Smarandache Triple Tripotents in $\mathbb{Z}_n$

and in Group Ring $\mathbb{Z}_2G$

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Abstract

In this paper, we study tripotent elements and Smarandache triple tripotents (S-T. tripotents) in $\mathbb{Z}_n$, the ring of integers modulo $n$, and in group ring $\mathbb{Z}_2G$ where $G$ is a cyclic group of order $2n$ ($n$ is an odd number).

Keywords: Tripotent, Smarandache triple tripotent

Introduction

The concept of m-idempotent was introduced by H. Chaoling and G. Yonghua at 2010,[2]. Smarandache concepts introduced by Florentin Smarandache [7]. Smarandache idempotent element in rings defined by Vasantha Kandasamy [8]. This paper has two sections. In section one we introduce the concept of Smarandache triple tripotent in rings (S-T. tripotent). We find the number of tripotents and S-T. tripotents and their forms in $\mathbb{Z}_n$, the ring of integers modulo $n$. In section two, we study tripotents and S-T. tripotents in the group ring $\mathbb{Z}_2G$, where $G$ is a cyclic group of order $2n$ ($n$ is an odd number), in particular, when $n$ is a Mersenne prime that is a prime of the form $2^k - 1$ for some prime $k$, and we obtain their numbers.

§1. Tripotents and S-T. tripotents in the ring $\mathbb{Z}_n$.

In this section the concept of S-T. tripotent introduced. We study tripotents and S - T. tripotents in $\mathbb{Z}_n$, for $n = 2^k$, $pq$, $pqr$, for distinct primes $p$, $q$ and $r$, we find the number of tripotents and S-T. tripotents and their forms.
Definition 1.1.[2]. An element $\alpha$ of a ring $R$ is called tripotent ($3$–idempotent), if $\alpha^3 = \alpha$. A tripotent element is said to be non-trivial tripotent if $\alpha^2 \neq \alpha$.

Now, we introduce the concept of S-T. tripotent.

Definition 1.2. Three distinct non-trivial tripotents $x$, $y$, $z$ in a commutative ring $R$ called Smarandache triple tripotent (S-T. tripotent) if $xy=z$, $xz=y$ and $yz=x$.

The proof of the following result is easy.

Proposition 1.3. In $\mathbb{Z}_n$, $n > 2$, the element $[n-1]$ (equivalence class of $n-1$) is a non-trivial tripotent (we write $n-1$ instead of $[n-1]$).

The following useful Lemma is needed.

Lemma 1.4. If $x$ is a non-trivial idempotent of $\mathbb{Z}_n$ and $x-1 \neq 2n$, then $x-1$ and $2x-1$ are non-trivial tripotents.

Proof: Let $x$ be a non-trivial idempotent of $\mathbb{Z}_n$ with $x-1 \neq \frac{n}{2}$. Then $x^2 \equiv x$ (mod $n$).

Now, $(x-1)^3 \equiv x-1$ (mod $n$), hence $x-1$ is a tripotent. We have to show that $(x-1)$ is not an idempotent. If $(x-1)^2 \equiv x-1$ (mod $n$), then $1-x \equiv x-1$ (mod $n$). Hence $2(x-1) \equiv 0$ (mod $n$). This means that $n|2(x-1)$. If $n$ is an odd number, then $n|(x-1)$, hence $x \equiv 1$ (mod $n$) which is a trivial idempotent. If $n$ is an even number, then $x-1 \equiv 0$ (mod $\frac{n}{2}$), so $x-1 \equiv \frac{n}{2}$ (mod $n$) which is a contradiction with the assumption. Therefore $x-1$ is a non-trivial tripotent. Similarly we can show $(2x-1)$ is a non trivial tripotent. ■

The converse of Lemma 1.4 is not true in general (if $y$ and $2y+1$ are non-trivial tripotents, then it is not necessary that $y+1$ is an idempotent and $y \neq \frac{n}{2}$).

Example 1.5. In $\mathbb{Z}_{60}$, the ring of integers modulo 60, take $y=4$, then $2y+1=9$. Clearly $y$ and $2y+1$ are non-trivial tripotents, but $y+1=5$ is not an idempotent.

In the following result, a condition under which the converse of Lemma 1.4 is true is given.

Proposition 1.6. Let $y$ and $2y+1$ be non-trivial tripotents in $\mathbb{Z}_n$, such that $(n, 12)=1$. Then $y+1$ is a nontrivial idempotent and $y \neq \frac{n}{2}$ (mod $n$).

Proof: From the assumption we have $y^3 \equiv y$ (mod $n$) and $(2y+1)^3 \equiv 2y+1$ (mod $n$). This implies that $n|12(y^2+y)$. But $(n,12)=1$, so $y^2+y\equiv 0$ (mod $n$). Consequently $(y+1)^2 \equiv y+1$ (mod $n$). Hence $(y+1)$ is a non trivial idempotent, and clearly $y \neq \frac{n}{2}$ (mod $n$). ■
Proposition 1.7. The ring \( \mathbb{Z}_{2^n} \), \( n > 2 \) has exactly three non trivial tripotents, furthermore they forms a S-T. tripotent.

Proof: By Proposition 1.3, the element \((2^n-1)\) is a non trivial tripotent, and easily one show that \(2^{n-1} - 1, 2^{n-1} + 1\) are non trivial tripotents, and that the triple \(2^n - 1, 2^{n-1} - 1, 2^{n-1} + 1\), forms a S-T. tripotent. Now, suppose that \(x\) is any other non trivial tripotent. Then \(x(x^2-1) \equiv 0 \pmod{2^n}\), so \(2^n \mid x(x^2-1)\). This means that, either \(2^n \mid x\) or \(2^n \mid x^2-1\). If \(2^n \mid x\), then \(x \equiv 0 \pmod{2^n}\) which is a contradiction with \(x \not\equiv 0 \pmod{2^n}\). Thus \(2^n \mid x^2-1\), hence \(x^2 \equiv 1 \pmod{2^n}\). This congruence has four solutions they are \(1, 2^{n-1} - 1, 2^{n-1} + 1\), \([6]\). The solution \(1\) is trivial and the others are the same as above. Hence \(\mathbb{Z}_{2^n}\) has exactly three non trivial tripotents, and it is easy to show that the triple \((2^{n-1} - 1), (2^{n-1} + 1), (2^n - 1)\) is a S-T. tripotent. ■

Proposition 1.8. Let \(p\) be an odd prime. Then \(\mathbb{Z}_{p^n}\), for \(n \geq 1\) has only one non trivial tripotent.

Proof: By Proposition 1.3, the element \(p^n - 1\) is a non trivial tripotent. Suppose \(x\) is any other non trivial tripotent in \(\mathbb{Z}_{p^n}\). Then \(x(x^2 - 1) \equiv 0 \pmod{p^n}\), this means \(p^n \mid x(x^2-1)\). If \(n=1\), \(p \mid x(x^2-1)\), then \(p \mid x\) or \(p \mid x^2-1\), but \(p \not\mid x\) because otherwise \(x \equiv 0 \pmod{p}\), hence \(p \mid x^2 - 1\), so \(x^2 \equiv 1 \pmod{p}\). The solutions of the congruence \(x^2 \equiv 1 \pmod{p}\) are \(1, p-1\) \([1]\), but \(1\) is a trivial idempotent and \(p-1\) is the same idempotent obtained by Proposition 1.3. Therefore there is exactly one nontrivial tripotent. Now, suppose \(n>1\) and that \(x \not\equiv p^n - 1\) is any non trivial tripotent. Then \(x(x^2 - 1) \equiv 0 \pmod{p^n}\). Since \(p\) is a prime, either \(p^n \mid x\) or \(p^n \mid x^2-1\). If \(p^n \mid x\), then \(x \equiv 0 \pmod{p^n}\) contradiction with \(x \not\equiv 0 \pmod{p^n}\), therefore \(p^n \mid x^2-1\), that means \(x^2 \equiv 1 \pmod{p^n}\), but this congruence has exactly two incongruent solutions \([6]\), either \(x \equiv 1 \pmod{p^n}\) which is a trivial idempotent or \(x \equiv p^n - 1 \pmod{p^n}\) which his the tripotent obtained from Proposition 1.3. Hence \(\mathbb{Z}_{p^n}\) has exactly one non trivial tripotent. ■

Recall that if \(a, b\) are positive integers with \((a, b) = d\), then the Diophantine equation \(xa + yb = c\) has infinite solutions if \(d \mid c\) and has no solution if \(d \nmid c\), we give the following result.

Theorem 1.9. Let \(n = pq\), where \(p\) and \(q\) are distinct odd primes. Then \(\mathbb{Z}_n\) has exactly five non trivial tripotents, and one S-T. tripotent.

Proof: By Proposition 1.3, the element \(pq - 1\) is a non trivial tripotent of \(\mathbb{Z}_n\). By Diophantine equation, there exist \(t, s \in \mathbb{Z}\), \(t > 0\) such that \(tq - sp = 1\) and there exist \(t_1, s_1 \in \mathbb{Z}\), \(t > 0\) such that \(t_1p - s_1q = 1\). It is shown in \([4]\), that \(tq\) and \(t_1p\) are non trivial idempotents (In fact \(tq = n+1 - tq \pmod{pq}\) ) of \(\mathbb{Z}_{pq}\). Then by Lemma 1.4 the elements \(tq - 1, 2tq - 1, n - tq\) and \(1 - 2tq\) are non trivial tripotents. So we get five non trivial tripotents. Suppose that \(x\) be any other non trivial tripotent, thus
let $x^2 \equiv x \pmod{pq}$, so $x(x^2 - 1) \equiv 0 \pmod{pq}$, which means $pq | x(x^2 - 1)$. There are three cases:

1. $p | x$ and $q | x^2 - 1$.
2. $p | x^2 - 1$ and $q | x$, and
3. $pq | x^2 - 1$.

In case (1), $x \equiv 0 \pmod{p}$, hence $x \equiv kp \pmod{pq}$, for some $k$, $0 \leq k \leq q - 1$, and

$q | x^2 - 1$, then $x^2 \equiv 1 \pmod{q}$, by [1], $x \equiv 1 \pmod{q}$ or $x \equiv q - 1 \pmod{q}$. If $x \equiv 1 \pmod{q}$, hence $x \equiv 1 + rq \pmod{pq}$ for some $r$, $0 \leq r \leq p - 1$, then $kp-rq=1$ which means $x=kp$ is an idempotent (it is a trivial tripotent). When $x \equiv q - 1 \pmod{q}$, we get $x \equiv s_3 q - 1 \pmod{pq}$ for some $s_3$, $0 \leq s_3 \leq p - 1$. Therefore $s_3 q - kp = 1$, this means $x=kp$ is a non trivial tripotent which is obtained before.

Case (2) is similar.

Case (3) $pq | x^2 - 1$, then $x^2 \equiv 1 \pmod{pq}$. This congruence has four solutions 1, 1-2tq, 2tq-1 and pq-1, [6]. The solution 1 is trivial, the others was obtained before. Therefore $Z_{pq}$ has exactly five non trivial tripotents, and a simple calculation shows that the triple $(n-1), (1-2tq), (2tq-1)$ is a S-T. tripotent.

Proposition 1.10. Let $p$ be an odd prime. Then $Z_{2p}$ has exactly two non trivial tripotents.

Proof: It is shown in [4], that $Z_{2p}$ has only two non trivial idempotents namely $p$ and $p+1$. Then by Lemma 1, the elements $p - 1$ and $2p - 1$ are non trivial tripotents in $Z_{2p}$. But $2p - 1$ is the same tripotent obtained by Proposition 1,3. Hence $Z_{2p}$ has two non trivial tripotents. Suppose that $x$ is any other non trivial tripotent, then $x(x^2 - 1) \equiv 0 \pmod{2p}$, there are two cases:

1. $2 | x$, and $p | x^2 - 1$.
2. $2 | x^2 - 1$ and $p | x$.

In case (1), $x \equiv 0 \pmod{2}$, hence $x \equiv 2t_1 \pmod{2p}$, for some $0 \leq t_1 \leq p - 1$, and

$p | x^2 - 1$, then $x^2 \equiv 1 \pmod{p}$. The congruent $x^2 \equiv 1 \pmod{p}$ has exactly two solutions 1, $1 \pmod{p}$, if $x \equiv 1 \pmod{p}$, then $x \equiv 1 + kp \pmod{2p}$ for some $0 \leq k \leq 2p - 1$, hence $2t_1 - kp = 1$ which means $x = 2t_1$ is an idempotent. When $x \equiv p - 1 \pmod{2p}$, hence $x \equiv s_1 p - 1 \pmod{2p}$ for some $0 \leq s_1 \leq 2p - 1$. Therefore $s_1 p - 2t_1 = 1$ this means $x = 2t_1$ is a non trivial tripotent which is obtained before.

Case (2), is similar.

Hence $Z_{2p}$ has exactly two non trivial tripotents.

Theorem 1.11. Let $n = p^a q$, where $p$, $q$ are distinct odd primes. Then $Z_n$ has exactly five non trivial tripotents, and one S-T. tripotent.

Proof: By Diophantine equation, there exist $t, s \in \mathbb{Z}$, $t > 0$ such that $tq - sp^a = 1$.

By similar method used in the proof of Theorem 1,9 one can show that $p^a q - 1, tq - 1, 2tq - 1, n - tq$ and $1 - 2tq$ are non trivial tripotents in $Z_{p^a q}$, and it is easy to show that the triple $n - 1, 1 - 2tq$ and $2tq - 1$, is a S-T. tripotent.
**Theorem 1.12.** Let \( n=2pq \) where \( p \) and \( q \) are distinct odd primes. Then \( \mathbb{Z}_n \) has exactly ten non trivial tripotents and two S-T. tripotents.

**Proof:** By Proposition 1.3, the element \( 2pq-1 \) is a non trivial tripotent. Suppose that \( p < q \). Then by Diophantine equation, there exist \( t, s \in \mathbb{Z}, t > 0 \) such that \( tq-sp=1 \) as \( (p, q)=1 \). As it is shown in [4], \( Z_n \) has exactly 6 non trivial idempotents they are \( pq, pq+1, tq, 2pq+1-tq, pq+1q \) and \( 1-tq+pq \). By Lemma 1.4 the elements \( pq-1, 2pq-1, tq-1, 2tq-1, 2pq-tq, 1-2tq \) and \( pq+1q-1 \) are non trivial tripotents. The element \( 2pq-1 \) is the same non trivial tripotent obtained by Proposition 1.3, so we obtain seven non trivial tripotents and it is not difficult to show that \( 1-2tq+pq, 2tq-1+pq \) and \( 2pq-tq+pq \) are also non trivial tripotents. Hence \( Z_{2pq} \) has ten non trivial tripotents. Suppose that \( x \) is any other non trivial tripotent for \( Z_{2pq} \). Then \( x (x^2-1) \equiv 0 \pmod{2pq} \), this means that \( pq \mid x(x^2-1) \). There are three cases:

1. \( pq \mid x \) or \( pq \mid x^2-1 \)
2. \( q \mid x \) and \( p \mid x^2-1 \)
3. \( q \mid x^2-1 \) and \( p \mid x \).

In case (1), \( x=pq \pmod{2pq} \), but \( pq \) is an idempotent, so it is a trivial tripotent. If \( pq \mid x^2-1 \), then \( x \equiv 1 \pmod{pq} \), this congruence has the following four solutions, \( 1 \) which is trivial, \( 2pq-1, 1-2tq \) and \( 2pq-1 \) are obtained before.

In case (2), \( x \equiv 0 \pmod{q} \), hence \( x \equiv t_1 q \pmod{2pq} \) for some \( 0 \leq t_1 \leq 2p-1 \), and \( p \mid x^2-1 \), then \( x^2 \equiv 1 \pmod{p} \), by [1] \( x \equiv 1 \pmod{pq} \), or \( x \equiv -1 \pmod{pq} \), are solutions of the congruence \( x^2 \equiv 1 \pmod{pq} \). If \( x \equiv 1 \pmod{pq} \), hence \( x \equiv 1+r \pmod{pq} \), for some \( 0 \leq r \leq 2q-1 \), then \( x \equiv 1 \pmod{pq} \) is an idempotent. When \( x \equiv -1 \pmod{pq} \), hence \( x \equiv 1 \pmod{pq} \) for some \( 0 \leq s_1 \leq 2q-1 \). Therefore \( x \equiv 1 \pmod{pq} \) is a non trivial tripotent which is obtained before. Case (3) is similar.

Hence \( Z_{2pq} \) has exactly ten non trivial tripotents.

Now, we show that the triple \( (2pq-1), (2tq-1), (1-2tq) \) is a S-T. tripotent.

\[
\begin{align*}
(2pq-1)(2tq-1) \pmod{2pq} &= 4tqpq - 2pq - 2tq + 1 \equiv 1-2tq \pmod{2pq},
(2pq-1)(1-2tq) \pmod{2pq} &= 2pq-1 \equiv 2pq-1 \pmod{2pq},
(2tq-1)(1-2tq) \pmod{2pq} &= 2pq-1 \equiv 2pq-1 \pmod{2pq}
\end{align*}
\]

Therefore \( (2pq-1), (2tq-1) \) and \( (1-2tq) \) is a S-T. tripotent. Similarly \( (pq-1), (1-2tq +pq), (2tq-1+pq) \) forms a S-T. tripotent. Hence \( Z_{2pq} \) has two S-T. tripotents.

The following example illustrates the above results.

**Example 1.13.**

1) The non trivial tripotents of \( \mathbb{Z}_8 \) are \( 3, 5 \) and \( 7 \). The triple \( 3, 5, 7 \) is a S-T. tripotent, (proposition 1.7).

2) \( Z_{243} \) has only one non trivial tripotent, namely \( 242 \), (proposition 1.8).

3) Consider \( Z_n, n=3\cdot 7=21 \). Now, \( 1(7) -2(3) =1 \) by Theorem 1.10, the tripotents are \( 6, 14, 20, 8, 13 \), and the triple \( 20, 8, 13 \) is a S-T. tripotent, (Theorem 1.9).
4) $\mathbb{Z}_{10}$ has exactly two non trivial tripotents they are 4 and 9, (proposition 1.10)

5) In $\mathbb{Z}_{135}$, 11(5)-2(27)=1. By Theorem 1.11, $\mathbb{Z}_{135}$ has five non trivial tripotents 54, 80, 26, 109, 134, and the triple 26, 109, 134 is a S-T. tripotent.

6) In $\mathbb{Z}_{154}$, 2(11)-3(7)=1. By Theorem 1.12 the elements 76, 21, 132, 98, 153, 43, 55, 120, 111 and 34 are non trivial tripotents, the triples 153, 43, 111 and 76, 120, 34 are S-T. tripotents.

**Theorem 1.14.** Let $n=pqr$ for distinct odd primes $p$, $q$, $r$. Then $\mathbb{Z}_n$ has exactly 19 non trivial tripotents, and at least three S-T. tripotents.

**Proof:** By Proposition 1.3, the element $pqr-1$ is a non trivial tripotent. Since $(pq, r)=1$ there are $s, t \in \mathbb{Z}$ with $t > 0$ such that $spq-tr=1$, and there are $s_2, t_2$ such that $s_2pr-t_2q=1$. It is shown in [4], $\mathbb{Z}_{pqr}$ has 6 non trivial idempotents, they are $spq, s_1qr, s_2pr, pqr+1-spq, pqr+1-s_1qr$ and $pqr+1-s_2pr$. By Lemma 1.4 the elements $spq-1, 2spq-1, s_1qr-1, 2s_1qr-1, s_2pr-1, 1-2spq, 1-2s_1qr, 1-2s_2pr, pqr-spq, pqr-s_1qr$ and $pqr-s_2pr$ are non trivial tripotents. We can also show that the following six elements $spq-s_1qr, s_1qr-spq, s_2pr-spq, s_2pr-s_1qr$ and $s_2pr-s_1qr$ are also non trivial tripotents in $\mathbb{Z}_n$. Suppose that $x$ is any other non trivial tripotent of $\mathbb{Z}_n$, then $x(x^2-1)\equiv 0 \pmod{pqr}$. This means that $pqr \mid x(x^2-1)$. If $pqr \mid x$ then $x \equiv 0 \pmod{pqr}$, contradiction with $x \equiv 0 \pmod{pqr}$. So we have the cases:

(1) $pqr \mid x^2-1$.
(2) $p \mid x$ and $qr \mid x^2-1$.
(3) $p \mid x^2-1$ and $q \mid x$.
(4) $pq \mid x$ and $r \mid x^2-1$.
(5) $pq \mid x^2-1$ and $qr \mid x$.
(6) $pr \mid x^2-1$ and $q \mid x$.
(7) $pr \mid x^2-1$ and $q \mid x$.

In case (1), $pqr \mid x^2-1$, then $x^2 \equiv 1 \pmod{pqr}$. This congruence has 8 solutions they are 1, 2spq-1, 2s_1qr-1, 2s_2pr-1, 1-2spq, 1-2s_1qr, 1-2s_2pr and $pqr-1$, [6]. But all of them were obtained before.

In case (2), $x \equiv 0 \pmod{p}$, then $x \equiv t_3p \pmod{pqr}$, for some $t_3, 0 \leq t_3 \leq qr-1$, and $x^2 \equiv 1 \pmod{qr}$, hence $x^2 \equiv 1+s_3qr \pmod{pqr}$, for some $s_3, 0 \leq s_3 \leq p-1$, thus $(r(p))^2-s_3qr=1$, hence $x \equiv r(t(p))^2$ is an idempotent.

Case (3) $x^2 \equiv 1 \pmod{p}$. Then $x \equiv 1 \pmod{p}$ or $x \equiv p-1 \pmod{p}$. If $x \equiv 1 \pmod{p}$, hence $x \equiv 1+kp \pmod{pqr}$, for some $k, 0 \leq k \leq qr-1$. When $x \equiv p-1 \pmod{p}$, then $x \equiv k_1p-1 \pmod{pqr}$, for some $k_1, 1 \leq k_1 \leq qr-1$, and $x \equiv qr \pmod{pqr}$, hence $x \equiv jqr \pmod{pqr}$, for some $j, 1 \leq j \leq p-1$, therefore $jqr-kp=1$ which means $x=jqr$ is an idempotent, also $k_1p-jqr=1$, leads to a contradiction, since $x=jqr$ is a non trivial tripotent.

Cases (4) and (6) are similar to case (3).

Cases (5) and (7) are similar to case (2).

Hence $\mathbb{Z}_{pqr}$ has exactly 19 non trivial tripotents. One can show that the triples 2spq-1, 1-2spq, pqr-1; 1-2s_1qr, 2s_1qr-1, pqr-1 and 2s_2pr, 1-2s_2pr, pqr-1 are S-T. tripotents.

We have to mention here that in general there are more than three S-T. tripotents but we could not find their forms.
Example 1.15. The non trivial tripotents of \( Z_{105} \), are 20, 104, 90, 35, 171, 99, 29, 34, 64, 49, 76, 56, 6, 50, 55, 41, 14, 69 and the triples : 29, 41, 34; 69, 99, 6; 4, 49, 56; 20, 55, 50; 104, 41, 64; 71, 76, 41; 64, 71, 29; 64, 76, 34; 104, 71, 34; 104, 76, 24 are S-T. tripotents.

\[ \text{§2. Smarandache triple tripotents in the group ring } Z_2G \]

In this section we study tripotents and S-T. tripotents in the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) (\( n \) is an odd number) generated by \( g \), specially, when \( n \) is a Mersenne prime, and we obtain their numbers. For definition of group ring see[3]. We start by the following definition.

**Definition 2.1.[8]**. Let \( R \) be a ring. An element \( 0 \neq x \in R \) is a Smarandache idempotent (S-idempotent) of \( R \) if

1) \( x^2=x \).
2) There exists a \( a \in R \setminus \{0, 1, x\} \)
   i) \( a^2 = x \) and
   ii) \( xa = a \) (ax = a) or \( ax = x \) (xa = x).

\( a \) called the Smarandache co-idempotent (S-co-idempotent).

The following lemma is needed.

**Lemma 2.2.** Let \( \alpha \) be a S-idempotent of the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) (\( n \) is an odd number) generated by \( g \) and \( \beta \) be a S-co-idempotent of \( \alpha \) with \( \alpha \beta = \beta \). Then \( \beta, \alpha + \beta + g^\alpha + g^\beta \) and \( \alpha + \beta + 1 \) are non trivial tripotents.

**Proof:** Since \( \beta \) is a S-co-idempotent of \( \alpha \), we get \( \beta^2 = \alpha \neq \beta \), consequently \( \beta^3 = \beta \), hence \( \beta \) is a non trivial tripotent. Then \( (\alpha + \beta + g^n)^3 = \alpha + \beta + g^n \), hence \( \alpha + \beta + g^n \) is a non trivial tripotent. Similarly \( \alpha + \beta + 1 \) is a non trivial tripotent. \( \blacksquare \)

**Theorem 2.3.** In the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) (\( n \) is an odd number) generated by \( g \), for any \( k \) distinct integers \( t_1 < t_2 < \cdots < t_k \), 0 < \( k \), \( t_i \leq n-1 \) for each \( i \), \( g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^B+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k} \) and \( 1 + g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k} \) are non trivial tripotents. Moreover the number of non trivial tripotents is equal to \( \sum_{s=1}^{n-1} \binom{n-1}{s} + \sum_{s=1}^{n-1} \binom{n-1}{s} \).

**Proof:** Let \( t_1, t_2, \ldots, t_k \) be any \( k \) distinct integers with \( 0 < t_1 < t_2 < \cdots < t_k \), \( t_i \), \( k \leq n-1 \). Let \( d_k = g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^B+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k} \). Then \( d_k^2 = g^{2t_1}+g^{2t_2}+\cdots+g^{2t_k}+g^{2n+2t_1}+g^{2n+2t_2}+\cdots+g^{2n+2t_k} = 1 \neq d_k \). Hence \( d_k^3 = d_k \), so \( d_k \) is a non trivial tripotent. Using some known facts from probability theory, the number of such tripotents is \( \sum_{s=1}^{n-1} \binom{n-1}{s} \). Clearly \( g^n \) is also a non trivial tripotent we denote \( d_0 = g^n \).

Let \( f_k = 1 + g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k} \). Then \( f_k^2 = 1 + g^{t_1}+g^{t_2}+\cdots+g^{t_k}+g^{n+t_1}+g^{n+t_2}+\cdots+g^{n+t_k} = 1 \neq f_k \).
Hence \( f_k^3 = f_k \), so \( f_k \) is a non trivial tripotent. Also using some known facts from probability theory, we get the number of such tripotents is \( \sum_{n=1}^{n-1} \binom{n}{s} \). Hence the number of non trivial tripotents we obtain is \( \sum_{n=1}^{n-1} \binom{n}{s} + \sum_{n=1}^{n-1} \binom{n}{s} \).

**Remark 2.4.** If \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m \) are m non trivial tripotents of \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) ( \( n \) is an odd number) generated by \( g \), and \( \alpha_i \neq \beta \) for all \( i \), where \( \beta \) is a S-co-idempotent of the S-idempotent \( \alpha = g^2 + g^4 + \cdots + g^{2n-2}, \) \([5]\) where \( \alpha \neq \alpha_i \), then \( \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_m \) is a tripotent if \( m \) is an odd number, and \( 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_m \) is a tripotent if \( m \) is an even number.

**Proposition 2.5.** In the group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) ( \( n \) is an odd number) generated by \( g \), if \( \alpha_1, \alpha_2 \) are any two non trivial tripotents in \( Z_2G \), then the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a S-T. tripotent.

**Proof:** Since in the group ring \( Z_2G \) the tripotents obtained are of the form \( d_k \) or \( f_k \) given in Theorem 2.3, then we have the following cases:

**Case 1:** \( \alpha_1, \alpha_2 \) are of the type \( d_k \). Let
\[
\alpha_1 = d_e = g^{\ell_1} + g^{\ell_2} + \cdots + g^{\ell_e} + g^{n+\ell_1} + g^{n+\ell_2} + \cdots + g^{n+\ell_e},
\]
and
\[
\alpha_2 = d_h = g^{s_1} + g^{s_2} + \cdots + g^{s_h} + g^{n+s_1} + g^{n+s_2} + \cdots + g^{n+s_h},
\]
where \( \ell_1, \ell_2, \ldots, \ell_e \) and \( s_1, s_2, \ldots, s_h \) are \( e \) and \( h \) distinct integers respectively, \( \ell_i \leq n-1, s_i \leq n-1 \) for each \( i, j \). By Remark 2.4, \( 1 + \alpha_1 + \alpha_2 \) is also a non trivial tripotent. We claim that the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a S-T. tripotent. For this purpose we describe the multiplication \( \alpha_1 \alpha_2 \) in the following array say \( A \):

\[
\begin{pmatrix}
g^{l_1+s_1} & g^{l_1+s_2} & \cdots & g^{l_1+s_h} & g^{n+l_1+s_1} & g^{n+l_1+s_2} & \cdots & g^{n+l_1+s_h} 
g^{l_2+s_1} & g^{l_2+s_2} & \cdots & g^{l_2+s_h} & g^{n+l_2+s_1} & g^{n+l_2+s_2} & \cdots & g^{n+l_2+s_h} 
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots 
g^{l_e+s_1} & g^{l_e+s_2} & \cdots & g^{l_e+s_h} & g^{n+l_e+s_1} & g^{n+l_e+s_2} & \cdots & g^{n+l_e+s_h} 
g^{n+l_1+s_1} & g^{n+l_1+s_2} & \cdots & g^{n+l_1+s_h} & g^{2n+l_1+s_1} & g^{2n+l_1+s_2} & \cdots & g^{2n+l_1+s_h} 
g^{n+l_2+s_1} & g^{n+l_2+s_2} & \cdots & g^{n+l_2+s_h} & g^{2n+l_2+s_1} & g^{2n+l_2+s_2} & \cdots & g^{2n+l_2+s_h} 
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots 
g^{n+l_e+s_1} & g^{n+l_e+s_2} & \cdots & g^{n+l_e+s_h} & g^{2n+l_e+s_1} & g^{2n+l_e+s_2} & \cdots & g^{2n+l_e+s_h}
\end{pmatrix}
\]
\( A = \left[ a_{ij} \right]_{(2e+1) \times (2h+1)} \) where \( a_{ij} \) is the summand of \( \alpha_1 \alpha_2 \) which is equal to the product of the \( ith \) summand of \( \alpha_1 \) with \( jth \) summand of \( \alpha_2 \). Considering the first and the \((e+2)th\) rows of this array we see that if \( g^i \) occurs in one of them it occurs in both of them for each \( i \) except \( (i = n+1, t_i) \), (as \( g^{2n+1} = g^{t} \)). By adding the terms of these two rows it remains only \( g^{t} + g^{n+1} \) (observing that the coefficient of each \( g^i, i=1, \cdots, 2n-1 \) is in \( Z_2G \)). Again by adding the second and the \((e+3)th\) rows in this array, according to the same argument it remains only \( g^{t} + g^{n+1} \). Proceeding in this manner we will get the \((e)th\) and the \((2e+1)th\) rows, adding there terms it remains only \( g^{t} + g^{n+1} \). So by adding all terms of this array we get, \( 1 + g^{t} + g^{n+1} \). By the same way we get \( \alpha_1 (1 + \alpha_1 + \alpha_2) = \alpha_2 \) and \( \alpha_2 (1 + \alpha_1 + \alpha_2) = \alpha_1 \). Therefore the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a S-T. tripotent.

**Case 2:** \( \alpha_1, \alpha_2 \) are of the type \( f_k \).
Let \( \alpha_1 = f_1 = 1 + g^{s_1} + g^{s_2} + \cdots + g^{r_i} + g^{n+r_1} + g^{n+r_2} + \cdots + g^{n+r_i}, and \)
\( \alpha_2 = f_j = 1 + g^{m_1} + g^{m_2} + \cdots + g^{m_j} + g^{n+m_1} + g^{n+m_2} + \cdots + g^{n+m_j}, \)
such that \( r_i, r_2, \ldots, r_i \) and \( m_1, m_2, \ldots, m_j \) are \( i \) and \( j \) distinct integers respectively, \( r_i \leq n-1, m_i \leq n-1 \) for each \( k, t, \) so by Remark 2.4, \( 1 + \alpha_1 + \alpha_2 \) is also a non trivial tripotent. By using same method as in case 1, we get that the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a S-T. tripotent.

**Case 3:** \( \alpha_1 \) of the type \( d_k \) and \( \alpha_2 \) of the type \( f_k \), where
\( \alpha_1 = d_h = g^{s_1} + g^{s_2} + \cdots + g^{s_h} + g^n + g^{n+s_1} + g^{n+s_2} + \cdots + g^{n+s_h}, \)
\( \alpha_2 = f_e = 1 + g^{t_1} + g^{t_2} + \cdots + g^{t_e} + g^{n+t_1} + g^{n+t_2} + \cdots + g^{n+t_e}, \)
such that \( s_1, s_2, \ldots, s_h \) and \( t_1, t_2, \ldots, t_e \) are \( h \) and \( e \) distinct integers respectively, \( s_i \leq n-1, t_j \leq n-1 \) for each \( i, j, \) then by Remark 2.4 the element \( 1 + \alpha_1 + \alpha_2 \) is also a non trivial tripotent. If \( 1 + \alpha_1 + \alpha_2 \) belongs to first type, then we get case 1 if it belongs to second type, then we get case 2. Hence the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a S-T. tripotent.

**Theorem 2.6.** The group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2n \) (n is an odd number) has at least \( 2^n \) non trivial tripotents and \( \left( \begin{array}{c} 2n-1 \end{array} \right) + \frac{1}{3} \left( \begin{array}{c} 2n-1 \end{array} \right) \) S-T. tripotents.

**Proof:** By Theorem 2.3, the group ring \( Z_2G \), has \( 2^{n-1} + 2^{n-1} - 1 = 2^n - 1 \) non trivial tripotents. It is shown in [5], that if \( G \) is generated by \( g \), then \( \alpha = g^2 + g^4 + \cdots + g^{n-1} + g^{n+1} + \cdots + g^{2n-2} \) is a \( S \)-idempotent and \( \beta = g + g^2 + \cdots + g^{n-2} + g^{n+2} + \cdots + g^{2n-1} \) is a \( S \)-co-idempotent. By Lemma 2.2, \( \beta \) is also a non trivial tripotent. Then \( Z_2G \) has at least \( 2^n \) non trivial tripotents. By Proposition 2.5, for any two non trivial tripotents \( \alpha_1, \alpha_2 \) in \( Z_2G \), the triple \( \alpha_1, \alpha_2, 1 + \alpha_1 + \alpha_2 \) is a
S-T. tripotent. Using some probability theory we get that, the number of such S-T. tripotents is \( \binom{2^{n-1}}{2} + \frac{1}{2}\binom{2^{n-1}-1}{2} \). ■

**Example 2.7.** Consider the group ring \( Z_2G \), where \( G = \langle g \mid g^{10}=1 \rangle \) is a cyclic group of order 10, generated by \( g \). Then by Theorem 2.6, the group ring \( Z_2G \) has 32 non trivial tripotents and the number of S-T. tripotents is 155. We list some of non trivial tripotents and S-T. tripotents:

\[ g^5, g^5+g^6+g^7, g^2+g^3+g^6+g^7+g^8, g^2+g^4+g^6+g^7+g^8+g^9 \]

\[ 1+g+g^6, 1+g+g^6+g^7, 1+g^2+g^4+g^7+g^9, 1+g^2+g^4+g^7+g^9, 1+g+g^3+g^6+g^7+g^9 \]

\[ g+g^2+g^3+g^4+g^6+g^7+g^9, g+g^2+g^3+g^4+g^6+g^7+g^9 \]

\[ 1+g+g^6 \text{ and } 1+g+g^6, 1+g^4+g^9, 1+g+g^4+g^6+g^9 \] are S-T. tripotents.

**Theorem 2.8.** The group ring \( Z_2G \), where \( G \) is a cyclic group of order \( 2p \) (\( p \) is Mersenne prime) has at least \( 2^m + 2^{m-2} \) non trivial tripotents and \( \binom{2^{p-1}}{2} + \frac{1}{2}\binom{2^{p-1}-1}{2} \) S-T. tripotents, where \( m = \frac{p-1}{2} \).

**Proof:** By Theorem 2.6, the group ring \( Z_2G \) has at least \( 2^p \) non trivial tripotents. It is shown in [5] that if \( G \) is generated by \( g \), then every element of the form \( \alpha = g^{2\ell} + g^{2\ell+2} + g^{2\ell+3} + \cdots + g^{2k\ell} \) is a S-idempotent of the group ring \( Z_2G \), where \( \ell \) is an odd number less than \( p \), and \( \beta = g^{2\ell} + g^{2\ell+2} + g^{2\ell+4} + \cdots + g^{2k\ell-1} + g^{2k\ell} \) is a S-co-idempotent of \( \alpha \) with \( \alpha \beta = \beta \alpha \), where \( t_i \) is defined by

\[ t_i = \begin{cases} \frac{x_i}{2} & \text{if } \frac{1}{2}x_i \text{ is odd} \\ \frac{x_i}{2} + p & \text{if } \frac{1}{2}x_i \text{ is even} \end{cases} \]

and \( x_i, i \geq 2 \) is the smallest positive integer such that \( x_i < 2p \).

Thus \( x_i \equiv 2^i \ell \pmod{2p} \), this means \( x_i = 2^i \ell - 2p \), for some \( r \in Z^+ \). S-idempotents of the form \( \alpha \ell \) called basic S-idempotents. Moreover it is shown that the sum of any number S-idempotents is also a S-idempotent, also it is proved that if \( \alpha \) is any such S-idempotent and \( \beta \) is a S-co-idempotent of \( \alpha \), then \( \alpha \beta = \beta \). By Lemma 2.2, S-co-idempotent are non trivial tripotents. Since the number of such S-co-idempotents is \( 2^{m-1} \), each of which is a non trivial tripotent. But one of these \( 2^{m-1} \) S-co-idempotents namely \( \beta = g^{2m} + \cdots + g^{2m-2} + g^{2m+2} + \cdots + g^{2n-1} \) is one of the \( 2^p \) non trivial tripotents obtained from Theorem 2.6, and no three of them form S-T. tripotent. Therefore the number of non trivial tripotents we obtained is \( 2^p + (2^{m-2}) \) and the number of S-T. tripotents obtained is \( \binom{2^{p-1}}{2} + \frac{1}{2}\binom{2^{p-1}-1}{2} \). ■

**References**

Smarandache triple tripotents


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