is also a solution to the integration. Both solutions will satisfy the equation:

$$\mathbf{V} = \frac{d\mathbf{X}}{d\mathbf{T}}$$

The author anticipates that this can be applied to such concepts as quantum spin and uncertainty.

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