Interesting problems in Geometry (2013-2014)

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Part II: convex geometry

Abstract: In this paper, I list some interesting problems I proposed in 2013-2014, which are the conclusion of the series papers “The analysis techniques for convexity”. I published this series work on scientific journal and in these problems, I try to sketch a research plan for the technical difficulties we meet in convex bodies and CAT spaces. So I hope readers will give me some advices for whether the outline appear in this paper is feasible and what is their meaningful.

1. convex bodies

Problem 1.1: When can we view the rotation of a convex body as the rotation of a line as its dimension tends to infinity?

Here, we find an interesting phenomenon in the rotation of convex body, such that, under some special condition, the rotation of a convex body with its dimension tends to infinity can be viewed as the rotation of a line.

We define a class of mutually disjoint sets firstly (page 2 formula (6)), and then we define the rotation map on these mutually disjoint sets as (page 2 formula (8)). So by the definition we make above and also relate them with the property of the rotation group (see Ludwig and Reitner’s paper), we can introduce the gauss curvature to analyze the area element, which imply that we can also select the infinite-small as \( t = \frac{1}{3 \pi^d (d - 1)^2} \), to make the rotation angle be zero as its dimension tends to infinity, so that, our conclusion seems feasible.

For more detail, you can go to: http://www.rxiv.org/abs/1404.0122 (page 1-5)

Problem 1.2: Is there a way to approximate the volume of a convex body?

In this problem, we find a method to approximate the volume of a convex body. Firstly, we introduce the expansion function to analyze the self-adjoint operator, hence, we can use the \( \Gamma \) function to approximate the periodicity of this expansion function. Nextly, we will introduce our approximation method in 5 examples.

1. Use the lesbegue theorem to study the blaschke-santalo inequality and get our bound for \( |K| \) (page 10 formula (14))
2. Define a support function to make the ratio between the volume of a convex body and its polar body tends to epsilon.
3. To analyze the natural logarithm, and to get our limit: \( \sqrt[n]{k} \to \infty \)

Therefore by the clamping force principle, we can obtain \( e = \frac{1}{n^{\infty}} \), which deduce our limit for the vector (page 14 formula (11))
4. we construct standard orthogonal sequence and use the uniform bounded theorem to get :
\[ |K| \leq \frac{1}{e} \]

5. to study the convex floating body, then by our approximation of the periodicity to the expansion function, we have
\[ |K| \leq C r^{-n-1} \]
next, by example 2 and 3, we can approximate to the radius of the convex body, also note that, by example 4, there exists
\[ |K| \leq \frac{1}{e} \]
so we can use the exponential function to approximate the volume (page 16 formula (6) and (7)). that is,
\[ |K| \leq \sqrt{\frac{x}{\log \log x}}^2 \]
where,
\[ x^{e \cdots e} = e^{x \cdot e \cdots e} \]
is it feasible? i need some help, thank you!
for more detail, you can refer to:
http://www.rxiv.org/abs/1404.0122 (page 5-16)

Problem 1.3: how to use a convex function to approximate the radius of a convex body?
here, we study the relation between the radius and the volume of a convex body. our main idea is:
1. firstly, we use the Morse-Sard theorem to construct infinitesimal cubes to approximate the main curvature of a floating body associated with the Riemann space.
2. substitute our approximation to the affine surface area (page 18, formula (21)), which lead us to get the relation formula for extrinsic curvature:
\[ K^2 - 2HK + \kappa = 0 \]
then, we can apply this relation to theorem 3.2 (page 16), and we get :
\[ LN - M^2 = LG - 2MF + NE \]
consequently, by the fact that the dimension \( n \) is odd, we can restrict the range of the main curvature in barbie's theorem as a constant.
here, we use the results in section 1 "when can we view the rotation of a convex body as the rotation of a line as its dimension tends to \( \infty \)" (page 1-5) to handle the curvature in affine surface area, which ensure us to assume a hypersurface which can intersect the bodies, and the measure of intersection tends to \( \epsilon \) (page 19, formula (25) (26) (27)), even lead us to get our relations above.
3. using the convexity of the brunn-minkowski theorem, we can calculate the relation between the radius and the volume of a convex body, that is:
\[ (m^{n} e^{m}) |K| \rightarrow \frac{\left(\alpha + \beta\right)}{(\alpha x + \beta y)^2} R_{\text{radius}} \]
here \( m \) represent the number of cubes, and we also set the dimension \( n = 3 \).
for more detail, you can go to:
http://www.vixra.org/abs/1404.0122 (page 1-5 16-21)
Problem 1.4: Any thought about the mean curvature of a convex body?

Here, we get a new estimate for the mean curvature for a convex body, such that:

\[
\frac{(1/8) \ln^2 (x^2 + 1)}{n} \leq (1 + rH)^n
\]

where \( x \leq \frac{\pi (t - 1/2) - a}{1 - a} \)

Our procedure is as follows:

1. Use the separable property of Euler characteristic to define mutually disjoint sets in Hadwiger’s theorem; then, by the differential mean value theorem, we can rearrange the sequence and get our first key equality (page 8, formula (10)).

Consequently, we can define a cut-off function (page 10, formula (11)), by this cut-off function and the key equality we get above, we can use the Minkowski inequality and the property of concave function to calculate the radius \( r \) in Hadwiger’s theorem, then we get the following result (page 9, formula (18) (19)).

2. By Theorem 2.1.3 (page 7), we can define the F-space and its auto-isomorphism; then we consider the support measure on convex ring, our method is to define a new ring, so that we can combine the auto-isomorphism and the support measure in stochastic geometry, to improve our estimate in step 1 (page 11, formula (7) (8)).

Here, the cut-off function can be used to relate the elements in convex ring and the radius in Hadwiger’s theorem.

3. Lastly, by the Steiner formula, we can use our result above to estimate the distribution function, which deduce an inequality for the relation between the radius in Hadwiger’s theorem and the main curvature in distribution function (page 11, formula (2)). Now our work is to use the average inequality to handle the inequality above, even to replace the main curvature by a natural logarithm function; and then we can get our last estimate for mean curvature as we desired at the beginning.

Two crucial points here are:

1. The possibility that we can arrange the sequence to make \( C_i \) mutually disjoint (page 11-12, remark), which ensure us to define the sets in Hadwiger’s theorem as (page 8, formula (8)).

2. The breakpoint at origin in the cut-off function we defined, which ensure us to use the property of random convex body to handle the mean curvature.

For more detail, you can go to:
http://www.vixra.org/abs/1404.0132 (page 7-12)

Problem 1.5: Some ideas about the Steiner symmetrization and a convex body

Problem 1.5.1: Use the Sobolev inequality to estimate the integral lower bound of convex function

Here we use the Sobolev inequality to get an estimate for convex function.

See: http://vixra.org/abs/1404.0132 (page 1-3)

Firstly we denote a borel set by Steiner symmetrization. Then if we apply the Steiner symmetrization to a convex body, the Lipschitz condition lead us to rewrite the n-order convex differential equation as the form (page 2, formula (10)). Nextly we try to use the Sobolev inequality
to handle the n-order convex differential equation to get this integral lower bound for an arbitrary convex function. (see page 3, formula (16))

Problem 1.5.2: how to use the support function to construct convex extension?
here we use the support function to extend convexity.
see: http://vixra.org/abs/1404.0132 (page 4-7)
at first, we try to use the unit vector to select the surface area as (page 4, formula (5)). then we can restrict the ratio of the intersection between the hypersurface and the convex body. our next step is to use the convex function we studied in problem 1.5.1, to handle the support function. then we can make: \( \kappa_n \to 1 \). lastly, we can introduce the Cauchy-Riemann condition to extend convexity as (page 7, formula (24)).

Problem 1.6: use the property of higher dimensional symmetric body to estimate the Euclidean norm
here we try to use the property of higher dimensional symmetric body to estimate the Euclidean norm. and our main idea is to use the fourier inequality to handle the Dvoretzky theorem at first. then by the property of trigonometric and exponential function, we can get our last estimate for Euclidean norms (page 3, formula (15))
see: http://vixra.org/abs/1404.0131 (page 1-3)

Problem 1.7: the relation between the lower bound of the volume and the curvature of a convex body
here we try to study the relation between the lower bound of the volume and the curvature of a convex body. and our main idea is as follow :firstly, we use the von-neumann theorem to handle the operator, which lead us to construct the rigid rotation for a convex body. then by the limit of the surface area (page 5, formula (6)), we can strengthen the rotation of Euclidean norm to Banach norm. lastly, if we apply the Waris formula to Blaschke-Santalo inequality to get the new relation between the lower bound of the volume and the curvature of a convex body (see page 6, formula (16) (17)).
For more detail, please search : http://vixra.org/abs/1404.0131 (page 4-6)

2. CAT-spaces
Problem 2.1: how about the symmetry of CAT(0) space?
here, we find a symmetric bound for the homomorphism on CAT(0), such that, there exists :
\[
(1 + \|y\|^2)^2 \geq 1 + 2(\sqrt{2} - 1) \frac{\pi^4}{R^4} (1 - \frac{1}{R^2})
\]
we get this result by 3 steps:
1. we define the vectors on geodesic space and use the inequality :
\[
\left| a^2 + b^2 + c^2 + 3abc \right| \leq \max( |a + bc|, |b + ac|, |c + ab| )
\]
to choose the infinite-small of the tangent cone; then we use the fact that  $A = \{ 2l/\pi - [2l/\pi] \}$ is dense in $[0,1]$ to divide $[0,1]$ into $1 + [1/\varepsilon]$ intervals, which imply that the map between $M \rightarrow R$, which map the riemann space into $[0,1]$, is also equivalent with the map $(1 - \frac{\pi}{\delta R})$, which map the Riemann manifold into $(1,\infty)$ oppositely.

2. we define a special probability measure equipped with ultrametric (page 4, formula (4) and (5)) to study the Gromov-Hausdorff metric on CAT(0), which make us can also define a reciprocal map between cube and the Euclid space, then we can replace the $\pi$ in this map by the gauss curvature.

3. by step 1, we can use the poisson equation to study the Riemann metric, the results imply that, we can paste the reciprocal maps together (in step 2), and get our last symmetric bound for the functions $\|x\|$ in CAT(0).

here we use the property of a base point in Riemann manifold.

for more detail, please go to :http://www.rxiv.org/abs/1404.0406 (page 1-7)

Problem 2.2: any thought about the distance between arbitrary two points on the CAT-Alexandrov space?

here we study the distance between arbitrary two points on the CAT-Alexandrov space. our main idea can be expressed as three steps:

1. use the property of distance convex to introduce the dual space, and then we can get a bound for the norm $\|x\|$. (see page 8, formula (5))

2. introduce the function : $F(x) = e^x - ax^2 - bx - c$ to study the lipschitz function, and then, we can get an estimate for the inner product in CAT space. (see page 10, formula (17) (18))

3. we make an assumption that the distance of arbitrary two points in the sequence is a rational number, and so on, we can make the infinitesimal $\varepsilon$ is in accordance with $\varepsilon_n$, by this result, and also combine it with step 1 and 2, we can get a new bound for the distance $\|uv\|$ (see page 14, formula (25))

for more detail, please refer to : http://www.rxiv.org/abs/1404.0406 (page 8-14)

Problem 2.3 : how about the form of the fixed point in hadamard space?

here, we study the structure of hadamard space. firstly, we introduce isomorphism to weaken the metric space $X$ to be a group at infinity; then, we define a special norm on convex set induced the distance $\|f - g\|$, the induction fails which imply that the scalar $:\lambda_\theta = \left| \frac{\nu_x}{\nu_x} \right| \rightarrow 1$.

next, we construct normal subgroup which contain the kernel, and we apply this homomorphism to the induced norm we defined above, also to approximate the limit in our weaken group at
infinity. that is: \( \partial_x \Phi \in [0, c \frac{\sqrt{\pi}}{2}] \).

Lastly, we define the cyclic and its centralizer in Gromov hyperbola group as:

\[
1 \log s_1 + 2 \log s_2 + \ldots + k \log s_k = k.
\]

To search the fixed point in Hadamard space, we guess the fixed point has the form, which is similar with:

\[
\Phi(x) \sim \frac{\sqrt{\pi}}{m \xi^m / k}, \text{ where, } m \in (2, n)
\]

For more detail, you can go to: http://www.rxiv.org/abs/1404.0408 (page 1-4)

**Problem 2.4: how to explain the homotopy property of the hyperbolic space?**

Here, we study the homotopy property of the CAT-hyperbolic space. And our step is as follow:

1. Introduce the Cauchy sequence in quotient space, then by the property of the lift in Lie group, we can define the inner automorphism.
2. By the property of the Coxeter complex, we can introduce the homotopy, and we can also define the road map on Gromov hyperbolic metric space.
3. By the metric inequality in Hadamard space, we can combine step 1 and 2, even to approximate the limit point of \( \| v \| \), consequently, we can write down the range of the road map in homotopy (see page 9, formula (7) and (8)).

For more detail, you can go to:

http://www.rxiv.org/abs/1404.0408 (page 4-9)

**Problem 2.5: how about the relation between the energy equation and the characteristic of the abel group in CAT spaces?**

Here we find a method to represent the characteristic of the abel group by the functions in energy equation, such that:

\[
\lambda_i = \frac{f^n (1 - C \rho)}{C \rho}
\]

Where, \( C = \frac{b}{1/2 (b^2 - a^2) (b - a)} \), \( \sum \lambda_i = 1 \), \( n = \sum r^k \).

Firstly, we divide the set \( E^n \) into 2 parts \( E^+_n, E^-_n \), and we use the property of infinite group to homology, which gives the Poincare characteristic; then we use the property of derived group to represent the elements in this group, and we can replace the characteristic in Poincare number by the elements in derived group next; lastly, we define a Steinhaus set function to analyze the infinite-small in energy function, by the property of Laplacian operator in polar coordinate, we achieve our results for the relation between the energy equation and the characteristic.

One point here is crucial that, since the error in the Laplacian operator is a component of \( \partial u \), so we should define the function in energy equation as the Steinhaus set function, which possess a shift can make the function \( f \) tend to infinite-small, then we can construct the \( \text{min set} \{ f \} \).
which also ensure the error exists and do not rely on the parameter \((c, d)\), which are out of control. Therefore, we can represent the characteristic by the function \(f\) in energy equation.

For more detail, you can refer to: http://www.rxiv.org/abs/1404.0408 (page 9-13)

**Problem 2.6: how to use the Ptolemy inequality to study the geodesic angle?**

Here, we try to use the Ptolemy inequality to study the geodesic angle. And we introduce 2 preliminary results:

1. The Ptolemy theorem in Euclid spherical geometry

By studying the matrix of 4 quadruple points on the Euclid spherical, we find the fact that, since there are 6 lines which connect these 4 points, so if 5 of the 6 lines are equal, then we will get that the sixth line's length is double as the other equally 5 lines; this result imply that the angle between the lines is included in \(\frac{\sqrt{6}}{2} \in (2k \pi / 6, 2k \pi + 5 \pi / 6)\); then we substitute this result into the discriminant, and we get an inequality about the radius in \(n + 1\) dimension:

\[
op_{n+1} \geq \frac{r^2}{t^2} \left(1 + \frac{r^2}{n}\right) - \frac{r^2}{n}
\]

2. Ptolemy inequality in Minkowski geometry

Here, we use the centroid method to study the n-polygon problem in Minkowski geometry. Firstly, we introduce a well known problem such that:

If every angle of a polygon is equal, and the sidelines are \(1^2, 2^2, \ldots, N^2\), then \(\sum n_j e^{\omega j} = 0\)

By this theorem, we can factorize the mass on the vertex into pairs and the number of pairs is primes \(N/2 = \prod p_i^{s_i}\), the weight of each pair is \(\sum 4k - 1\), then we can divide these pairs into groups and every group has prime points too (page 21 in [1]); consequently we can divide the sidelines into 2 parts: \(1^2, 3^2, \ldots\) and \(2^2, 4^2, \ldots\);

The next step is to construct regular n-polygon and use the Ptolemy inequality to make a regulation for the average of the sum of the mass in different group, and we can rearrange these groups of mass to ensure the first part \(1^2, 3^2, \ldots\) is larger than the average and the second part \(2^2, 4^2, \ldots\) is less than it.

Therefore we can apply this average of sum to the distance formula in Minkowski geometry in polar coordinate (page 24 in [1]).

Here we also use combinatorics method (result we get in step 2) to study the natural logarithm in distance formula of Minkowski geometry (page 25).

Our goal is to represent the polar angle in Minkowski geometry as the product of the mass lie on different vertex (page 26 in [1]).
so our question is how to apply the 2 results above to the geodesic angle?

by the inequality for the radius in $n + 1$ dimension euclid spherical, we can ensure $v_{n+1} \geq 0$; consequently we can substitute the representation of the polar angle in step 2 to spherical equation, which imply that we can also restrict the range of $\cos^2 \varphi$, that is $:[\sqrt{33} - 41, 3]$.

lastly, we apply the property of ptolemy space to get our estimate for geodesic angle, the bounde is:

$$1 - 4e^{2t} / 3 + 2 / (3 - 4e^{2t} / 3)$$

is this method feasible? for more detail, you can refer to:

http://www.rxiv.org/abs/1401.0224 (page 15-27)

http://www.rxiv.org/abs/1404.0125

**Problem 2.7 :** how to use the property of constant width to approximate the volume of the great ball in Riemann space?

here we try to use the property of constant width to approximate the volume of the great ball in Riemann space.

see: http://vixra.org/abs/1404.0406 (page 14-17)

we use the property of constant width to approximate the convex integral function at first. then we can bound the gradient operator (see page 15, formula (1)). our next step is to select the curvature as:

$$\frac{\sqrt{c}K}{\pi} \to 0$$

to construct great ball. then by the upper bound of the gradient operator, we can get our last volume (page 17, formula (10))

also note that here we use the condition of the geodesic curvature to approximate the volume of this special ball.

**Problem 2.8 :** the existence of a special equality in alexandrov space

here we try to prove the existence of a special equality in alexandrov space.

see: http://vixra.org/abs/1404.0408 (page 13-14)

firstly, we try to use the property of convex integral function to rewrite the semigroup operator in dual from. Then we apply the random measure to support function and we compute the convolution to prove the existence of this equality for alexandrov space.

also note that, here we use the fact that the interior of the random space with semigroup operator is convex and compact, which allow us to introduce the property of convex body.

**Comprehensive problem**

**Problem 3 :** is this outline for convex body feasible?

here, we give an outline for convex body. see: http://www.rxiv.org/abs/1412.0121 (page 1-5)

**Problem 3.1 :** the $L_p$ minkowski problem

firstly, we introduce problem 4.1 (see page 2) to handle the epsilon term in convex function. then by the bound of the sobolev operator, we can construct unit mixed volume body.
our next step is to use the fact that the difference between the upper and lower limit of the convex function is less than the radius of the body. so by problem 2 (see page 2), we can select \( n \to \infty \) to approximate the volume of the convex body, such that, we can make \( V(K) \) tends to infinite small. then by the property of convexity extension, we can let \( |p| = 1 \), to construct the convex function \( \frac{n - 1}{n}(1 - C_{x, p} x) \sim x(1 - x) \).

**Problem 3.2: boundary condition and periodicity solution**

firstly, we use problem 5 (see page 2), to construct the bound for symmetric function \( H(x) \). then we select the function \((1/8)h^{-1}(x^2 + 1)\) in problem 5 as a suitable value, to get the periodicity condition. then if we let \( t \to \infty \) and \( h = 1/2 \), we can get the damped oscillation condition for dirichlet problem. related it with the symmetric function, we can also get another condition that \( \min\{g\} \to 0 \).

**Problem 3.3: convex body in hyperbolic space**

at first, we can introduce problem 3 (see page 2), to select the convex function as:

\[
f^{-1}(p) = \frac{\alpha + \beta}{(\alpha x + \beta y)^2}
\]

then by the results we get in 2 above, we can deduce \( \max H(x) \geq 0 \), so if we select the angle in problem 6 as a suitable value, we can let \( t \to \infty \) and get the inequality:

\[
\frac{HK}{H - \sqrt{H - K}} \leq C_a
\]

then we can get 2 condition \( K < 0 \) or \( K > H^{-2} \). since the second condition contradict the definition of gauss curvature and mean curvature, we can get the characteristic of a hyperbolic space.

**Problem 4: is this outline for CAT space feasible?**

here, we give an outline for CAT space. see: http://www.rexiv.org/abs/1412.0121 (page 6-9)

**Problem 4.1: comparison theorem and growth mode**

firstly, we use problem 8 (see page 7) to ensure the curvature bound for great ball and the unit hyper-ball are in accordance. then we can compare their volume by Toponogov theorem. since we can select \( x \to 0 \) in Euclidean sphere by the positive definite condition. so by the structure of the volume of the great ball (see problem 3 ,page 6), we can introduce problem 1 to let \( f \to \frac{\pi}{R} \).
then we can get the estimate for the Ricci curvature and the radius:

\[ \sqrt{\frac{\pi}{\text{Ricci}}} \sim r, \] which imply that the base point exists (see problem 1, page 6).

**Problem 4.2: energy equation and singular point**

at first, we use problem 4 (page 6) to construct the lipschitz function. then by problem 6 (page 7), we can get the bound for the characteristic value, that is: \( \lambda_i \leq \frac{1}{rx} \). so if we introduce the limit for the density (see problem 3 below), we can replace the density in energy equation with the radius. then by the homotopy property (problem 5, page 6), we can let \( \log h \to \ln 2 \). therefore we prove that the energy accumulated by the growth of radius slow down as the time lapsed.

**Problem 4.3: approximate the limit of the density**

firstly, by the property of alexandrof space: \( c(n) \leq T \leq C(n) \), we can introduce the Gromov hyperbolic space to make \( z \) tends to the base point \( p \).

then if we introduce problem 2 (page 6), we can compare the series \( e^{rx} \) and \( e^{\epsilon} \). since in problem 2, we can let \( \epsilon \) in accordance with \( \epsilon_1 \). so if we select \( n = 1, d = \sqrt{t} \) by problem 7 (page 7), we can restrict the \( T \) be greater than \( t \), so that we can approximate the density as desired.

**Reference:**