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A note on the harmonic series and the logarithm

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Abstract

A relationship between the harmonic series and the logarithm is presented.
The formula H(n)-log(n) for the Euler-Mascheroni constant is adopted accordingly.

$$\gamma \quad = \quad \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right) \quad - \quad \left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots\frac{1}{n+n^2}\right)$$

Figure 1: Illustration of H(n) and log(n) as part of H(n+n^2)

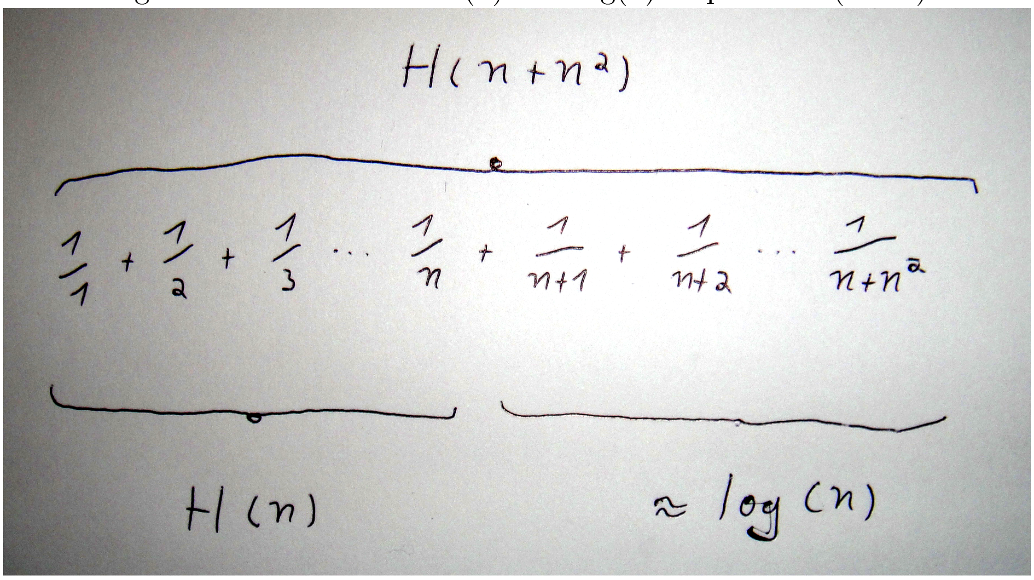
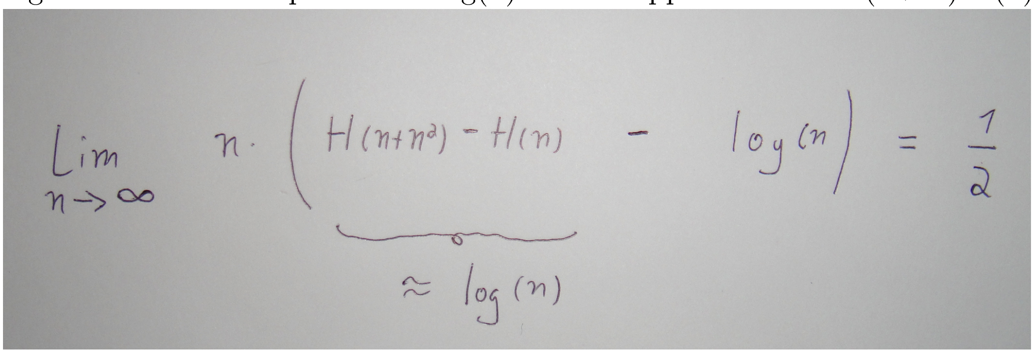


Figure 2: Relationship between log(n) and its approximation H(n+n^2)-H(n)



Mathematica Codes:

```
Limit[2*HarmonicNumber[n] - HarmonicNumber[n + n*n], n -> Infinity]
Limit[n*(HarmonicNumber[n + n*n] - HarmonicNumber[n] - Log[n]), n -> Infinity]
```

Generalization:

The logarithm can be understood as part(s) of the harmonic series.
This is illustrated in Formula 1.

$$\lim_{N\to\infty} \underbrace{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{N}}_{H_N} + \underbrace{\frac{1}{N+1}\cdots\frac{1}{N+N^2}}_{\approx\ log(N)} + \cdots + \underbrace{\frac{1}{N+N^{N-1}+1}\cdots\frac{1}{N+N^N}}_{=\ log(N)} \tag{1}$$

And further illustrated in simplified form in Formula 2.

$$\lim_{N\to\infty} \underbrace{\frac{1}{1}+\cdots\frac{1}{N^1}}_{H_N} + \underbrace{\cdots\frac{1}{N^2}}_{\approx\ log(N)} + \underbrace{\cdots\frac{1}{N^3}}_{\approx\ log(N)} + \underbrace{\cdots\frac{1}{N^4}}_{\approx\ log(N)} + \quad\cdots\quad + \underbrace{\cdots\frac{1}{N^N}}_{=\ log(N)} \tag{2}$$

Note on the Gamma approximation quality by harmonic log approximations:

$\gamma \approx H(n) - (H(n+n^2) - H(n))$

works better than any

$\gamma \approx H(n) - (H(n+n^3) - H(n+n^2))$

$\gamma \approx H(n) - (H(n+n^4) - H(n+n^3))$

$\gamma \approx H(n) - (H(n+n^5) - H(n+n^4))$

$\vdots$

$\gamma \approx H(n) - log(n)$