The paradox in general relativity

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Abstract. This is a research paper which review the former series “From ptolemy theorem to non-Euclidean geometry”. In this paper, we introduce the geometric model basing on relativity theory of that series. We also apply the trigonometry techniques in projective geometry, to give some new estimate for field equation. And our final goal is to propose a reasonable explanation for the essential property of the 4 dimension space-time in general relativity, especially for the well-known twin and clock paradox.

Introduction

General relativity is the geometric theory of gravitation published by Albert Einstein in 1916, and the current description of gravitation in modern physics. General relativity generalizes special relativity and Newton’s law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime. In particular, the curvature of space-time is directly related to the energy and momentum of whatever matter and radiation are present. The relation is specified by the Einstein field equations, a system of partial differential equations.

In physics, the twin paradox is a thought experiment in special relativity involving identical twins, one of whom makes a journey into space in a high-speed rocket and returns home to find that the twin who remained on Earth has aged more. This result appears puzzling because each twin sees the other twin as moving, and so, according to an incorrect naive application of time dilation and the principle of relativity, each should paradoxically find the other to have aged more slowly.

Preliminary

Firstly, we introduce our 10 new results for the geometric relativity model as the form of questions (see [1])

Problem 1: The discriminant in special relativity

$$\beta^2 = 1 - \frac{(vt \pm \Delta)^2}{4(x-vt)^2}$$

Where, \( \Delta = \sqrt{(vt)^2 + 4x(x-vt)} \)

Problem 2: Whether we can use the area of a triangle and its Apollonius circle’s radius to represent the length contraction in special relativity?

$$x_i = \frac{ct - (2 - 2\sqrt{2})e - (3 - 2\sqrt{2})RC^2 t - (2\sqrt{2} - 3)Rt}{(6 - 4\sqrt{2})r}$$

In which, \( C^2 = (\Delta + s - (3/2)c^2 + 2R\sqrt{r}) - (3/2)as \) is the radius ratio between the Apollonius circle and the circumcircle.

Problem 3: The limit of the Apollonius circle’s radius and the speed of light

For example:

$$\frac{1}{3\sqrt{3R_{Apollonius}}} = -2\sqrt{c^2 \varepsilon^2}$$
Problem 4: The power in relativity theory and the Steiner ellipse

For example: \( W = m^2 c^2 (r^2 v^2 + c^2) = m^2 c^2 (\frac{R^2 (t^2 + (1-t)^2)}{R^2 (t^2 + (1-t)^2) - t(1-t)}) v^2 + c^2 \)

Problem 5: How to use the property of circumcircle to handle the ratio between length and velocity in relativity effect?

For instance: \( \frac{l}{v} \rightarrow 1 \pm \sqrt{k} \) (Here, \( k \) is the slope in parametric circle equation)

Problem 6: Kiepert hyperbola and its application in general relativity

\[
26 \cdot (2Rk)^2 = 1 + \frac{l(6 - 4 \sqrt{k2})}{(3 - 2 \sqrt{k2})} - \frac{ct - (2 - 2 \sqrt{k2})c - (2 \sqrt{k2} - 3)Rt}{(3 - 2 \sqrt{k2})t}
\]

Here, we select that: \( RC^2 \leq \frac{ct - (2 - 2 \sqrt{k2})c - (2 \sqrt{k2} - 3)Rt}{(3 - 2 \sqrt{k2})t} \leq \frac{l(6 - 4 \sqrt{k2})}{(3 - 2 \sqrt{k2})t} \)

Problem 7: Kiepert parabola and its application in general relativity

\[
l \rightarrow \pm \sqrt{2ki - k - 2(1-t)R}
\]

We can choose the sideline as: \( a = \frac{1}{2}(-1 \pm \sqrt{2k}) = k \pm \sqrt{2ki} \) (where, \( R^2 \leq \frac{c^2}{2} - \frac{4}{c^2} \))

Problem 8: Lorentz transformation and pedal triangle

\[
L(u) = ur^2 (r + 1) \rightarrow 12 ut = 12 \cdot \frac{a^2}{(12 + 8 \sqrt{2})a} = 12 \cdot \frac{4^2}{\sqrt{2}(12 + 8 \sqrt{2})c}
\]

Problem 9: Jerabek hyperbola and the differential form in fields equation

\[
dx^2 = Edx^2_1 + 2 Fdx_1 dx_2 + Gdx^2_2
\]

In which,

\[
E = c^2 \frac{c^2 - x^2_2}{(c^2 - r^2)^2} \rightarrow \frac{9 \sqrt{3}}{16 t^2 (c^2 - r^2)^2}
\]

\[
F = c^2 \frac{x_1 x_2}{(c^2 - r^2)^2} \rightarrow \frac{c^4 t^2}{(c^2 - r^2)^2}
\]

\[
G = c^2 \frac{c^2 - x^2_1}{(c^2 - r^2)^2} \rightarrow \frac{9 \sqrt{3}}{16 t^2 (c^2 - r^2)^2}
\]

Problem 10: The charge and the temperature in relativity thermodynamics

\[
e^2 c^2 = CR^3 T \sqrt{1 - \beta^2} = CR^3 T \left[ \frac{R^2 (t^2 + (1-t)^2) - t(1-t)}{R^2 (t^2 + (1-t)^2)} \right]
\]
Remark: Here, we use the parametric method to handle the equation of conics. At first, we use series to expand the conic equation; then we use the function of eccentricity to handle the sectional radius $r$, and we can solve the equation and get its roots $x$ (we can also use the sideline to represent the root $x$). Lastly, we give the relation between the root and the slope $k$.

Application

1. Rotating frequency and parallax distance
1.1 Estimate the length contraction and rotating frequency

Firstly, we introduce problem 2: $t \leq \frac{c t - (2 - 2\sqrt{2}) c - (3 - 2\sqrt{2}) R C \gamma t - (2\sqrt{2} - 3) R t}{6 - 4\sqrt{2}}$. This is about the length contraction. Where, $C^2 = \left( \frac{R_{\text{Apollonius}}}{R} \right)^2 = (\Delta + s - (3/2)(4s^2 + 2Rr) - (3/2)as')^2$.

Then, we can select the Euler angle by problem 2 as: $OA + OB + OC \leq \frac{3\sqrt{2}(a + b + c)}{4(\sin \alpha + \sin \beta + \sin \gamma)}$.

Next, we try to apply the property of Fermat point and we select again: $C_\gamma = C_\gamma = C_\gamma \varepsilon$. Now it is clear that:

$$\frac{2\sqrt{2} + 1}{3}(x + y) \leq \frac{3\sqrt{2} c t}{2(\sin \alpha + \sin \beta + \sin \gamma)} = \frac{3\sqrt{2} c t}{2}.
$$

Recall the results in [2], the lorentz rotating matrix is: $A = BR$. Now we can use the condition above to choose the rotation matrix as: $R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Next, we try to substitute the matrix into angular velocity: $\Omega_{(x,y)} = -R^{-1} \gamma^2 P^2 (x,t) \Omega_y R \rightarrow 0$. In which, $F(x,t) = 1 + \frac{\gamma \sqrt{r - 1}}{\sqrt{r + 1}} \varepsilon$ (here, we apply problem 1 to set $\beta = \frac{v}{c} \rightarrow 0$). Then we introduce problem 3:

$$\frac{1}{3\sqrt{3} R_{\text{Apollonius}}} = -2\sqrt{7} i R^{-2} \rightarrow -2\sqrt{7} i c^2 \varepsilon^2$$. The limit of the angular velocity can be seen:

$$R^{-1}(1 + \frac{\gamma^2 (r - 1)}{\gamma + 1}) R = \lim_{C_{1x}} \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]^{-1} \left(1 + \frac{R}{2c} R_{\text{Apollonius}} \right) \Omega_y R \Rightarrow \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

Next step is to use the parametric variant $x, y$. Firstly, we assume that: $x = R \pm \sqrt{kR} = k - \sqrt{k^2 - y^2}$

Then we can make a substitution: for $C_\gamma \varepsilon : x = h(R) = R \pm \sqrt{kR}$. From the results we get above, we have:
\[ \frac{C_3}{c} \left( h + i \sqrt{h^2 - R^2} \right) \leq t . \text{ Where, } x + y = h + i \sqrt{h^2 - 2kh} = h + i \sqrt{h^2 - Rh} . \]

Now the following estimate holds:

\[ l + C_5 \cdot \frac{c}{t} \leq \frac{e(l - 3 - 2\sqrt[2]{2})c - (3 - 2\sqrt[2]{2})RC^2 - t - (2\sqrt[2]{2} - 3)Rt}{(6 - 4\sqrt[2]{2})} = \left( C_{4.5}c - C_{4.6}RC^2 - C_{4.7}R \right) \]

Then, we can introduce the property of the YFF center to handle the radius ratio between Apollonius circle and circumcircle: \( C^2 = \left( \frac{R_{\text{Apollonius}}}{R} \right)^2 = (1/2 + s - (3/2)4s^2 - (3/2)s^2)^2 = 36 \).

The following relation is obviously now:

\[ C_{4.3}c - C_{4.4}RC^2 - C_{4.5} R = C_{4.3}c - C_{4.4} R \Rightarrow l \leq \left( C_{4.5} - C_{4.6} / t \right)c - C_{4.7} R . \]

Consequently, \( \lim C_{5.1} \left( 1 + \frac{R}{2c} \right)_i \frac{1}{R_{\text{Apollonius}}} \Omega = 1 \Rightarrow \lim \Omega = \frac{Re}{C_5 \left( 1 + \frac{R}{2c} \right)_i} \)

Where, \( x + y = h + i \sqrt{h^2 - 2kh} = h + i \sqrt{h^2 - Rh} . \)

On the other hand, we take the radius ratio into consideration, that is : \( \frac{r_1}{r_2} = \frac{x - x_1}{x - x_2} \) \( \Rightarrow C_{5.1} \). This fact will encourage us to introduce: \( 4 \Delta^2 = \left( 2 - \frac{2}{bc} \right) b^2 (1 - y)^2 \cdot (b_1 + (c_1 - b_1) x)^2 \) and select \( k = 1 \). To ensure : \( x = R \pm R \). Then \( x = 2R \), \( y = 2i \sqrt{R^2 - R} \). Where, \( R = C_{5.2} i \).

1.2 clock and twin paradox

At first, we try to study the rotating frequency (see [3]): \( \Omega = \frac{8w}{3r_6} \cdot \frac{2GM}{c^2} \Rightarrow w = \frac{C_5 R^3 c^5}{GM \left( 1 + \frac{R}{2c} \right)_i} \).

on the other hand, we imagine an infinite large shell. By using the fact: \( J = mR \times v \), we deduce that :

\[ \Omega = \frac{2GJ}{c^2 \left( R^3 + \left( R + \frac{2GM}{c^2} \right) \cdot \frac{J^2}{M^2 c^2} \right)} \]

Next, we introduce the results in [4]: \( \frac{(1 - R^2 w^2 / c^2)^{1/2} \cos \alpha}{1 + (Rw / c) \sin \alpha} = \frac{1}{2} \Rightarrow Rw = \frac{\sqrt{3}}{3} \). Combine [2] and [3], we obtain: \( p_i = c \cdot \left( 1 + \frac{2gR}{c^2} \right) \). Relate this with the results in [5], we get a further result, that is :
\[ p_s = |F| < \frac{v_i}{K \sigma a^2 \Omega} \left( \sqrt{1 + (K^2 - 1)a^2} - 1 \right) \leq \frac{v_i e}{a^2 \Omega} \Rightarrow p_s < \frac{(C_{+s} - C_s(l) c - C_{+s} R)}{C_{+s} R c^3 t^2} \cdot \varepsilon \]

\[ \Rightarrow p_s = c \cdot \left(1 + \frac{2gR}{c^2}\right) < \frac{(C_{+s} - C_s(l) c - C_{+s} R)}{ct} = \frac{C_x}{t} - \frac{C_y}{t^2} - C_c \]

From [8], we have: \( d \tau < \sqrt{2} f (c, t) dt \). Where, \( f (c, t) = \frac{C_x}{ct} - \frac{C_y}{ct^2} \). In this way, we can get a bound for the proper time: \( \Delta \tau < \frac{C_{+s}}{c} (Int - \frac{1}{t}) \). On the other hand, from [7], we can write down the bound for interval time, that is: \( \Delta t = \tau (\gamma - 1) < \frac{C_{+s}}{c} (Int - \frac{1}{t}) \gamma = \sqrt{1 - v^2 / c^2} \).

After we study the result in [6][9] carefully, we can find another frequency. The following calculation holds:

\[ \Omega_x = \frac{2GJ}{c^2 (R^2 + (R + \frac{2GM}{c^2}) \cdot \frac{J^2}{M^2 c^2})} = \frac{2GM e}{c^2 (R^2 + (R + \frac{2GM}{c^2}) \cdot \frac{(M e)^2}{M^2 c^2})} \]

\[ = \frac{2GM e}{c^2 R^2 + (R + \frac{2GM}{c^2}) e^2} \Rightarrow t \rightarrow C_{+s} t_R^2 + C_{+s} (1 + \frac{2GM}{c^2 R^2}) \cdot \frac{1}{c^4} (*) \]

By using the fact: \( x = 2R, y = 2i \sqrt{R^2 - R} \). and relate it with the following relation, we deduce that:

\[ \frac{Rct}{c^2 R^2 + (R + \frac{2GM}{c^2}) e^2} \rightarrow C_{+s} \cdot \text{we can now transform the upper bound for proper time} \]

\[ \frac{C_{+s}}{c} (Int - \frac{1}{t}) \]

to the roots of the function: \( f (t) = Int - \frac{1}{t} \). If we can also find the relation: \( x = k \left( \frac{x - vt}{\gamma} + v \tau \right) = 2R \)

\[ \Rightarrow \frac{1}{\gamma} = \frac{\Delta \tau - \frac{1}{c}}{ke} \rightarrow \frac{\Delta \tau - \frac{1}{c}}{\Delta t - \frac{1}{c}} \Rightarrow \gamma - 1 \rightarrow \frac{\Delta t - \frac{1}{c}}{\Delta \tau - \frac{1}{c}} - 1 = \frac{\Delta t}{\Delta \tau} \Rightarrow \tau \rightarrow \frac{1}{c} \cdot \text{Then, we can get the} \]
periodic equation for the intersection points in twin paradox, such that: \( g(t) = \text{Int} - \frac{1}{t} + C_r = 0 \).

On the other hand, if we can substitute above into the expression of time: \( y = \frac{\alpha (1 - x^2)}{x \alpha + \sqrt{1 + \alpha^2}} \). The limit is:

\[
t \to C_{k_1} c R^2 + C_{k_1} (1 + \frac{2GM}{c^2 R}) \cdot \frac{1}{c^3} = C_{k_1} c R^2 + \frac{C_{k_2} + C_{k_3} Ri}{c^3 w \gamma} = C_{k_1} c R^2 + \frac{C_{k_2} + C_{k_3} Ri}{c^2 \gamma} + C_r = 0
\]

Here, we try to use the rotating frequency \( w \) and the parallax distance \( x \) to calculate the intersection point for 2 spaceships which carry a pair of twins after a short proper time.

2. Spacetime interval and gravitational constant

2.1 The power and momentum in relativity effect

At first, we introduce the power in problem 4: \( W = m \frac{2}{0} c \frac{2}{0} \frac{R^2 (t^2 + (1-t)^2)}{R^2 (t^2 + (1-t)^2) - t(1-t)} v^2 + c^2 \).

Where, \( c^2 = \frac{R^2 (t^2 + (1-t)^2)}{2t(1-t)} \). The Steiner ellipse read: \( x_1 = \frac{x}{2}, y_1 = \frac{y}{2} = \frac{c^2 - R^2 (t^2 + (1-t)^2)}{2t(1-t)} \).

By the formula of focal length: \( l = [1/f a - 2(b^2 (a + v - 2w) + c^2 (a + w - 2v) - 1)]/a^3 \) \( l \sin \theta \). We can assume that: \( a^2 = \frac{6 k}{x(1 - \cos \theta)} \to \infty \) \( (\cos \theta \to 1) \). Then we can introduce the instant momentum, that is: \( p = F v \cos \theta = F v \), with \( 2 m_u c v = F v \). By using the results in [5], we can obtain the limit below: \( c^2 \to C_k \left( \frac{1}{\gamma} + \frac{v^2}{c^2 \gamma^3} \right) \).

Our next step is to analyze the eccentricity for ellipse: \( x = \frac{1}{2} \left( \sqrt{2 k} \pm \sqrt{\frac{2}{k^2} - 4 \left( \frac{2}{k^2} - 2 \sqrt{2c} \right)} \right) \) \( \text{let us} \) \( \text{continue the discussion in} \ [5], \text{it makes us to assume:} \ x(t) = \frac{c^7 m}{F} \sqrt{1 + \left[ \frac{g(t - t_0)}{c} + \frac{C_1}{c} \right]^2 + C_2} \). After immediate calculation, we have: \( C_2 = \frac{\sqrt{2}}{2k}, C_1 = \frac{v u}{\gamma} \). Now it is clear to see:
\[
\sqrt{1 + \left(\frac{g(t-t_0)}{c}\right)^2 + \frac{C_1 v^2}{c^2}} = \frac{2c^2m}{F} \rightarrow C_1 \left(\frac{1}{\gamma} + \frac{v^2}{c^2\gamma^2}\right) \frac{m}{F}
\]

Where \( x \rightarrow v_0 + g(t-t_0), x \rightarrow g = F/m \)

Lastly, we can find the relation:
\[
\frac{2\sqrt{2c^2 - \frac{6}{k^2}}}{1 + \frac{g(t-t_0)}{c} + \frac{C_1 v^2}{c^2}} = \frac{2\sqrt{2c^2 - \frac{6}{k^2}}}{1 + \left(\frac{x}{c}\right)^2}
\]

the steiner ellipse to make substitution, we have:
\[
\frac{2\sqrt{2c^2 - \frac{6}{k^2}}}{1 + \left(\frac{x}{c}\right)^2} = \frac{4 \sqrt{c^2 - 6}}{1 + \left(\frac{x}{c}\right)^2}
\]

Now we finish section 2.1.

2.2 4-dimensional metric and spacetime interval

Here, we can introduce \([2]\) to handle the spacetime metric: \[
g_{rr} = c^2 - 2bX - a^2X^2 > 0
\]

Where, \( g_{rr} = \frac{v_j^2 - \sum \frac{v_r^2}{x^2}}{dt} (\frac{df(t)}{dt})^2 \rightarrow x \). Then if we introduce the example above for length contraction:

\[
\frac{x}{(\frac{df(t)}{dt})^2} \cdot g_{rr} = \frac{x}{(\frac{df(t)}{dt})^2} = c^2 - 2bX - a^2X^2 > v_j^2 - 2v_jX - X^2 = 0
\]

we have: \( v_j^2 - 2v_jX - X^2 = v_j^2 - 2v_jX - X^2 = 0 \Rightarrow X = (1 \pm \sqrt{2})v_j \). On the other hand, if we realized that: \( X \rightarrow \frac{c^2}{g} \cdot \left(\frac{gt/c}{2}\right) + x(1 + \left(\frac{gt/c}{2}\right))^2 \). Related it with problem 5, which means that we can write \( X \) in the form of \( l \). Consequently, \( \frac{c^2}{g} \cdot \left(\frac{gt/c}{2}\right) + x(1 + \left(\frac{gt/c}{2}\right))^2 \rightarrow ct \left(\frac{df}{dt}\right) \). If we substitute it into the space-time metric, we have: \( \frac{c^2}{g} \cdot F(g,t) + x(1 + F(g,t)) = c - \frac{c}{t} \).

with: \( F(g,t) = \left(\frac{gt/c}{2}\right)^2 \). By choosing an unit time ball, we can also deduce that: \( x = -\frac{c^2 F(g,t)}{1 + F(g,t)} \).

Next, we introduce the results in \([10]\) \([11]\) \([12]\). At first, we rewrite the space-time interval as:
\[ c^2 d\tau^2 = F_\mu^2 \left( c^2 (1 + \frac{g x}{c})^2 dt^2 - dx^2 - dy^2 - dz^2 \right). \] Where, \( \Delta \tau = F_\mu^2 (1 + \frac{g x}{c^2}) \Delta t \)

We then introduce the results in problem 9:

\[ F_\mu^2 = \frac{9 \sqrt[3]{3}}{16 t^2 (c^2 - r^2)^2} = \left( \frac{1}{(\gamma - 1)(1 + \frac{g x}{c^2})} \right)^2 \Rightarrow t(c^2 - r^2) \rightarrow C(\gamma - 1)(1 + \frac{g x}{c^2}) \]

As the same manner the authors did in [11] and [12], we obtain:

\[ ds^2 = rd\tau^2 - \frac{1}{r} dr^2 = (1 - w^2 r^2) dt^2 - dr^2 \Rightarrow r^2 = \frac{1}{1 + w^2} \]

That is: \( t(c^2 - \frac{1}{1 + w^2}) \rightarrow C(\gamma - 1)(1 + \frac{g x}{c^2}) \)

### 2.3 Gravitational constant

Here, we discuss [13] and [14]. Firstly, we consider the equation:

\[ x = \frac{2 g}{c^2 + g x} \cdot \frac{2}{1} - \frac{g(1 + \frac{g x}{c^2})}{c^2} \]

By observation, we have:

\[ \frac{g}{c^2 + g x} = \frac{1}{(1 + F(g, t))^4} = (1 + \frac{g x}{2c^2} + g x). \]

Continue it with problem 5, we get:

\[ R = C x. \]

Then, \[ \sqrt{\frac{4 R c - 6 c}{k^2 x}} \rightarrow C_x \frac{1}{g y + \frac{v^2}{c^2 \gamma^2}} \frac{1}{g} \Rightarrow T = \frac{1}{g y} + \frac{v^2}{g c^2 \gamma^2} \rightarrow \frac{1}{x} \]

Now, we can introduce the proper time:

\[ d\tau = 1 - \frac{3}{2} \frac{GM}{c^2 r} - \frac{9}{8} \left( \frac{GM}{c^2 r} \right)^2 \pm \frac{3 GM}{c^3 r} \sqrt{wT} \] on the other hand, recall that: \( ds^2 = -c^2 d\tau^2 \). From: \( r^2 = \frac{1}{1 + w^2} \), we have:

\[ \frac{c^2}{\gamma - 1} (1 - \frac{3 GM}{2 c^2 r} - \frac{9 (GM)}{8 c^2 r} \pm \frac{3 GM}{c^3 r} \sqrt{wT}) = r \Rightarrow \]

\[ \pm \frac{3 GM}{c^3 r} \sqrt{wT} = \frac{r(\gamma - 1)}{c^2} + \frac{3 GM}{2 c^2 r} + \frac{9 (GM)}{8 c^2 r} \]

Also note that, in problem 9, we use the fact that the speed of light in vacuum do not change. So we can properly adjust the order of the speed of light as:

\[ \pm \sqrt{wT} = \frac{1}{c^2} (\gamma - 1) + \frac{c}{3 wr T} + \frac{3 c}{2} + \frac{c}{8} - \frac{c}{3 wr^2 T} \sim 0. \]

Now, we try to discuss the existence of the roots for the gravitational equation above, that is:

\[ \frac{GM}{c^2 r} = -\frac{3}{2} / 2 + \sqrt{(3/2)^2 - (9/2)(r/c^2) - 1} \Rightarrow r = \text{ci} / 2 \quad (\text{Where, } GM = wc^2 r^3 T) \]

For another, by using the results in 2.1 and 2.2, we can select: \( c^2 = gx \) and the following relation holds:
\[
\frac{w_r^3 T}{GM} \cdot [\left( \gamma - 1 \right) + g_x \left( \frac{T}{2} + \frac{3 w_r^2 T^2}{8} - \frac{1}{3 w_r^2 T} \right)] = 0.
\]

Finally, we deduce with the results we get above:

\[
\frac{w_r^3}{3 w_g x r T} \cdot \left[ \frac{1}{8} \right] + g_x \left( \frac{T}{2} + \frac{3 w_r^2 T^2}{8} - \frac{1}{3 w_r^2 T} \right) = 0
\]

\[
\Rightarrow \frac{1}{3 g x r} = - g_x w \left( \frac{T}{2} + \frac{3 w_r^2}{8} - \frac{1}{3 w_r^2 T^2} \right) \Rightarrow \frac{1}{3 g x r} = \frac{1}{8} \Rightarrow \frac{1}{3 g x r} = \frac{8}{8} \Rightarrow
\]

\[
\frac{T}{3 g x r} = \frac{c^2}{3 w_T} - \frac{3 w_r^3 c^2 T}{8} \Rightarrow \frac{T^2}{3 g x r} = \frac{c^2}{3 w_T} - \frac{3 GMT}{8} \Rightarrow \frac{GMT}{8} = \frac{c^2}{3 w_T} - \frac{T^2}{3 g x r} \Rightarrow
\]

\[
\frac{9 GM}{8} = \frac{c^2}{3 w_T} - \frac{T}{g x r} = \frac{1}{w} \left( \frac{g x}{r T} - \frac{T}{g x} \right). \text{(In which, } T = \frac{1}{g} + \frac{v^2}{g c^2 r})
\]

3. Lorentz transformation and curvature tensor

At first, we apply problem 8 to study the lorentz matrix in [15] and [16]: \( L(u) \leq C_1 \left( \frac{1}{ct} + \frac{1}{(ct)^3} \right) \).

We denote: \( \gamma = \frac{1}{c} + \frac{\gamma^2 v^2}{c^2 (1 + \gamma^2)} \). On the other hand, if we can introduce the imaginary axis to calculate:

\[
U_\beta = \left\{ \begin{array}{ccc}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{array} \right\} \Rightarrow \left\{ \begin{array}{ccc}
1 & 0 & 2 \\
2 & 1 & 0 \\
0 & 2 & 1 - i \beta
\end{array} \right\} \Rightarrow \left\{ \begin{array}{ccc}
1 & 0 & 2 \\
2 & 1 & 0 \\
0 & 2 & 1 + i \beta
\end{array} \right\} \rightarrow \text{Our next ext step is to analyze the angle } \phi \text{ (see [7][11]): } \cos \theta = \cosh \phi. \text{ Consequently, } x U_\beta^T x = x^T \left( 1 - \frac{1}{\cosh \phi} \right) / (\cos \theta - 1);
\]

\[
\sinh \phi = \frac{x y - x y}{k} (\cos \theta - 1). \text{ Then from [7], we can set: } \frac{\sqrt{1 - \beta^2} \sin \theta}{\cos \theta + \beta} \approx \sqrt{\frac{1 - \beta^2}{1 + \beta}} \theta. \text{ Related it with [11], we obtain: } e^{i \phi} = \Psi; v \cos \phi + iv \sin \phi = v \Psi = ve^{i \phi}. \text{ That is: }
\]

\[
\theta = i \phi \Rightarrow \cos \theta \rightarrow 1 \Rightarrow x U_\beta^T x = x^T x
\]

Next, we can calculate the matrix: \( (3 + i \beta)x = x^T ; (3 - i \beta)y = y^T \), which means that:

\[
(3 - i \beta) x y - (3 + i \beta) x y = -2 i \beta x y = k
\]
Here, $k$ is a real number.

The results in problem 6, 7 is: $2e \cdot (2Rk)^3 = 1 + \frac{l(6 - 4\sqrt{2})}{(3 - 2\sqrt{2})} - \frac{ct - (2 - 2\sqrt{2})c - (2\sqrt{2} - 3)Rt}{(3 - 2\sqrt{2})t}$.

Immediate calculation shows that: $2e \cdot (2Rk)^3 = 1 + C_i l - C_2 c + \frac{C_1 c}{t} + C_4 R$. On the other hand, by

$$
\frac{\cos Bc}{K} \cdot \left( \frac{1 - \frac{\sin A}{\csc(\frac{A + \pi}{3})}}{\sqrt{3} - 1} \right) + \left( \frac{\sin A}{\csc(\frac{A + \pi}{3})} \right) \to 0 . \text{we can assume}
$$

the limit below:

$$\lim_{l \to \frac{\pm \sqrt{2}k i - k - 2(1-t)R}{2t-1}} = 0 . \text{That is: } R^2 \leq \frac{c^2}{2} - \frac{4}{c^2} \text{ (Where,}
$$

$c$ is the sideline). For another, from the limit of length, we can select: $-2i\beta xy = K = i \Rightarrow xy = -\frac{1}{2\beta}$.

Then, we try to write out the estimate for problem 6: $16 iR^3 = \frac{1}{e} \left( 1 + C_i l - C_2 c + \frac{C_4 c}{t} + C_4 R \right)$.

If we put the length out to calculate, we have: $l = \frac{-16 iR^3 - c + C_2 c^2 - C_4 c^2 / t - Rc}{C_1}$

$1 - C_2 c - C_4 c / t = ct$. (Here, we make a projection from real axis to imaginary axis). Then, we finish our preparation for example 3.

Now, we can write out the lorentz transformation: $L(u) = \gamma^2 (1 - \beta^2)$, and we can also introduce the results in example 1, 2 above, to deduce: $\frac{1}{ct} < 1 + \frac{2gr}{c^2} \Rightarrow \gamma^2 (1 - \beta^2) < C_4 \left((1 + \frac{2gr}{c^2}) + (1 + \frac{2gr}{c})^3 \right)$;

In which, $x = -\frac{c^2 F (g,t) / g}{1 + F (g,t)}; F (g,t) = \left( \frac{gt / c}{} \right)^2$.

On the other hand, if we consider the property of Malfatti circle (see problem 8), we can choose:

$1 + \frac{2gr}{c^2} = 1 + 2gr \cdot (c^2 (1-t) + \frac{1}{4}) = 1 + C_2 gr \Rightarrow \gamma^2 (1 - \beta^2) < C_4 (gR + (Rg)^3)$

Then, by using the results in (2) to analyze the curvature tensor, we have: $c / 2 \cdot (x + y) < l$.

Let us combine it with above: $xy = -1 / 2\beta$. We then obtain: $\frac{c}{2} \cdot \left( x - \frac{1}{2\beta x} \right) < l \Rightarrow \frac{1}{2} \cdot \left( x - \frac{1}{2\beta x} \right) < r$

Where, $x = \frac{1}{2} \left( \frac{\sqrt{2}}{k} \pm \sqrt{\frac{2}{k^2} - 4 \left( \frac{2}{k^2} + 2\sqrt{2}c \right)} \right)$. 

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Taking \( \mathbf{17} \mathbf{18} \mathbf{19} \mathbf{20} \) into account, we further discuss the field equation. And we write out the form of tensor at first (see \( \mathbf{17} \)):

\[
L = \frac{1}{2} g_{\mu \nu} x^\mu x^\nu + A_\mu x^\mu = -\frac{c^2}{2} = -\frac{1}{2} ((1 + \frac{gx}{c^2})^2 c^2 \tau^2 + x^2). \]

From \( f = \frac{1}{2} (x + \frac{c^2}{g}) \), we know: \( x(0) = -\frac{c^2}{g} \). On the other hand, \( \mathbf{18} \) gives: \( x = \frac{gt}{\sqrt{1 + g^2 \tau^2}} \). From \( \mathbf{19} \), the expression of \( L(t) \), we realized that: when the limit \( t \to 0 \), we have: \( x(0) = -\frac{c^2}{g} \). Which is equivalent with \( L(t) = 0 \)

or \( L(t) = \frac{c^2}{g} (\sqrt{1 + 2ct} - \frac{1}{\sqrt{1 + 2ct}}) \). Here, we use the last one to approximate. If we also related it with the results in \( \mathbf{20} \), we can get: \( L(t) = \frac{c^2}{g} (\sqrt{1 + 2ct} - \frac{1}{\sqrt{1 + 2ct}}) = 2 \gamma \pi r \).

By \( \mathbf{21} \mathbf{22} \mathbf{23} \), we also know that: \( \tau = -\frac{\nu^2}{2c^2 \gamma} \). (Here, we try to analyze the situation in epsilon time ball)

Lastly, we estimate its lagrange form, the curvature tensor is:

\[
L = -\frac{c^2}{2} = -\frac{1}{2} ((1 + \frac{gx}{c^2})^2 c^2 \tau^2 + x^2) \leq \frac{1}{2} ((1 + \frac{gx}{c^2})^2 c^2 \tau^2 + 1) . \]

On the other hand, \( \gamma = \frac{1}{c} + \frac{\gamma^2 \nu^2}{c^3 (1 + \gamma)} \leq C_i \frac{1}{\sqrt{gR}} (gR)^3 \). We observed that:

\[
\tau = -\frac{\nu^2}{2c^2 \gamma} \leq \frac{C_i}{\sqrt{gR}} (gR)^3 - \frac{1}{c} . \]

The situation in epsilon time can be viewed as:

\[
(C_i \frac{1}{\sqrt{gR}} (gR)^3 - \frac{1}{c}) \cdot \frac{t}{2} \cdot \frac{c (1 + \gamma)}{\gamma} \sim (C_i \frac{1}{\sqrt{gR}} (gR)^3 - \frac{1}{c})(ct) \sim C_i \frac{1}{\sqrt{gR}} (gR)^3 (ct)
\]

For another:

\[
\frac{c^2}{2} (\sqrt{1 + 2ct} - \frac{1}{\sqrt{1 + 2ct}}) = r > \frac{1}{2} (x - \frac{1}{2 \beta x}) \Rightarrow \frac{1}{\pi} (\sqrt{1 + 2ct} - \frac{1}{\sqrt{1 + 2ct}}) > \frac{gx}{c^2} - \frac{g}{2 \beta x^2} \sim \frac{gx}{c^2} - \frac{g}{2 xc} \Rightarrow \frac{1}{\pi} (\sqrt{1 + 2ct} - \frac{1}{\sqrt{1 + 2ct}}) + \frac{g}{2 xc}
\]

Therefore, we deduce that:

\[
L \leq \frac{1}{2} ((1 + \frac{gx}{c^2})^2 c^2 \tau^2 + 1) \sim \frac{C_i}{2} \frac{1}{\sqrt{gR}} (gR)^3 \sim 1 = \frac{C_i}{2} \frac{1}{\sqrt{gR}} (gR)^3 + 1 + \frac{g}{2 xc}
\]

Finally, we can write down the
estimate for tensor equation with lorentz matrix: \( L \leq C \sqrt{\det B} \left[ (\frac{R}{2x})^2 + \frac{RE}{2x} \right]. \)

Where, \( L = (1/2) g_{\mu\nu} x^\mu x^\nu + A_{\mu} x^\mu \) and \( B \) is the lorentz matrix.

4. Fields equation and quantum mechanics

Here, we introduce \([24] [25] [26]\) and relate them with 3 examples above, to estimate the fields equation.

\[
\tau = 2\left[ \frac{d}{\gamma^v} + \frac{D - 2d}{a} \right] \Rightarrow \tau = 2\left[ \frac{t}{\gamma} + \frac{l_o - 2t^2}{\gamma} \right] \Rightarrow \tau = 2\left[ \frac{t}{\gamma} + \frac{l_o}{\gamma} - 2f^2 \right] \sqrt{1 + r^2 \Omega^2}}.\]

Next, the results in \([27] [28]\) gives: \( E dE = \frac{1}{\beta} - \frac{l_0}{\beta vt} \), with \( l_0 = gt^2 \left( \frac{1}{4(1 + C\beta r)} - \frac{1}{\sqrt{1 + C\beta r}} + 3 \right) \).

We deduce that: \( \Rightarrow l_0 = \left( \frac{g}{c^2}(x + \frac{c^2}{g^2}) - \frac{c^2}{g} \left( \frac{1}{4(1 + C\beta r)} - \frac{1}{\sqrt{1 + C\beta r}} + 3 \right) \).

Our next step is to introduce problem 10 and the Schrödinger (see \([29] [30]\)). Firstly, from \([31]\) we have:

\[
e^2 c^2 \approx CR^2 \sqrt{\frac{R^2 (t^2 + (1-t)^2) - t(1-t)}{R^2 (t^2 + 1-t)^2}} \Rightarrow e^2 c^2 = (\gamma - 1) r^3 \sqrt{1 - \beta^2} = (\gamma - 1) r^3.
\]

Then, we handle the Schrödinger equation as below. The following relation is clearly now:

\[
2[\cosh \lambda E - \cosh \lambda m] = \frac{\sinh \frac{2}{g} \sinh \frac{2}{g} \lambda m}{e^{i\lambda E}} = \frac{\sinh \frac{2}{g} \lambda m}{e^{i\lambda E}} = \frac{\sinh \frac{2}{g} \lambda m}{m^2 (1 - \lambda E)} \Rightarrow
\]

\[
(e^{i\lambda E} + \frac{1}{e^{i\lambda E}}) - (e^{im} + \frac{1}{e^{im}}) = \frac{(e^{i\lambda m} - \frac{1}{e^{i\lambda m}})^2}{4e^{i\lambda E}} \cdot \frac{p^2 (1 - \lambda m)^2}{m^2 (1 - \lambda E)^2}
\]

We now can substitute it into the energy equation \( dE \). It is: \( dE = \frac{1}{\lambda E} \sqrt{1 - 2 \cosh(\lambda m) e^{-2iE} + e^{-2iE}} \).

By observation, we discover that:

\[
\frac{(e^{i\lambda m} - \frac{1}{e^{i\lambda m}})^2}{4e^{i\lambda E}} \cdot \frac{p^2 (1 - \lambda m)^2}{m^2 (1 - \lambda E)^2} = 1 - 2 \cosh(\lambda m) e^{-iE} + e^{-2iE}
\]

\[
1 - \frac{1}{e^{2iE}} - \frac{(e^{i\lambda m} - \frac{1}{e^{i\lambda m}})^2}{4e^{i\lambda E}} \cdot \frac{p^2 (1 - \lambda m)^2}{m^2 (1 - \lambda E)^2} + \frac{1}{e^{2iE}} \Rightarrow
\]

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\[ dE = \frac{1}{\beta} \sqrt{1 - 2 \cosh(\lambda m) e^{-\lambda E} + e^{-2\lambda E}} + \frac{1}{\beta} \left( \frac{e^{\lambda m} - 1}{2 e^{\lambda E/2}} \right) \frac{p(1 - \lambda m)}{m(1 - \lambda E)} \]

On the other hand, by the preparation we did above, we can find:

\[ \sqrt{E}dE = \frac{1}{\beta} \frac{l_0}{\beta vt} \Rightarrow \sqrt{E} \left( \frac{e^{\lambda m} - 1}{2 e^{\lambda E/2}} \right) \frac{p(1 - \lambda m)}{m(1 - \lambda E)} = \frac{1}{\beta} \frac{l_0}{\beta vt} = \]

\[ \frac{1}{\beta} \frac{1}{\beta vt} \left( \frac{g}{c^2} (x + \frac{c^2}{g})^2 - \frac{c^2}{g} \right) \left( \frac{1}{4(1 + Cr)} - \frac{1}{\sqrt{1 + Cr}} + 3 \right) \]

In which, \( c^2 = \frac{(\gamma - 1)r^3}{e^2} \).

After immediately calculation, we have:

\[ \frac{g}{c^2} (x + \frac{c^2}{g})^2 - \frac{c^2}{g} \left( \frac{1}{4(1 + Cr)} - \frac{1}{\sqrt{1 + Cr}} + 3 \right) \approx \frac{g}{(r \cdot 1_x)^2} (x + \frac{(r \cdot 1_x)^2}{g})^2 - \frac{(r \cdot 1_x)^2}{g} \approx \]

\[ \frac{\beta \sqrt{E}}{\beta \sqrt{E}} \left( \frac{e^{\lambda m} - 1}{2 e^{\lambda E/2}} \right) \frac{p(1 - \lambda m)}{m(1 - \lambda E)} \leq 1 - \frac{C_x}{vt} \left( \frac{g}{r^2} \left( x + \frac{r^2}{g} \right)^2 - \frac{r^2}{g} \right) \]

Then, we take the limit: \( \lambda \to 0 \) and we can write out the estimate for energy at last, that is:

\[ \frac{\beta \sqrt{E}}{2m} \leq 1 - \frac{C_x}{vt} \left( \frac{g}{r^2} \left( x + \frac{r^2}{g} \right)^2 - \frac{r^2}{g} \right) = 1 - \frac{C_x}{d} \left( \frac{gx^2}{r^2} + 2x \right) \]

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