# A Fuzzy-Cautious OWA Approach with Evidential Reasoning


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***Abstract*—Multi-criteria decision making (MCDM) is to make decisions in the presence of multiple criteria. To make a decision in the framework of MCDM under uncertainty, a novel fuzzy - Cautious OWA with evidential reasoning (FCOWA-ER) approach is proposed in this paper. Payoff matrix and belief functions of states of nature are used to generate the expected payoffs, based on which, two Fuzzy Membership Functions (FMFs) representing optimistic and pessimistic attitude, respectively can be obtained. Two basic belief assignments (bba's) are then generated from the two FMFs. By evidence combination, a combined bba is obtained, which can be used to make the decision. There is no problem of weights selection in FCOWA-ER as in traditional OWA. When compared with other evidential reasoning-based OWA approaches such as COWA-ER, FCOWA-ER has lower computational cost and clearer physical meaning. Some experiments and related analyses are provided to justify our proposed FCOWA-ER.**



***Index Terms*—Evidence theory, OWA, belief function, uncertainty, decision making, information fusion.**


## I. Introduction

In real-life situations, decision making always encounters difficult multi-criteria problems [1]. In classical Multi-Criteria Decision Making (MCDM) framework, the ordered weighted averaging (OWA) approach proposed by Yager [2] has been increasingly used in wide range of successful applications for the aggregation of decision making problems such as image processing, fuzzy control, market prediction and expert systems, etc [3]. OWA is a generalized mean operator providing flexibility in the aggregation. Thus the aggregation can be bounded between minimum and maximum operators. This flexibility of the OWA operator is implemented by using the concept of orness (optimism) [4], which is a surrogate for decision maker's attitude. One important issue in the OWA aggregation is the determination of the associated weights. Many approaches [5]–[10] have been proposed to determine the weights in OWA. See the related references for details.

In multi-criteria decision making, decisions are often made under uncertainty, which are provided by several more or less reliable sources and depend on the states of the world: decisions can be taken in certain, risky or uncertain environment. To implement the decision making under uncertainty, many approaches were proposed including DS-AHP [11], DSmT-AHP [12] and ER-MCDA [13], etc. Especially for the OWA under uncertainty, Yager proposed an OWA approach with evidence reasoning [14]. In our previous work, a cautious OWA with evidential reasoning (COWA-ER) was proposed to take into account the imperfect evaluations of the alternatives and the unknown beliefs about groups of the possible states of the world. COWA-ER mixes MCDM principles, decision under uncertainty principles and evidential reasoning. There is no step of weights selection in COWA-ER, which is good for the practical use. Recently, we find that there also exists drawbacks in COWA-ER. More precisely, the computational cost of the combination of different evidences by COWA- ER highly depends on the number of alternatives we encounter in decision making. When the number of alternatives is large, the computational cost will increase significantly.

In this paper, we propose a modified COWA-ER approach, called Fuzzy-Cautious OWA with Evidential Reasoning (FCOWA-ER), by using a different way to manage the uncertainty caused by weights selection. Payoff matrix together with the belief structure (knowledge of the states of the nature) are used to generate two Fuzzy Membership Functions (FMFs) representing the optimistic and pessimistic attitude, respectively. Then two bba's can be obtained based on the two FMF's by using $\alpha$-cut approach. Based on evidence combination, the combined bba can be obtained and the final decision can be made. The FCOWA-ER approach doesn't need a (ad-hoc) selection of weights as in the traditional OWA. When compared with COWA-ER, FCOWA-ER has less computational cost and clearer physical meaning because it requires only one combination operation regardless of the number of alternatives. The proposed FCOWA-ER can be seen as a trade-off between the optimistic and the pessimistic attitudes. The preference of the two attitudes can be adjusted by the users using discounting factors in the combination of evidences. Some experiments and related analyses are provided to show the rationality and efficiency of this new FCOWA-ER approach.

## II. Multi-Criteria Decision Making under Uncertainty

Multi-criteria decision making (MCDM) refers to making decisions in the presence of multiple, usually conflicting or discordant, criteria. Consider the following matrix $C$ provided to a decision maker:

$$\begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & \cdots & S_j & \cdots & S_n \end{array} \\ \left[\begin{array}{ccccc} C_{11} & \cdots & C_{1j} & \cdots & C_{1n} \\ \vdots & & \vdots & & \vdots \\ C_{i1} & & C_{ij} & & C_{in} \\ \vdots & & \vdots & & \vdots \\ C_{q1} & \cdots & C_{qj} & \cdots & C_{qn} \end{array}\right] \end{array} = C$$

In the above each $A_i$ corresponds to a possible alternative available to the decision maker. Each $S_j$ corresponds to a possible value of the variable called the state of nature. $C_{ij}$ corresponds to the payoff to be received by the decision maker if he selects action $A_i$ and the state of nature is $S_j$. The problem encountered by the decision maker in MCDM is to select the action which gives him the optimum payoff.

Among all the available MCDM approaches, Ordered Weighted Averaging (OWA) is a very important one, which is introduced below.

### A. Ordered Weighted Averaging (OWA)

OWA was proposed by Yager in [2]. An OWA operator of dimension $n$ is a function $F : \mathbb{R}^n \to \mathbb{R}$ that has associated with a weighting vector[1] $W = [w_1, w_2, ..., w_n]^T$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. For any set of values $a_1, ..., a_n$

$$F(a_1, ..., a_n) = \sum\nolimits_{i=1}^n (w_i \cdot b_i) \tag{1}$$

where $b_i$ is the $i$th largest element in the collection $a_1, ..., a_n$. It should be noted that the weights in the OWA operator are associated with a position in the ordered arguments rather than a particular argument.

The OWA operator depends on the associated weights, hence the weights determination is very crucial. Some commonly used weights selection strategies are as follows [14]:

1) *Pessimistic Attitude:* If $W = [0, 0, ..., 1]^T$, then

$$F(a_1, a_2, ..., a_n) = \min_j[a_j].$$

2) *Optimistic Attitude:* If $W = [1, 0, ..., 0]^T$, then

$$F(a_1, a_2, ..., a_n) = \max_j[a_j].$$

3) *Hurwicz Strategy:* If $W = [\alpha, 0, ..., 1 - \alpha]^T$, then

$$F(a_1, a_2, ..., a_n) = \alpha \cdot \max_j[a_j] + (1 - \alpha) \cdot \min[a_j].$$

4) *Normative Strategy:* If $W = [1/n, 1/n, ..., 1/n]^T$, then

$$F(a_1, a_2, ..., a_n) = (1/n) \cdot \sum\nolimits_{i=1}^n a_i.$$

The OWA operator can be seen as the decision-making under ignorance, because in classical OWA, there is no knowledge about the true state of the nature but that it belongs to a finite set. It should be noted that the pessimistic and optimistic strategies provide limited classes of OWA operators. There also exist other strategies to determine the weights, e.g., the weights generation based on entropy maximization. See related references [5]–[10] for details.

[1]where $X$ is a vector or a matrix and $X^T$ denotes the transpose of $X$.

Based on such OWA operators, for each alternative $A_i$, $i = 1, \ldots, q$, we can choose a weighting vector $W_i = [w_{i1}, w_{i2}, \ldots w_{in}]$ and compute its OWA value $V_i \triangleq F(C_{i1}, C_{i2}, \ldots, C_{in}) = \sum_j w_{ij} \cdot b_{ij}$ where $b_{ij}$ is the $j$th largest element in the collection of payoffs $C_{i1}, C_{i2}, \ldots, C_{in}$. Then, as for decision-making under ignorance, we choose $A^* = A_{i^*}$ with $i^* \triangleq \arg\max_i\{V_i\}$.

### B. Uncertainty in MCDM context

Decisions are often made based on imperfect information and knowledge (imprecise, uncertain, incomplete) provided by several more or less reliable sources and depend on the states of the world: decisions can be taken in certain, risky or uncertain environment [15]. In a MCDM context, the decision under uncertainty means that the evaluations of the alternative are dependent on the state of the world.

Introducing the ignorance and the uncertainty in a MCDM process consists in considering that consequences of alternatives ($A_i$) depend on the state of nature represented by a finite set $S = \{S_1, S_2, ..., S_n\}$. For each state, the MCDM method provides an evaluation $C_{ij}$. We assume that this evaluation $C_{ij}$ done by the decision maker corresponds to the choice of $A_i \in \{A_1, ..., A_q\}$ when $S_j$ occurs with a given (possibly subjective) probability. The evaluation matrix is defined as $C = [C_{ij}]$ where $i = 1, ..., q$ and $j = 1, ..., n$.

Since the payoff to the decision maker depends upon the state of nature, his procedure for selecting the best alternative depends upon the type of knowledge he has about the state of nature. For representing the uncertainty for the state of nature, the belief functions introduced in Dempster-Shafer Theory (DST) [16] (known also as the Evidence Theory) can be used. This is briefly introduced below.

### C. Basics of Evidence Theory

In DST, the elements in the frame of discernment (FOD) denoted by $\Theta$ are mutually exclusive and exhaustive. Suppose $2^\Theta$ denotes the powerset of FOD. One defines the function $m : 2^\Theta \to [0, 1]$ as the basic belief assignment (bba, also called mass function) if it satisfies:

$$\sum\nolimits_{A \subseteq \Theta} m(A) = 1, \ m(\emptyset) = 0 \tag{2}$$

The belief function ($Bel$) and the plausibility function ($Pl$) are defined below, respectively:

$$Bel(A) = \sum\nolimits_{B \subseteq A} m(B) \tag{3}$$

$$Pl(A) = \sum\nolimits_{A \cap B \neq \emptyset} m(B) \tag{4}$$

Let us consider two bba's $m_1(.)$ and $m_2(.)$ defined over the FOD $\Theta$. Their corresponding focal elements[2] are $A_1, ..., A_k$ and $B_1, ..., B_l$. If $k = \sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) < 1$, the function $m : 2^\Theta \to [0, 1]$ denoted by

$$m(A) = \begin{cases} 0, \ A = \emptyset \\ \frac{\sum\limits_{A_i \cap B_j = A} m_1(A_i)m_2(B_j)}{1 - \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j)}, A \neq \emptyset \end{cases} \tag{5}$$

[2]a focal element $X$ of a bba $m(.)$ is an element of the power set of the FOD such that $m(X) > 0$.

is also a bba. The rule defined in Eq. (5) is called Dempster's rule of combination.

### D. MCDM with belief structures

Yager proposed an approach for decision making with belief structures [14]. One considers a collection of $q$ alternatives belonging to $A = \{A_1, A_2, ..., A_q\}$ and a finite set $S = \{S_1, S_2, ..., S_n\}$ of states of the nature. We assume that the payoff/gain $C_{ij}$ of the decision maker in choosing $A_i$ when $S_j$ occurs are given by positive (or null) numbers. The payoffs $q \times n$ matrix is defined by $C = [C_{ij}]$ where $i = 1, ..., q$ and $j = 1, ..., n$ as in eq. (2). The decision-making problem consists in choosing the alternative $A^* \in A$ which maximizes the payoff to the decision maker given the knowledge on the state of the nature and the payoffs matrix $C$. $A^* \in A$ is called the best alternative or the solution (if any) of the decision-making problem.

In Yager's approach, the knowledge on the state of the nature is characterized by a belief structure. Clearly, one assumes that a priori knowledge on the frame $S$ of the different states of the nature is given by a bba $m(.) : 2^S \to [0, 1]$. Decision under certainty is characterized by $m(S_j) = 1$; Decision under risk is characterized by $m(S_j) > 0$ for some states $S_j \in S$; Decision under full ignorance is characterized by $m(S_1 \cup S_2 \cup ... \cup S_n) = 1$, etc. Yager's OWA for decision making under uncertainty combines the schemes used for decision making under risk and ignorance. It is based on the derivation of a generalized expected value $C_i$ of payoff for each alternative $A_i$ as follows:

$$C_i = \sum\nolimits_{k=1}^{r} m(X_k) V_{ik} \tag{6}$$

where $r$ is the number of focal elements of the belief structure. $m(X_k)$ is the mass assignment of the focal element $X_k \in 2^S$. $V_{ik}$ is the payoff we get when we select alternative $A_i$ and the state of nature lies in $X_k$. The derivation of $V_{ik}$ is done similarly as for the decision making under ignorance (i.e., the procedure of OWA) when restricting the states of the nature to the subset of states belonging to $X_k$ only. One can choose different strategies to determinate the weights. Actually, $C_i$ is essentially the expected value of the payoffs under $A_i$. Select the alternative with highest $C_i$ as the optimal one.

### E. Cautious OWA with Evidential Reasoning

Yager's OWA approach is based on the choice of a given attitude measured by an optimistic index in [0, 1] to get the weighting vector $W$. How to choose such an index/attitude? This choice is ad-hoc and very disputable for users. In our previous work [15] we have only considered jointly the two extreme attitudes (pessimistic and optimistic ones) jointly and developed a method called Cautious OWA with Evidential Reasoning (COWA-ER) for decision under uncertainty based on the imprecise evaluation of alternatives.

In COWA-ER, the pessimistic and optimistic OWA are used respectively to construct the intervals of expected payoffs for different alternatives. For example, if there exist $q$ alternatives, the expected payoffs are as follows.

$$E[C] = \begin{bmatrix} E[C_1] \\ E[C_2] \\ \vdots \\ E[C_q] \end{bmatrix} = \begin{bmatrix} [C_1^{\min}, C_1^{\max}] \\ [C_2^{\min}, C_2^{\max}] \\ \vdots \\ [C_q^{\min}, C_q^{\max}] \end{bmatrix}$$

Therefore, one has $q$ sources of information about the parameter associated with the best alternative to choose. For decision making under imprecision, the belief functions framework is used again. COWA-ER includes four steps:

- Step 1: normalization of imprecise values in $[0, 1]$;
- Step 2: conversion of each normalized imprecise value into elementary bba $m_i(.)$;
- Step 3: fusion of bba $m_i(.)$ with some combination rule;
- Step 4: choice of the final decision based on the resulting combined bba.

In step 2, we convert each imprecise value into its bba according to a very natural and simple transformation [17]. Here, we need to consider the finite set of alternatives $\Theta = \{A_1, A_2, \dots, A_q\}$ as the frame of discernment and the sources of belief associated with them are obtained from the normalized imprecise expected payoff vector $E^{Imp}[C]$. The modeling for computing a bba associated to $A_i$ from any imprecise value $[a; b] \subseteq [0; 1]$ is simple and is done as follows:

$$\begin{cases} m_i(A_i) = a, \\ m_i(\bar{A}_i) = 1 - b \\ m_i(A_i \cup \bar{A}_i) = m_i(\Theta) = b - a \end{cases} \tag{7}$$

where $\bar{A}_i$ is the $A_i$'s complement in $\Theta$. With such a conversion, one sees that $Bel(A_i) = a$, $Pl(A_i) = b$. The uncertainty is represented by length of the interval $[a; b]$ and corresponds to the imprecision of the variable (here the expected payoff) on which the belief function for $A_i$ is defined.

## III. A Novel Fuzzy-COWA-ER

The COWA-ER has its rationality and can well process the MCDM under uncertainty. However the complexity and the computational time of the combination of COWA-ER method is highly dependent on the number of alternatives used for decision-making. When the number of alternatives is large, the computational cost will increase significantly. In COWA-ER, each expected interval is used as the information sources, however, these expected intervals are jointly obtained and thus these information sources are relatively correlated, which is harmful for the followed evidence combination. In this paper, we propose modified COWA-ER called Fuzzy-COWA-ER. Before presenting the principle of FCOWA-ER, we first recall that the pessimistic and optimistic OWA versions are used respectively to construct the intervals of expected payoffs for different alternatives as follows:

$$E[C] = \begin{bmatrix} E[C_1] \\ E[C_2] \\ \vdots \\ E[C_q] \end{bmatrix} = \begin{bmatrix} [C_1^{\min}, C_1^{\max}] \\ [C_2^{\min}, C_2^{\max}] \\ \vdots \\ [C_q^{\min}, C_q^{\max}] \end{bmatrix}$$

### A. Principle of FCOWA-ER

In COWA-ER, each row of the expected payoff $E[C]$ is used as information sources while in FCOWA-ER, we consider the two columns of $E[C]$ as two information sources, representing the pessimistic and the optimistic attitude, respectively. The column-wise normalized expected payoff is

$$E^{Fuzzy}[C] = \begin{bmatrix} N_1^{\min}, N_1^{\max} \\ N_2^{\min}, N_2^{\max} \\ \vdots \\ N_q^{\min}, N_q^{\max} \end{bmatrix}$$

where $N_i^{\min} \in [0,1]$ $(i = 1, ..., q)$ represents the normalized value in the column of pessimistic attitude and $N_i^{\max} \in [0,1]$ represents the normalized value in the column of optimistic attitude. The vectors $[N_1^{\min}, ..., N_q^{\min}]$ and $[N_1^{\max}, ..., N_q^{\max}]$ can be seen as two fuzzy membership functions (FMFs) representing the possibilities of all the alternatives: $A_1, ..., A_q$.

The principle of FCOWA-ER includes the following steps:

- Step 1: normalize each column in $E[C]$, respectively, to obtain $E^{Fuzzy}[C]$;
- Step 2: conversion of two normalized columns, i.e., two FMFs into two bba's $m_{Pess}(.)$ and $m_{Opti}(.)$;
- Step 3: fusion of bba's $m_{Pess}(.)$ and $m_{Opti}(.)$ with some combination rule;
- Step 4: choice of the final decision based on the resulting combined bba.

In Step 2, we implement the conversion of the FMF into the bba by using $\alpha$-cut approach as follows:

Suppose the FOD is $\Theta = \{A_1, A_2, ..., A_q\}$ and the FMF is $\mu(A_i), i = 1, ..., q$, the corresponding bba introduced in [18] is used to generate $M$ $\alpha$-cut $(0 < \alpha_1 < \alpha_2 < \cdots < \alpha_M \leq 1)$, where $M \leq |\Theta| = n$.

$$\begin{cases} B_j = \{A_i \in \Theta | \mu(A_i) \geq \alpha_j\} \\ m(B_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M} \end{cases} \tag{8}$$

$B_j$, for $j = 1, ..., M$, $(M \leq |\Theta|)$ represents the focal element. For simplicity, here we set $M = q$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_q \leq 1$ as the sort of $\mu(A_i)$.

### B. Example of FCOWA-ER versus COWA-ER and OWA

**Example 1**: Let's take states $S = \{S_1, S_2, S_3, S_4, S_5\}$ with the associated bba $m(.)$ given by:

$$\begin{cases} m(S_1 \cup S_3 \cup S_4) = 0.6 \\ m(S_2 \cup S_5) = 0.3 \\ m(S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) = 0.1 \end{cases}$$

Let's also consider alternatives $A = \{A_1, A_2, A_3, A_4\}$ and the payoffs matrix:

$$C = \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \tag{9}$$

*1) Implementation of OWA:* The $r = 3$ focal elements of $m(.)$ are $X_1 = S_1 \cup S_3 \cup S_4$, $X_2 = S_2 \cup S_5$ and $X_3 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$. $X_1$ and $X_2$ are partial ignorance and $X_3$ is the full ignorance. One considers the following submatrix (called bags by Yager) for the derivation of $V_{ik}$, for $i = 1, 2, 3, 4$ and $k = 1, 2, 3$.

$$M(X_1) = \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \\ M_{41} \end{bmatrix} = \begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix}$$

$$M(X_2) = \begin{bmatrix} M_{12} \\ M_{22} \\ M_{32} \\ M_{42} \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix}$$

$$M(X_3) = \begin{bmatrix} M_{13} \\ M_{23} \\ M_{33} \\ M_{43} \end{bmatrix} = \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} = C$$

• Using pessimistic attitude, and applying the OWA operator on each row of $M(X_k)$ for $k = 1$ to $r$, one gets finally: $V(X_1) = [V_{11}, V_{21}, V_{31}, V_{41}]^T = [7, 5, 3, 6]^T$, $V(X_2) = [V_{12}, V_{22}, V_{32}, V_{42}]^T = [5, 2, 9, 4]^T$ and $V(X_3). = [V_{13}, V_{23}, V_{33}, V_{43}]^T = [5, 2, 3, 4]^T$. Applying formula (6) for $i = 1, 2, 3, 4$ one gets finally the following generalized expected values using vectorial notation:

$$[C_1, C_2, C_3, C_4]^T = \sum_{k=1}^{r=3} m(X_k) \cdot V(X_k) = [6.2, 3.8, 4.8, 5.2]^T$$

According to these values, the best alternative to take is $A_1$ since it has the highest generalized expected payoff.

• Using optimistic attitude, one takes the max value of each row, and applying OWA on each row of $M(X_k)$ for $k = 1$ to $r$, one gets: $V(X_1) = [V_{11}, V_{21}, V_{31}, V_{41}]^T = [13, 12, 10, 15]^T$, $V(X_2) = [V_{12}, V_{22}, V_{32}, V_{42}]^T = [6, 10, 13, 9]^T$, and $V(X_3) = [V_{13}, V_{23}, V_{33}, V_{43}]^T = [13, 12, 13, 15]^T$. One finally gets $[C_1, C_2, C_3, C_4]^T = [10.9, 11.4, 11.2, 13.2]^T$ and the best alternative to take with optimistic attitude is $A_4$ since it has the highest generalized expected payoff. Then we have expected payoff as

$$E[C] = \begin{bmatrix} E[C_1] \\ E[C_2] \\ E[C_3] \\ E[C_4] \end{bmatrix} \subset \begin{bmatrix} [6.2; 10.9] \\ [3.8; 11.4] \\ [4.8; 11.2] \\ [5.2; 13.2] \end{bmatrix}$$

*2) Implementation of COWA-ER:* Let's describe in details each step of COWA-ER. In step 1, we divide each bound of intervals by the max of the bounds to get a new normalized imprecise expected payoff vector $E^{Imp}[C]$. In our example, one gets:

$$E^{Imp}[C] = \begin{bmatrix} [6.2/13.2; 10.9/13.2] \\ [3.8/13.2; 11.4/13.2] \\ [4.8/13.2; 11.2/13.2] \\ [5.2/13.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [0.47; 0.82] \\ [0.29; 0.86] \\ [0.36; 0.85] \\ [0.39; 1.00] \end{bmatrix}$$

In step 2, we convert each imprecise value into its bba according to a very natural and simple transformation [17]. Here, we need to consider the finite set of alternatives $\Theta =$

$\{A_1, A_2, A_3, A_4\}$ as FOD. The sources of belief associated with them are obtained from the normalized imprecise expected payoff vector $E^{Imp}[C]$. The modeling for computing a bba associated to the hypothesis $A_i$ from any imprecise value $[a; b] \subseteq [0; 1]$ is very simple and is done as in (7). where $\bar{A}_i$ is the complement of $A_i$ in $\Theta$. With such a simple conversion, one sees that $Bel(A_i) = a$, $Pl(A_i) = b$. The uncertainty is represented by the length of the interval $[a; b]$ and it corresponds to the imprecision of the variable (here the expected payoff) on which the belief function for $A_i$ is defined. In the example, one gets:

TABLE I
BBA'S OF THE ALTERNATIVES USED IN COWA-ER.

| Alternatives $A_i$ | $m_i(A_i)$ | $m_i(\bar{A}_i)$ | $m_i(A_i \cup \bar{A}_i)$ |
|---|---|---|---|
| $A_1$ | 0.47 | 0.18 | 0.35 |
| $A_2$ | 0.29 | 0.14 | 0.57 |
| $A_3$ | 0.36 | 0.15 | 0.49 |
| $A_4$ | 0.39 | 0 | 0.61 |

In step 3, we use Dempster's rule of combination to obtain[3] the combined bba, which is listed in Table II.

TABLE II
FUSION OF 4 BBA'S WITH DEMPSTER'S RULE FOR COWA-ER.

| Focal Element | $m_{Dempster}(.)$ |
|---|---|
| $A_1$ | 0.2522 |
| $A_2$ | 0.1151 |
| $A_3$ | 0.1627 |
| $A_4$ | 0.1894 |
| $A_1 \cup A_4$ | 0.0087 |
| $A_2 \cup A_4$ | 0.0180 |
| $A_3 \cup A_4$ | 0.0137 |
| $A_1 \cup A_2 \cup A_4$ | 0.0368 |
| $A_1 \cup A_3 \cup A_4$ | 0.0279 |
| $A_2 \cup A_3 \cup A_4$ | 0.0576 |
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.1179 |

In step 4, we use Pignistic Transformation to obtain the bba's corresponding pignistic probability listed in Table III. More efficient (but complex) transformations, like DSmP, could be used instead [19]. Based on the pignistic probability obtained, the decision result is $A_1$.

TABLE III
PIGNISTIC PROBABILITY BASED ON COWA-ER.

| Focal Element | $BetP(.)$ |
|---|---|
| $A_1$ | 0.3076 |
| $A_2$ | 0.1851 |
| $A_3$ | 0.2275 |
| $A_4$ | 0.2798 |

*3) Implementation of FCOWA-ER:* In step 1 of FCOWA-ER, we normalize each column in $E[C]$, respectively. In our example, one gets:

$$E^{Fuzzy}[C] = \begin{bmatrix} [6.2/6.2; 10.9/13.2] \\ [3.8/6.2; 11.4/13.2] \\ [4.8/6.2; 11.2/13.2] \\ [5.2/6.2; 13.2/13.2] \end{bmatrix} \approx \begin{bmatrix} [1.0000; 0.8258] \\ [0.6129; 0.8636] \\ [0.7742; 0.8485] \\ [0.8387; 1.0000] \end{bmatrix}$$

[3]Other combination rules can be used instead to circumvent the limitations of Dempsters rule discuseed in [19], [20].

Then we obtain two FMFs, which are
$\mu_1 = [1, 0.6129, 0.7742, 0.8387]$;
$\mu_2 = [0.8258, 0.8636, 0.8485, 1.0000]$.

In step 2, by using $\alpha$-cut approach, $\mu_1$ and $\mu_2$ are converted into two bba's $m_{Pess}(.)$ and $m_{Opti}(.)$ as listed in Table IV. In Step 3, we use Dempster's rule[4] to combine $m_{Pess}(.)$ and $m_{Opti}(.)$ to get $m_{Dempster}(.)$ as listed in Table V.

TABLE IV
THE TWO BBA'S TO COMBINE OBTAINED FROM FMFS.

| Focal Element | $m_{Pess}(.)$ | Focal Element | $m_{Opti}(.)$ |
|---|---|---|---|
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.6129 | $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.8257 |
| $A_1 \cup A_3 \cup A_4$ | 0.1613 | $A_2 \cup A_3 \cup A_4$ | 0.0227 |
| $A_1 \cup A_4$ | 0.0645 | $A_2 \cup A_4$ | 0.0152 |
| $A_1$ | 0.1613 | $A_4$ | 0.1364 |

TABLE V
FUSION OF TWO BBA'S WITH DEMPSTER'S RULE FOR FCOWA-ER.

| Focal Element | $m_{Dempster}(.)$ |
|---|---|
| $A_1$ | 0.1370 |
| $A_4$ | 0.1227 |
| $A_1 \cup A_4$ | 0.0549 |
| $A_2 \cup A_4$ | 0.0096 |
| $A_3 \cup A_4$ | 0.0038 |
| $A_1 \cup A_3 \cup A_4$ | 0.1370 |
| $A_2 \cup A_3 \cup A_4$ | 0.0143 |
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.5207 |

In step 4, we use again the Pignistic Transformation to get the pignistic probabilities listed in Table VI. Based on these probabilities, the decision result is also $A_1$. The decision results of COWA-ER and FCOWA-ER are the same.

TABLE VI
PIGNISTIC PROBABILITY BASED ON FCOWA-ER.

| Focal Element | $BetP(.)$ |
|---|---|
| $A_1$ | 0.3403 |
| $A_2$ | 0.1397 |
| $A_3$ | 0.1826 |
| $A_4$ | 0.3374 |

## IV. ANALYSES ON FCOWA-ER

### A. On computational complexity

In FCOWA-ER, only two bba's are involved in the combination. That is to say only one combination step is needed. Whereas in the original COWA-ER, if there exists $q$ alternatives, there should be $q-1$ evidence combination operations to do. Furthermore, the bba's obtained in the FCOWA-ER by using $\alpha$-cut are consonant support (nested in order). This will bring less computational complexity when compared with the bba's generated in the original COWA-ER. In summary, it is clear that the new proposed FCOWA-ER has lower computational complexity.

[4]In fact, and more generally the choice of a rule of combination is entirely left to the preference of the user in our FCOWA-ER methodology.

### B. On physical meaning

In this new FCOWA-ER approach, the two different information sources are pessimistic OWA and optimistic OWA. The combination result can be regarded as a tradeoff between these two attitudes. the physical or practical meaning is relatively clear. Furthermore, if the decision-maker has preference on pessimistic or optimistic attitude, one can use discounting in evidence combination to satisfy one's preference. We can set the preference of pessimistic attitude as $\lambda_p$ and set the the preference of optimistic attitude as $\lambda_o$. Then the discounting factor can be obtained as

$$\begin{cases} \beta = \lambda_o/\lambda_p, & \lambda_o \leq \lambda_p \\ \beta = \lambda_p/\lambda_o, & \lambda_p < \lambda_o \end{cases} \tag{10}$$

Then according to the discounting method [16], one will take:

$$\begin{cases} m_\beta(X) = \beta \cdot m(X), & \text{for } X \neq \Theta \\ m_\beta(\Theta) = \beta \cdot m(\Theta) + (1-\beta) \end{cases} \tag{11}$$

If $\lambda_o \leq \lambda_p$, then $m(.)$ in (11) should be $m_{Opti}(.)$; If $\lambda_p \leq \lambda_o$, then $m(.)$ in (11) should be $m_{Pess}(.)$; By using the discounting and choosing a combination rule, the decision maker's has a flexibility in his decision-making process.

### C. On management of uncertainty

In the FCOWA-ER, we first define the bba vertically taking into account the uncertainty between alternatives for the pessimist attitude and for the optimistic attitude. Then we combine two columns. The uncertainty incorporated in the FMF obtained, which represents the possibility of each alternative to be chosen as the final decision result. Based on $\alpha$-cut approach, the FMF is transformed into bba. The uncertainty is thus transformed to the bba. Although based on each column, only the information of pessimistic or optimistic is used, the combination operation followed can use both the two information sources (pessimistic and optimistic attitudes). Thus the available information can be fully used in FCOWA-ER. In COWA-ER, the modeling for each row (interval) takes into account the true uncertainty one has on the bounds of payoff for each alternative, then after modeling each bba $m_i(.)$, one combines them "vertically" to take into account the uncertainty between alternatives.

Although the ways to manage the uncertainty incorporated in are different for COWA-ER and FCOWA-ER, they both utilize (differently) the whole available information.

### D. On robustness to error scoring

Based on many experiments, we have observed that almost all the decision results given by FCOWA-ER agree[5] with COWA-ER results and are rational. However when the difference among the values in payoff matrix is significant, the COWA-ER can yield to wrong decisions whereas FCOWA-ER yields to rational decisions as illustrated in Example 2 below.

[5]when using the same rule of combination in steps 3, and the same probabilistic transformation in steps 4.

**Example 2** Let's take states $S = \{S_1, S_2, S_3, S_4, S_5\}$ with associated bba $m(S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) = 1$, and consider alternatives $A = \{A_1, A_2, A_3, A_4\}$ and the payoffs matrix:

$$C = \begin{bmatrix} 12 & 11 & 10 & 120 & 7 \\ 9 & 10 & 6 & 110 & 3 \\ 7 & 13 & 5 & 100 & 6 \\ 6 & 2 & 3 & 150 & 4 \end{bmatrix} \tag{12}$$

We see that the difference between max value and min value of each line is significant. For example, in the fourth row of $C$, only $S_4$ brings extremely high score for $A_4$ whereas other states bring homogeneous low score values for $A_4$. Whatever state of nature we consider $S_1$, $S_2$,..., or $S_5$, $A_1$ is either the top 1 or top 2 choice according to the ranks of the alternatives for states $S_i$, $i = 1, \ldots, 5$ as shown below:

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| $S_1$ | 1 | 2 | 3 | 4 |
| $S_2$ | 2 | 3 | 1 | 4 |
| $S_3$ | 1 | 2 | 3 | 4 |
| $S_4$ | 2 | 3 | 4 | 1 |
| $S_5$ | 1 | 4 | 2 | 3 |

So, intuitively, according to the principle of majority voting, the decision result should be $A_1$ but not $A_4$. According to rank-level fusion, the decision result should also be $A_1$.

The expected payoffs are:

$$E[C] = \begin{bmatrix} [7, & 120] \\ [3, & 110] \\ [5, & 100] \\ [2, & 150] \end{bmatrix}$$

- Using COWA-ER, one has

$$E^{Imp}[C] = \begin{bmatrix} [0.0467, & 0.8000] \\ [0.0200, & 0.7333] \\ [0.0333, & 0.6667] \\ [0.0133, & 1.0000] \end{bmatrix}$$

The bba's to combine are listed in Table VII and the combination results by using Dempster's rule are in Table VIII.

TABLE VII
EXAMPLE 2: BBA'S OF THE ALTERNATIVES USED IN COWA-ER.

| Alternatives $A_i$ | $m_i(A_i)$ | $m_i(\bar{A}_i)$ | $m_i(A_i \cup \bar{A}_i)$ |
|---|---|---|---|
| $A_1$ | 0.0467 | 0.2000 | 0.7533 |
| $A_2$ | 0.0200 | 0.2667 | 0.7133 |
| $A_3$ | 0.0333 | 0.3333 | 0.6334 |
| $A_4$ | 0.0133 | 0 | 0.9867 |

The pignistic probabilities listed in IX indicate that the decision result[6] of COWA-ER is $A_4$.

- Using FCOWA-ER, one has

$$E^{Fuzzy}[C] = \begin{bmatrix} [1.0000, & 0.8000] \\ [0.4286, & 0.7333] \\ [0.7143, & 0.6667] \\ [0.2857, & 1.0000] \end{bmatrix}$$

In FCOWA-ER, the bba's to combine are listed in Table X and their Dempster's combination is listed in Table XI.

TABLE VIII
EXAMPLE 2: DEMPSTER'S FUSION OF 4 BBA'S FOR COWA-ER.

| Focal Element | $m_{Dempster}(.)$ |
|---|---|
| $A_1$ | 0.0438 |
| $A_2$ | 0.0182 |
| $A_3$ | 0.0309 |
| $A_4$ | 0.0297 |
| $A_1 \cup A_4$ | 0.0664 |
| $A_2 \cup A_4$ | 0.0471 |
| $A_3 \cup A_4$ | 0.0335 |
| $A_1 \cup A_2 \cup A_4$ | 0.1775 |
| $A_1 \cup A_3 \cup A_4$ | 0.1261 |
| $A_2 \cup A_3 \cup A_4$ | 0.0895 |
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.3373 |

TABLE IX
EXAMPLE 2: PIGNISTIC PROBABILITY BASED ON COWA-ER.

| Focal Element | $BetP(.)$ |
|---|---|
| $A_1$ | 0.2625 |
| $A_2$ | 0.2152 |
| $A_3$ | 0.2038 |
| $A_4$ | 0.3185 |

TABLE X
EXAMPLE 2: BBA'S OF THE ALTERNATIVES USED IN FCOWA-ER.

| Focal Element | $m_{Pess}(.)$ | Focal Element | $m_{Opti}(.)$ |
|---|---|---|---|
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.2857 | $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.6667 |
| $A_1 \cup A_2 \cup A_3$ | 0.1429 | $A_1 \cup A_3 \cup A_4$ | 0.0667 |
| $A_1 \cup A_3$ | 0.2857 | $A_3 \cup A_4$ | 0.0667 |
| $A_1$ | 0.2857 | $A_1$ | 0.1999 |

TABLE XI
EXAMPLE 2: DEMPSTER'S FUSION OF THE TWO BBA'S FOR FCOWA-ER.

| Focal Element | $m_{Dempster}(.)$ |
|---|---|
| $A_1$ | 0.3223 |
| $A_4$ | 0.0667 |
| $A_1 \cup A_2$ | 0.0111 |
| $A_1 \cup A_3$ | 0.2222 |
| $A_1 \cup A_4$ | 0.0222 |
| $A_1 \cup A_2 \cup A_3$ | 0.1111 |
| $A_1 \cup A_2 \cup A_4$ | 0.0222 |
| $A_1 \cup A_2 \cup A_3 \cup A_4$ | 0.2222 |

The pignistic transformation of $m_{Dempster}(.)$ yields to the pignistic probabilities listed in Table XII.

TABLE XII
EXAMPLE 2: PIGNISTIC PROBABILITY BASED ON FCOWA-ER.

| Focal Element | $BetP(.)$ |
|---|---|
| $A_1$ | 0.5500 |
| $A_2$ | 0.1056 |
| $A_3$ | 0.2037 |
| $A_4$ | 0.1407 |

Based on the pignistic probabilities, the decision result obtained with FCOWA-ER is now $A_1$, which is the correct one. In this example, FCOWA-ER shows its robustness when compared with COWA-ER.

### *E. On the normalization procedures*

In fact, there exist at least three normalization procedures that we briefly recall below. Suppose $\mathbf{x}$ is the original vector input, $\mathbf{x_i}$ represents the $i$th dimension of $\mathbf{x}$. $y_i$ represents the $i$th dimension of the normalized vector $\mathbf{y}$, then we examine the following three types of normalization:

1) Type I: $y_i = x_i / \max(\mathbf{x})$
2) Type II: $y_i = (x_i - \min(\mathbf{x}))/(\max(\mathbf{x}) - \min(\mathbf{x}))$
3) Type III: $y_i = x_i / \sum_j (x_j)$

To verify wether the decision results of COWA-ER and the new FCOWA-ER can be affected by the normalization procedure, we did some tests as follows. We randomly generate payoff matrices and use all the three types of normalization approaches in COWA-ER and FCOWA-ER respectively. Then we make comparisons among the results obtained. We repeat the experiment 50 times (50 Monte-Carlo runs). Based on our simulation results, we find that normalization approaches can affect the decision results of COWA-ER and FCOWA-ER, although the ratio of disagreement among different normalization approach is small (about 1 to 2 times of disagreement out of 50 experiments in average). Example 3 is a case where the disagreement occurs due to the different types of normalization.

**Example 3**: We consider the following payoff matrix

$$C = \begin{bmatrix} 15 & 5 & 30 \\ 5 & 40 & 40 \\ 40 & 30 & 30 \\ 15 & 10 & 40 \end{bmatrix}$$

The bba is $m(X_1) = 0.5439$, $m(X_2) = 0.3711$, $m(X_3) = 0.0849$, where $X_1 = S_2 \cup S_3$, $X_2 = S_1 \cup S_2$ and $X_3 = S_1 \cup S_3$.

• Using COWA-ER, based on normalization Type I, Type II and Type III, we can obtain the corresponding expected payoffs

$$E_I[C] = \begin{bmatrix} [0.1462, & 0.6108] \\ [0.6009, & 1.0000] \\ [0.7500, & 0.8640] \\ [0.2606, & 0.7680] \end{bmatrix}$$

$$E_{II}[C] = \begin{bmatrix} [0.0000, & 0.5442] \\ [0.5326, & 1.0000] \\ [0.7072, & 0.8407] \\ [0.1340, & 0.7283] \end{bmatrix}$$

$$E_{III}[C] = \begin{bmatrix} [0.0292, & 0.1221] \\ [0.1202, & 0.2000] \\ [0.1500, & 0.1728] \\ [0.0521, & 0.1536] \end{bmatrix}$$

Then we obtain the pignistic probabilities listed in Table XIII. From Table XIII, one sees that the decision result with Type III normalization is $A_2$ while those of Type I and Type II yields $A_3$.

TABLE XIII
EXAMPEL 3: PIGNISTIC PROB. FOR TYPES I, II & III AND COWA-ER.

| Focal Element | $BetP_I(.)$ | $BetP_{II}(.)$ | $BetP_{III}(.)$ |
|---|---|---|---|
| $A_1$ | 0.0587 | 0.0388 | 0.1690 |
| $A_2$ | 0.3203 | 0.3324 | 0.3180 |
| $A_3$ | 0.5223 | 0.5444 | 0.2920 |
| $A_4$ | 0.0987 | 0.0844 | 0.2210 |

[6]based on max of $BetP(.)$.

• Using FCOWA-ER, based on normalization of Type I, Type II and Type III, we get the corresponding expected payoffs

$$E_I[C] = \begin{bmatrix} [0.1950, & 0.6108] \\ [0.8013, & 1.0000] \\ [1.0000, & 0.8640] \\ [0.3475, & 0.7680] \end{bmatrix}$$

$$E_{II}[C] = \begin{bmatrix} [0.0000, & 0.0000] \\ [0.7531, & 1.0000] \\ [1.0000, & 0.6506] \\ [0.1894, & 1.4040] \end{bmatrix}$$

$$E_{III}[C] = \begin{bmatrix} [0.0832, & 0.1884] \\ [0.3419, & 0.3084] \\ [0.4267, & 0.2664] \\ [0.1483, & 1.2368] \end{bmatrix}$$

Then we can get the pignistic probabilities listed in Table XIV. From Table XIV, one sees that the decision with Type II normalization is $A_2$ while those of Type I and Type III yields $A_3$.

TABLE XIV
EXAMPLE 3: PIGNISTIC PROBABILITY BASED ON FCOWA-ER

| Focal Element | $BetP_I(.)$ | $BetP_{II}(.)$ | $BetP_{III}(.)$ |
|---|---|---|---|
| $A_1$ | 0.0306 | 0.0000 | 0.0306 |
| $A_2$ | 0.4118 | 0.5421 | 0.4118 |
| $A_3$ | 0.4763 | 0.4300 | 0.4763 |
| $A_4$ | 0.0813 | 0.0279 | 0.0813 |

So in a little percentage of cases, we must be cautious when choosing a normalization procedure and so far there is no clear theoretical answer for the choice of the most adapted normalization procedure. We prefer the Type I normalization procedure since it is very simple and intuitively appealing.

## V. CONCLUSION

In this paper, we have proposed a fuzzy cautious OWA method using evidential reasoning (FCOWA-ER) to implement the multi-criteria decision making, where evidence theory, fuzzy membership functions and OWA are used jointly. This method has less computational complexity and has clearer physical meaning. Furthermore, it is more robust to the error scoring in MCDM. Experimental results and related analyses show that our FCOWA-ER is interesting and flexible because its three main specifications can be adapted easily for working: 1) with other rules of combination than Dempster's rule, 2) with other probabilistic transformations than BetP, and 3) with different normalization procedures. Of course the performances of FCOWA-ER depend on the choice of these three main specifications taken by the MCDM system designer. The method to generate the bba from the FMF based on $\alpha$-cut depends on the selection of the parameter vector of $\alpha$. The impact of the choice of the specifications as well as $\alpha$ to evaluate the performance of FCOWA-ER will be further analyzed in our future works. This paper was devoted to the theoretical developemnt of FCOWA-ER and its evaluation for applications to real MCDM problems is part of our future research works.


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