I envisage an application of the theory of the decomposed extended Lie’s products to a simplified version of the Ising’s model (anisotropic quantum Heisenberg models).

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Content

A physical and concrete context for the theory ................................................................. 1
Entering in the domain of validity of the theory of the (E) question ..................................... 2
Definition 01: extended Lie’s product (or bracket) ............................................................... 2
A statement and a guess ....................................................................................................... 2
Analyzing the coherence of the guess ................................................................................ 3
Important precisions .......................................................................................................... 4
The concept of determinant ............................................................................................... 4
A better definition for extended Lie’s products ................................................................. 5
A better definition of the trivial split for block matrices .................................................... 5
“How can the initial theorem be applied to ELPs involving block matrices?” ..................... 6
Conclusion ......................................................................................................................... 6
Annex: the initial theorem ................................................................................................ 6
Bibliography ....................................................................................................................... 9

A physical and concrete context for the theory

A theoretical model is actually equivalent to a waste of time or to a mental masturbation if it doesn’t bring immediately a concrete application with it. This is the motivation for that document. The application will be a peculiar model of anisotropic quantum Heisenberg model (AQHM), namely the one of Ising. As explained in [01; p. 2 (An article to which I entirely loan the starting part of work)], the AQHM (citation): “… consists of N spin-1/2 degrees of freedom, each of which is interacting with every other at equal strength. The corresponding Hilbert space $\mathcal{H} = (\mathbb{C}^2)^\otimes N$ is the tensor product of N copies of the spin $\frac{1}{2}$ Hilbert space $\mathbb{C}^2$, and the Hamilton operator is given by [01; p. 2, (1)]’’… The Ising’s model just corresponds to a peculiar case for which that generic Hamilton’s operator is reduced to:

\[
H_H = -\frac{\lambda}{2N} \sum_{k=1}^{N} \sigma_3^k \cdot \sigma_3^l - h. \sum_{k=1}^{N} \sigma_3^k
\]

Here, $h$ is the strength of an external magnetic field orientated along the 3 axis; the $\sigma_3^k$ are N operators on $\mathcal{H}$ and act like the third Pauli’s spin $\frac{1}{2}$ matrix on the $k^{th}$ factor of the tensor product space $\mathcal{H}$, and like identity operators on all the other factors. The commutation relation is:

\[
[\sigma_3^k, \sigma_3^l] = 0
\]
Furthermore, $\lambda_3$ is the coupling strengths in the third (arbitrary) spatial directions and allows us to adjust the degree of anisotropy. Note that it is explicitly shown in [02] that anisotropic quantum Heisenberg models are among the systems that can be engineered with cold polar molecules in optical lattices. Short said: the Ising’s model (a) concerns a linear chain of molecules extending along the axis determinate by an external magnetic field and (b) the third Pauli’s spin $\frac{1}{2}$ matrix on the $k^{th}$ factor commutes with the third Pauli’s spin $\frac{1}{2}$ matrix on the $l^{th}$ factor. Here ends my debt relatively to [01].

### Entering in the domain of validity of the theory of the (E) question

**Definition 01: extended Lie’s product (or bracket)**

$$ \forall \mathbf{u}, \mathbf{w} \in E_3(\mathbb{C}), \left[ \mathbf{u}, \mathbf{w} \right]_{\nabla A} = \sum_{\alpha} \sum_{\beta} (\mathbf{u}^\alpha \cdot \mathbf{w}^\beta - \mathbf{w}^\alpha \cdot \mathbf{u}^\beta) \cdot A_{\alpha\beta} \cdot \mathbf{e}_i $$

The discussion here is developed in a three dimensional space. Extended Lie’s products (short: ELPs) are, *per definition*, built on anti-symmetric cubes ($A_{ab}^c + A_{ba}^c = 0$). The anti-symmetry reduces drastically the number of the eventually non-vanishing components. In a three dimensional space, the 27 components of a “normal” cube, usually denoted $\triangledown A^{(3)}$ are reduced to a matrix $A^{(3)}$ of $M_3(\mathbb{C})$. This fact motivates the notation $\left(A^{(3)}\right)^\wedge \left(p, t\right)$ instead of $\triangledown_{\nabla A}^{(3)} \left(p, t\right)$ characterizing the ELPS defined in that context. This modest change is not only a question of style. It will help later to understand the link with the concept of Lie derivative. (Remark: The extended exterior and the extended Lie’s products are related to each other by a factor $\frac{1}{4}$).

### A statement and a guess

The theory of the (E) question (TEQ) is concerned with the introduction of extended Lie’s products (ELPs) into physical domains of applications. Its initial context is some vector space. As it is already known $\mathbb{C}$ is such a space and the tensor product of two such spaces is such a space again; consequently: $\mathcal{H}$ is a space vector and we can hope to work with, with the spirit characterizing the TEQ.

Because of the so-called initial theorem (see *annex*), the second important ingredient allowing an introduction of the ELPS in a given theory is a polynomial of degree two, say something like: $f^{(3)}(\mathbf{t}) = u_{ik} \cdot t^i \cdot t^k + u_{ik} \cdot t^k + u^i$ (Einstein’s convention for the repeated indices). This being said, the reader may already guess where I am going.

Although the following limitation will immediately come in contradiction with the concept of long-range system, I shall reduce my first exploration to the case $N = 3$. This “a priori” has a pedagogical motivation. Indeed, within that restricted context, the relation (01) becomes:

(03) $$ H_3 = - \frac{\lambda_3}{6} \sum_{k,l=1}^3 \sigma^k_3 \cdot \sigma^l_3 - h \cdot \sum_{k=1}^3 \sigma^k_3 $$

And this is obviously a polynomial of degree two (PP2) inside $\mathcal{H}$. More precisely, this is a PP2 such that the three third Pauli’s spin $\frac{1}{2}$ matrices are its solutions: (04) $$ \left(\mathbf{t}\right) = (\sigma^1_3, \sigma^2_3, \sigma^3_3) $$

where the index $\ldots, 1, 2, 3$ are in fact labeling the molecule (first, second and third one) on the third axis.
The theory of the (E) question

Introducing decomposed extended Lie's products into the Ising's model – v1


\[ f((3)^t) = \lambda \sum_{k,l=1}^{3} \sigma_k^3 \cdot \sigma_l^3 + h \cdot \sum_{k=1}^{3} \tilde{d}_3^k + H_h = [0] \in M_2(\mathbb{C}) \]

\[ \forall k, l = 1, 2, 3: u_{kl} = \lambda \in \mathbb{C}; \forall k = 1, 2, 3: u_k = h \text{ (the magnetic field)} \text{ and } u = H_h \in M_2(\mathbb{C}) \]

Thanks the so-called initial theorem (see annex), the existence of that PP2 is a sufficient argument to believe/suspect that, eventually, there also exists an ELP such that:

\[ |(3)^t > = |A\Phi((3)^t), (3)^t > + |(3)^z > \]

Where I have made two implicit assumptions:

1. \([A]\Phi((3)^t)\) is the representation in \(M_3(M_2(\mathbb{C}))\) of the trivial split of that ELP;
2. the Hamilton's operator \(H_h\) is supposedly the representation in \(M_2(\mathbb{C})\) of the “determinant” of the divisor \([H_h]\) of that ELP; the divisor is itself an element of \(M_3(M_2(\mathbb{C}))\).

Analyzing the coherence of the guess

This is the place where the greatest attention must be given to the coherence of the mathematical notation because the latter can be misleading. The \(\sigma_k^3\)s and \(H_h\) are in fact (2-2) matrices. With the definition (04), the vector \((3)^t\) is now a triplet of (2-2) matrices. This implies no difficulty concerning the definition of the ELP itself; indeed I may write as usually:

\[ |(3)^t > = |\sum_{k,l=1}^{3} A_{kl}^m \cdot \sigma_k^3 \cdot \sigma_l^3 > \text{ where } A_{kl}^m \in \mathbb{C} \]

Each component of that new kind of ELP is a (2-2) matrix. A first specificity arises here because of pairs like:

\[ A_{kl}^m \cdot \sigma_k^3 \cdot \sigma_l^3 + A_{kl}^m \cdot \sigma_l^3 \cdot \sigma_k^3 = A_{kl}^m \cdot (\sigma_k^3 \cdot \sigma_l^3 - \sigma_l^3 \cdot \sigma_k^3) = [0] \]

Indeed, since (a) any ELP is, per construction, built on an anti-symmetric cube the set of all \(A_{kl}^m\) which is denoted \(\wedge A\); here \(A_{kl}^m + A_{lk}^m = 0\) and (b) the commutation relation (02) must hold true, all these pairs vanish

\[ A_{kl}^m \cdot \sigma_k^3 \cdot \sigma_l^3 + A_{lk}^m \cdot \sigma_l^3 \cdot \sigma_k^3 = A_{kl}^m \cdot \sigma_k^3 \cdot \sigma_l^3 - A_{lk}^m \cdot \sigma_l^3 \cdot \sigma_k^3 = A_{kl}^m \cdot (\sigma_k^3 \cdot \sigma_l^3 - \sigma_l^3 \cdot \sigma_k^3) = [0] \]

and that new kind of ELP, (07) is in fact reduced to:

\[ |(3)^t > = |\sum_{k=1}^{3} A_{kk}^m \cdot (\sigma_k^3)^2 > \]

Let me recall that the initial theorem has only been demonstrated for anti-symmetric cubes \(\wedge A\) with vanishing \(A_{kk}^m\). Consequently, in opposition with my classical approach concerning the ELPs, the complex scalars \(A_{kk}^m\) should perhaps not automatically vanish; in which case the initial theorem is no more automatically true. At this precise stage, the mathematician in me must make a choice: either I decide to behave classically and I consider that this new type of ELPs always vanish in these physical conditions or I try to explore a new corner of this theoretical construction, hoping that experiments
will give me “a posteriori” a justification for these extrapolations. Anyway, let me recall that a vanishing ELP may also be decomposed and, because of that remark, let me go further in that exploration in supposing that $A_{nk}^m = 0$.

$$|_{\{A\}^\wedge t}^{(3)}, (3)^t_{\{A\}^\wedge t} > = |^{(2)}[0], (2)[0], ^{(2)}[0] > = ^{(3)}0$$

The physical interpretation of that mathematical choice is clear: for triplets of molecules placed linearly under the influence of some external magnetic field $h$, the TEQ approach proposes the introduction of vanishing but decomposed ELPs inside the Ising’s model.

Important precisions

The next technical difficulty is: “How can I define a trivial split in that new mathematical context (= in respecting the prescriptions given after the relation (06))?” Believing to respect the spirit of my own work and observing (08), a first and naïve answer may be something like:

\[(09) \quad [A] \Phi ((3)^t) \leftrightarrow A_{nk}^m \cdot \sigma_3^k\]

For each value of $m$, the r.h.t is a (2-2) matrix, more precisely: nothing else but a linear combination of the three components of $(3)^t$ and, with that proposition, the quantity $|A_{nk}^m \cdot \sigma_3^k - Hh|$ can be calculated in a coherent way. But strong critics must be immediately done:

- First: there is at least a semantic difficulty related to the concept of “determinant”;
- Second: furthermore with (09), I don’t have one but three matrices (one for each value of $m$)!
- Third: the obstruction concerning the non-realization of the initial theorem has not been eliminated

Let me consider each point of that critic.

The concept of determinant

Well, in saying that the quantities $|A_{nk}^m \cdot \sigma_3^k - Hh|$, I just intuitively admit that the determinant of a (2-2) matrix can be calculated in a classical manner and, in extenso in that case, that the calculation yields a scalar in $\mathbb{C}$. But this is exactly the point where the mistake can be done because it is a priori not the same thing to calculate the determinant for a (2-2) matrix with entries in $\mathbb{C}$ and the expected determinant for a (3-3) matrix with entries in $M_2(\mathbb{C})$; w.o.w: this doesn’t yet say that I am able to write the r.h.t of (06) as a polynomial of degree three depending on the three (2-2) matrices $\sigma_3^k$. Consequently I first have to make the choice of another symbol for the notion of determinant and I should perhaps prefer the following notation, less confusing even if it demands attention:

\[(10) \quad f((3)^t) \leftrightarrow \text{Det}((3)^t)_{\{A\}^\wedge t} - [Hh] \in M_2(\mathbb{C})\]

The next logical question is thus: “How can I precisely and concretely calculate $\text{det} \{\ldots\}$?” Any element of $M_2(\mathbb{C})$ can be considered as an entity, a “block”, in fact exactly as if it would be a scalar. The classical rules can thus be transposed in $M_2(\mathbb{C})$. The unique remaining difficulty is a conceptual one. Why and how should I attribute the label “determinant” to an element of $M_2(\mathbb{C})$? This presumably presupposes the existence of some equivalence between a characteristic property of that element and some scalar taken in $\mathbb{C}$ that would have been a classical determinant for that element. This mathematical point is in need of clarification in my head.
A better definition for extended Lie’s products

Nevertheless all this suggests now a better formulation for the block matrices involved in that application of the TEQ; for example I may envisage block matrices like:

\[
\begin{pmatrix}
\sigma_3, \sigma_3 \\
\sigma_3, \sigma_3 \\
\sigma_3, \sigma_3 \\
\end{pmatrix}
= T_2(.)((3)t, (3)t) \in M_3(M_2(C))
\]

Such block matrices are a special realization of Pythagorean tables (matrices). Here the operation denoted by a point, ".", represents the multiplication of two matrices of \(M_2(C)\). In a more sophisticated way, I can also write:

\[
\begin{pmatrix}
A_{11} m, \sigma_3 \\
A_{12} m, \sigma_3 \\
A_{13} m, \sigma_3 \\
A_{21} m, \sigma_3 \\
A_{22} m, \sigma_3 \\
A_{23} m, \sigma_3 \\
A_{31} m, \sigma_3 \\
A_{32} m, \sigma_3 \\
A_{33} m, \sigma_3 \\
\end{pmatrix}
\in M_3(M_2(C))
\]

which may be read as:

\[
[A^m] \cdot T_2(.)((3)t, (3)t)
\]

where the symbol "•" denotes an operation which I shall call an “extern slide acting on the left" of \(M_3(M_2(C))\) matrices. This is nothing but a kind of ponderation of each element in \(M_2(C)\) by a complex number (an element of \(C\)) taken in a matrix of \(M_3(C)\); the subscript m is a counting label. Let me add one more operation:

\[
\forall (D)[U] = \ldots [u_{kl}] \in M_D(C): [U] \oplus = \sum_k \sum_l u_{kl}
\]

From these conventions it is easy to check that the definition (07) for the ELP defined in \(M_2(C)\) can be rewritten as:

\[
|A| \wedge (3)t, (3)t > = |\ldots ([A^m] \cdot T_2(.)((3)t, (3)t)) \oplus \ldots >
\]

That rephrasing makes it clear that an extended Lie’s product (ELP) is a deformation of a Pythagorean table. The deformation is realized by the (elements of the) matrix \((3)[A]\) resulting from the reduction of the anti-symmetric cube \((3)\nabla A\).

Here, as already mentioned, the anti-symmetry of \((3)\nabla A\) and the commutation relation (02) together reduce the proposed formalism in such a way that each matrix \([A^m] \cdot T_2(.)((3)t, (3)t)\) becomes a skew-symmetric matrix and that each component of the new ELP becomes just the trace of that matrix. With other words, exceptionally in that case:

\[
\forall m = 1, 2, 3: ([A^m] \cdot T_2(.)((3)t, (3)t))^\oplus = \text{Trace}([A^m] \cdot T_2(.)((3)t, (3)t))
\]

A better definition of the trivial split for block matrices

Anyway, the trivial split can be defined by analogy with the classical formulation of the TEQ as:
The theory of the (E) question

Introducing decomposed extended Lie’s products into the Ising’s model – v1


\[(15)\]

\[|A|\Phi((3)t) = \sum \begin{bmatrix} A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \\ A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \\ A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \end{bmatrix} \]

And this evidently yields:

\[(16)\]

\[|A\wedge ((3)t, (3)t) \geq |A|\Phi((3)t). \]

"How can the initial theorem be applied to ELPs involving block matrices?"

With the purpose to transpose the spirit of the classical considerations developed around the concept of ELP, let me propose to rewrite the relation \((10)\) as:

\[(17)\]

\[f((3)t) = \text{Det} \{ \sum_k \begin{bmatrix} A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \\ A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \\ A_{k1} \cdot \sigma_3^{k} & A_{k2} \cdot \sigma_3^{k} & A_{k3} \cdot \sigma_3^{k} \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \}

If the initial theorem holds true here, the transposition \((17)\) yields a polynomial of degree two involving matrices of \(M_2(C)\). There is no obvious obstruction against the hypothesis that that polynomial could eventually be the relation \((01)\) or \((04)\). But there is at least one obligatory condition for that:

\[(18)\]

\[\text{Det} \{ \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \} = H_h \in M_2(C) \]

Conclusion

In that document I have tried to extrapolate classical considerations concerning extended Lie’s products in such a manner that that extrapolation may eventually be applied to a simplified Ising’s model concerning only three aligned molecules. The attempt is formally acceptable provided theoretical developments will be made in the next future on the notion of determinant.

Annex: the initial theorem

Proposition

In a three dimensional space, \(E_3(C)\), where \(C\) is the set of all complex numbers, any decomposed extended exterior product is associable with a quadratic form.

Demonstration

Let us suppose that the following relation holds a priori true in \(E_3(C)\):

\[| \bigtriangleup_{\nabla_A} (3)\text{projectile, (3)target} > - \cdot (3)\text{[Divisor]_{ext}}. |(3)\text{target}> + |(3)\text{residual part}> = |0> \]

Since we have proven the existence of at least one trivial split for any extended exterior product, that relation can be systematically re-written:

\[\{ \nabla_A \Phi((3)\text{projectile}) > - (3)\text{[Divisor]_{ext}}. |(3)\text{target}> = |(0)\text{residual part}> \]
That new relation may be interpreted as a linear system depending on the components of the target. Looking for the solutions of that system needs the calculation of its discriminant:

\[ f((3)\text{projectile}) = |A\Phi((3)\text{projectile}) - [P]| = |A_{ab} \cdot (\text{Projectile})^c - p_{ab}| \]

Let us start with the calculations:

(A-01.01)

\[
\forall (3)x = \text{projectile}: (x^1, x^2, x^3) \in \mathbb{C}^3 \\
\begin{align*}
f((3)\text{projectile}) &= \\
&= \begin{vmatrix}
A_{k1}^1 \cdot x^k - p_{11} & A_{k2}^1 \cdot x^k - p_{12} & A_{k3}^1 \cdot x^k - p_{13} \\
A_{m1}^2 \cdot x^m - p_{21} & A_{m2}^2 \cdot x^m - p_{22} & A_{m3}^2 \cdot x^m - p_{23} \\
A_{n1}^3 \cdot x^n - p_{31} & A_{n2}^3 \cdot x^n - p_{32} & A_{n3}^3 \cdot x^n - p_{33}
\end{vmatrix} \\
&= (A_{k1}^1 \cdot x^k - p_{11}) \cdot \{(A_{m2}^2 \cdot A_{n3}^3 - A_{m3}^2 \cdot A_{n2}^3). x^m + (p_{23}. A_{m2}^2 + p_{32}. A_{m3}^2)\} \\
&\quad - (A_{k2}^2 \cdot x^k - p_{12}) \cdot \{(A_{m1}^2 \cdot A_{n3}^3 - A_{m3}^2 \cdot A_{n1}^3). x^m + (p_{23}. A_{m1}^2 + p_{31}. A_{m3}^2)\} \\
&\quad + (A_{k3}^3 \cdot x^k - p_{13}) \cdot \{(A_{m1}^3 \cdot A_{n2}^3 - A_{m2}^3 \cdot A_{n1}^2). x^m + (p_{22}. A_{m1}^3 + p_{32}. A_{m2}^3)\}
\end{align*}
\]

The terms that we are studying write:

\[ \{A_{k1}^1 \cdot (A_{m2}^2 \cdot A_{n3}^3 - A_{m3}^2 \cdot A_{n2}^3) - A_{k2}^2 \cdot (A_{m1}^2 \cdot A_{n3}^3 - A_{m3}^2 \cdot A_{n1}^3) + A_{k3}^3 \cdot (A_{m1}^3 \cdot A_{n2}^3 - A_{m2}^3 \cdot A_{n1}^2)\}. x^k. x^m. x^n \]

The coefficients \((k, m, n)\) are:

(1, 1, 1):

\[ 0. (A_{m2}^2 \cdot A_{n3}^3 - A_{m3}^2 \cdot A_{n2}^3) - A_{k1}^1 \cdot (0. A_{n3}^3 - A_{n2}^3 \cdot 0) + A_{k3}^3 \cdot (0. A_{n2}^3 - A_{m2}^3 \cdot 0) = 0 \]

For the same reasons, the coefficients \((2, 2, 2)\) and \((3, 3, 3)\) vanish.

(1, 2, 3):

\[ A_{11}^1 \cdot (A_{22}^2 \cdot A_{33}^3 - A_{23}^2 \cdot A_{32}^3) - A_{12}^2 \cdot (A_{21}^2 \cdot A_{33}^3 - A_{23}^2 \cdot A_{31}^3) + A_{13}^3 \cdot (A_{21}^2 \cdot A_{32}^3 - A_{22}^2 \cdot A_{31}^3) = \\
0. (A_{22}^2 \cdot A_{33}^3 - A_{23}^2 \cdot A_{32}^3) - A_{11}^1 \cdot (0 - A_{22}^2 \cdot A_{31}^3) + A_{12}^2 \cdot (A_{21}^2 \cdot 0 - A_{23}^2 \cdot A_{31}^3) + A_{13}^3 \cdot (A_{21}^2 \cdot A_{32}^3 - 0. A_{31}^3) = \\
A_{11}^1 \cdot A_{22}^2 \cdot A_{31}^3 + A_{12}^2 \cdot A_{31}^3 + A_{13}^3 \cdot A_{21}^2 \cdot A_{32}^3 \]

(2, 3, 1):

\[ A_{21}^1 \cdot (A_{32}^2 \cdot A_{13}^3 - A_{33}^2 \cdot A_{12}^3) - A_{22}^2 \cdot (A_{31}^2 \cdot A_{13}^3 - A_{33}^2 \cdot A_{11}^3) + A_{23}^3 \cdot (A_{31}^2 \cdot A_{12}^3 - A_{32}^2 \cdot A_{11}^3) = \\
A_{21}^1 \cdot (A_{32}^2 \cdot A_{13}^3 - A_{12}^3 \cdot 0. A_{13}^3) - A_{22}^2 \cdot (A_{31}^2 \cdot A_{12}^3 - A_{33}^2 \cdot A_{11}^3) + A_{23}^3 \cdot (A_{31}^2 \cdot A_{13}^3 - A_{32}^2 \cdot 0) = \\
A_{21}^1 \cdot A_{32}^2 \cdot A_{13}^3 + A_{22}^2 \cdot A_{13}^3 + A_{23}^3 \cdot A_{13}^3 \]
The theory of the (E) question

Introducing decomposed extended Lie’s products into the Ising’s model – v1


(3, 1, 2):
\[ A_{31}. (A_{12} \cdot A_{23}^2 - A_{13} \cdot A_{22}^3) - A_{32}. (A_{11}^2 \cdot A_{23}^3 - A_{13} \cdot A_{21}^3) + A_{33}. (A_{11} \cdot A_{22}^3 - A_{12} \cdot A_{21}^3) = A_{31}. (A_{12} \cdot A_{23}^3 - A_{13} \cdot A_{22}^3, 0) - A_{32}. (0, A_{23}^3 - A_{13} \cdot A_{21}^3) + 0, (A_{11}^2 \cdot A_{22}^3 - A_{12} \cdot A_{21}^3) = A_{31}. A_{12}^2 \cdot A_{23}^3 + A_{32}. A_{13}^2 \cdot A_{21}^3. \]

(3, 2, 1)
\[ A_{31}. (A_{22} \cdot A_{13}^3 - A_{23} \cdot A_{12}^3) - A_{32}. (A_{21}^2 \cdot A_{13}^3 - A_{23} \cdot A_{11}^3) + A_{33}. (A_{21} \cdot A_{12}^3 - A_{22} \cdot A_{11}^3) = A_{31}. (0, A_{13}^3 - A_{23} \cdot A_{12}^3) - A_{32}. (A_{21} \cdot A_{13}^3 - A_{23} \cdot 0) + 0, (A_{21}^2 \cdot A_{12}^3 - A_{22} \cdot A_{11}^3) = - A_{31}. A_{23}^2 \cdot A_{13}^3 - A_{32}. A_{21}^2 \cdot A_{13}^3. \]

(1, 3, 2)
\[ A_{13}. (A_{23}^2 \cdot A_{33}^3 - A_{23} \cdot A_{23}^3) - A_{12}. (A_{31}^2 \cdot A_{33}^3 - A_{33} \cdot A_{21}^3) + A_{13}. (A_{31} \cdot A_{23}^3 - A_{32} \cdot A_{21}^3) = 0, (A_{23}^2 \cdot A_{33}^3 - A_{33} \cdot A_{23}^3) - A_{12}. (A_{31} \cdot A_{33}^3 - 0, A_{21}^3) + A_{13}. (A_{31} \cdot 0 - A_{32} \cdot A_{21}^3) = - A_{12}. A_{31}^2 \cdot A_{23}^3 - A_{13}^3 \cdot A_{32}^2 \cdot A_{21}^3. \]

Since we are obliged to sum all the coefficients together and since the cube is anti-symmetric (and \((-1)^3 = -1\)), the above coefficients vanish collectively. Let us examine the others possible combinations:

(1, 2, 2)
\[- A_{12}. A_{21}^2 \cdot A_{23}^3 + A_{12}. A_{23}^2 \cdot A_{21}^3. \]

(2, 1, 2)
\[ A_{21}. A_{12}^2 \cdot A_{23}^3 - A_{23} \cdot A_{12}^2 \cdot A_{21}^3. \]

(2, 2, 1)
\[ A_{23}. A_{21}^2 \cdot A_{12}^3 - A_{21} \cdot A_{23}^2 \cdot A_{12}^3. \]

Since we are obliged to sum all the coefficients together and since the cube is anti-symmetric (and \((-1)^2 = 1\)), the above coefficients vanish collectively. The same holds for (1, 3, 3), (3, 1, 3) and (3, 3, 1); (2, 3, 3), (3, 2, 3) and (3, 3, 2).

Conclusion: the initial theorem

Any discriminant \((A-01.01)\) is a quadratic form and can be generically written:

\((A-01.02)\)
\[ \forall (3)^{x} = \text{projectile}: (x^1, x^2, x^3) \in C^3 \]
\[ f((3)^{x}) = |^n \Phi((3)^{x}) - (3)^{[P]}| = \sum a^i \cdot b^i \cdot x^i + \sum a^j \cdot b^j \cdot x^j - |P| \]
Bibliography