

# A note on the harmonic series and the logarithm

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## Abstract

A relationship between the harmonic series and the logarithm is presented. The formula  $H(n) - \log(n)$  for the Euler-Mascheroni constant is adopted accordingly.

$$\gamma = \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) - \left( \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n^2} \right)$$

Figure 1: Illustration of  $H(n)$  and  $\log(n)$  as part of  $H(n+n^2)$

$$H(n+n^2) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n^2}$$

$\underbrace{\hspace{15em}}_{H(n)} \quad \underbrace{\hspace{15em}}_{\approx \log(n)}$

Figure 2: Relationship between  $\log(n)$  and its approximation  $H(n+n^2) - H(n)$

$$\lim_{n \rightarrow \infty} n \cdot \left( \underbrace{H(n+n^2) - H(n)}_{\approx \log(n)} - \log(n) \right) = \frac{1}{2}$$

Mathematica Codes:

```
Limit[2*HarmonicNumber[n] - HarmonicNumber[n + n*n], n -> Infinity]
Limit[n*(HarmonicNumber[n + n*n] - HarmonicNumber[n] - Log[n]), n -> Infinity]
```

Generalization (with substitution of  $n + n^x$  by  $n^x$  for simplification)

Figure 3: Illustration of  $H(n)$  and  $\log(n)$  as part(s) of the Harmonic series

The diagram shows the harmonic series  $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  partitioned into four segments by vertical lines. Each segment is bracketed and labeled below:

- The first segment, containing  $\frac{1}{1} + \dots + \frac{1}{n}$ , is labeled  $H(n)$ .
- The second segment, containing  $\dots + \frac{1}{n^2} + \dots$ , is labeled  $\approx \log(n)$ .
- The third segment, containing  $\dots + \frac{1}{n^3} + \dots$ , is labeled  $\approx \log(n)$ .
- The fourth segment, containing  $\dots + \frac{1}{n^4} + \dots$ , is labeled  $\approx \log(n)$ .