The Class of *q*-Cliqued Graphs: Eigen-Bi-Balanced Characteristic,Designs and an Entomological Experiment

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Abstract

Much research has involved the consideration of graphs which have sub-graphs of a particular kind, such as cliques. Known classes of graphs which are eigen-bi-balanced, i.e. they have a pair a,b of non-zero distinct eigenvalues, whose sum and product are integral, have been investigated. In this paper we will define ta new class of graphs, called *q-cliqued* graphs, on q^2+1 vertices, which contain q cliques each of order q connected to a central vertex, and then prove that these *q*-cliqued graphs are eigen-bi-balanced with respect to a conjugate pair whose sum is -1 and product 1-q. These graphs can be regarded as design graphs, and we use a specific example in an entomological experiment.

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1. Introduction

There is much interest in considering graphs which have sub-graphs of a particular kind, such as cliques – see Babat and Sivasubramaniam [1], Graham and Hoffman [3], and Liazi, Milis, Pascual and Zissimopoulos [6]. Known classes of graphs which are eigen-bi-balanced are considered in Winter and Jessop [7].These graphs have an associated pair of (real) conjugate eigenvalues (from the graph's adjacency matrix) whose sum and product are integral. It appears that the conjugate pair arises out of the centrality of certain vertices of the graph, which are strongly connected (edgewise) to other vertices of the graph. For example, the wheel graph has a central vertex connected by its spokes to the remaining vertices of the graph. Bipartite graphs have two sets of vertices strongly connected to each other. The vertices of the complete graph are each strongly connected to each other. In this paper we will define ta new class of graphs, called *q-cliqued* graphs, on q^2+1 vertices, involving a central vertex connected to q cliques each of order q, and then prove that these *q*-cliqued graphs are eigen-bi-balanced with respect to a conjugate pair whose sum is -1 and product 1-q. These graphs can be regarded as design graphs, and we use a specific example (q=3) in an entomological experiment.

2. Construction of *q*-cliqued graphs

In this section, for $q \geq 2$, we construct a q-cliqued graph, labelled ${G_{K_q}}^*$, and find its associated adjacency matrix. We take q copies of the complete graph on q vertices K_q, together with a single vertex v. Generally, we label the vertices of the ith copy of $(K_q)^i$ as $v_1^i, v_2^i, \ldots, v_q^i$, for $i = 1,2,\ldots,q$.

2.1 For $q = 2$, the graph ${G_{K_2}}^*$

For $q = 2$, take 2 copies of K_2, namely $(K_2)^i$; $i = 1,2$ together with a single vertex v. Join v to v_1^i; $i = 1,2$, so that v has degree 2. More generally, join v to v_1^i; $1 \leq i \leq q$. so that v has degree *2* generally.

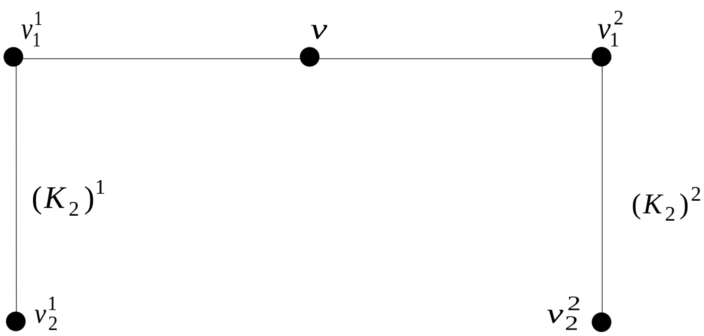

Figure 2.1.1: Construction of ${G_{K_2}}^*$ - (a)

Finally, join vertices v_2^1 *and* v_2^2 of $(K_2)^1$ and $(K_2)^2$ to form a 5-cycle.

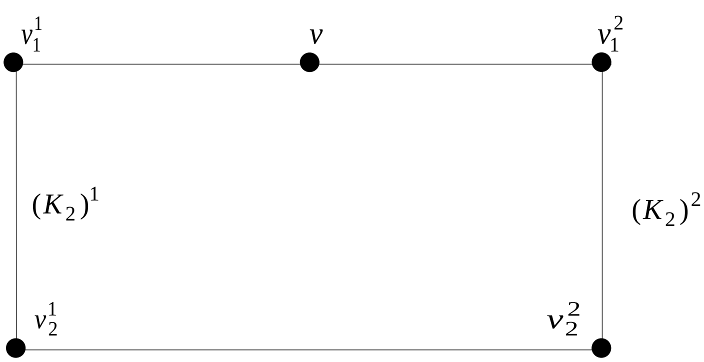

Figure2.1.2: Construction of ${G_{K_2}}^*$ - (b)

Label vertex v as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2, v_2^1 = v_3$, and $v_1^2 = v_4, v_2^2 = v_5$. This graph does not contain a 2-lantern sub-graph so it is a design graph, namely a 2-cliqued design graph.

Then the *5*x*5* adjacency matrix of $G_{K_2}{}^*$, where the rows are $v_1,..., v_5$ and the columns are $v_1,..., v_5$, is:

$$A\left(G_{K_2}{}^*\right)=\begin{bmatrix}0&1&0&1&0\\1&0&1&0&0\\0&1&0&0&1\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix}$$

By definitionof $\det\left(\lambda I - A\left(G_{K_2}{}^*\right)\right)$, the characteristic polynomial of $A\left(G_{K_2}{}^*\right)$ is $\lambda^5 - 5\lambda^3 + 5\lambda - 2$.

The eigenvalues of this adjacency matrix are: 2 (once); $\frac{-1+\sqrt{5}}{2}$ (twice) and $\frac{-1-\sqrt{5}}{2}$ (twice). The conjugate eigen-pairs are $\frac{-1\pm\sqrt{5}}{2}$.

2.2 For $q=3$, the graph $G_{K_3}{}^*$

For $q=3$, we take 3 copies of K_3, namely $(K_3)^1$, $(K_3)^2$, and $(K_3)^3$ together with a single central vertex *v*. Join *v* to v_1^i; $i=1,2,3$:Join the remaining vertices of the 3 copies of K_3 to form 3 5-cycles. i.e., v_3^1 and v_3^2 and v_2^3; v_3^3 and v_2^1.

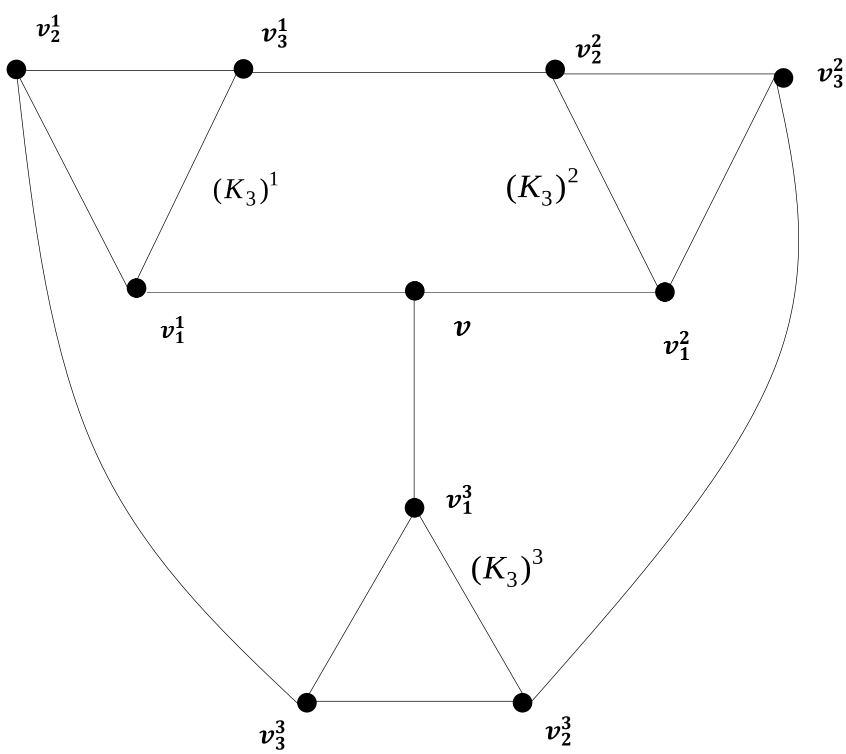

Figure 2.2.2: Construction of $G_{K_3}{}^*$ - (b)

Label central vertex *v* as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2$, $v_2^1 = v_3$, $v_3^1 = v_4$, and $v_1^2 = v_5$, $v_1^2 = v_6$, $v_1^2 = v_7$, and $v_1^3 = v_8$, $v_1^3 = v_9$, $v_1^3 = v_{10}$.

Then the 10*x*10 adjacency matrix of $G_{K_3}{}^*$, where the rows are $v_{1,}, v_2, ..., v_{10}$ and the columns are $v_{1,}, v_2, ..., v_{10}$ is:

$$A\left(G_{K_3}{}^*\right) = \begin{bmatrix} 0 & 1 & 0 & & 1 & & & 1 & & \\ 1 & 0 & 1 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & & 1 \\ & 1 & 1 & 0 & & 1 & & & & \\ 1 & & & & 0 & 1 & 1 & & & \\ & & & 1 & 1 & 0 & 1 & & & \\ & & & & 1 & 1 & 0 & & 1 & \\ 1 & & & & & & & 0 & 1 & 1 \\ & & & & & & 1 & 1 & 0 & 1 \\ & & 1 & & & & & 1 & 1 & 0 \end{bmatrix}$$

All blank elements are zero. Since no two columns are the same, the exclude 3-lantern condition holds.

The characteristic polynomial of the adjacency matrix for $q = 3$ is:

$$\lambda^{10} - 15\lambda^8 - 6\lambda^7 + 75\lambda^6 + 48\lambda^5 - 144\lambda^4 - 114\lambda^3 + 75\lambda^2 + 68\lambda + 12$$

The eigenvalues of this adjacency matrix are: 3, 1, -2, -2, 1.879, 1.879, -0.347, -0.347, -1.532, -1.532 .

The conjugate eigen-pair is $\frac{-1 \pm \sqrt{9}}{2}$.

2.3 For $q = n$, the general construction of graph $G_{K_n}{}^*$

The general construction of the $(1+n^2)x(1+n^2)$ adjacency matrix of $G_{K_n}{}^*$ where the rows are $v_1, v_2, ..., v_{1+n^2}$ and the columns are $v_1, v_2, ..., v_{1+n^2}$ is as follows:

$a_{i,i} = 0;\ 1 \le i \le (1+n^2)$

Join v **to** $v_1^i;\ 1 \le i \le n$ **:**

$a_{1,1+\lambda n+1} = 1;\ 0 \le \lambda \le n-1$

$a_{1+\lambda n+1,1} = 1;\ 0 \le \lambda \le n-1$

Sub-cliques:

$a_{1+\lambda n+k,1+\lambda n+l} = 0;\ 0 \le \lambda \le n-1;\ 1 \le k \le n;\ 1 \le l \le n;\ \ k = l$

$a_{1+\lambda n+k,1+\lambda n+l} = 1;\ 0 \le \lambda \le n-1;\ 1 \le k \le n;\ 1 \le l \le n;\ \ k \ne l$

v_n^i **of clique *i* (the *n*th vertex in clique** i**) joins to** v_2^{i+1} **(the *2*nd vertex in clique** $(i+1)$**):**

$a_{1+\lambda n+n,1+(\lambda+1)n+2} = 1;\ 0 \le \lambda \le n-1;$

$a_{1+(\lambda+1)n+2,1+\lambda n+n} = 1;\ 0 \le \lambda \le n-1;$

v_j^i **of clique *i* (the *j*th vertex in clique** i**) joins to** v_{j-1}^{i+1} **(the *(j-1)*th vertex in clique** $(i+1)$**):**

$a_{1+\lambda n+j,1+(\lambda+1)n+(j-1))} = 1;\ 0 \le \lambda \le n-1;\ 4 \le j \le n-1;\ \ j\ \ even$

$a_{1+(\lambda+1)n+(j-1)),1+\lambda n+j,} = 1;\ 0 \le \lambda \le n-1;\ 4 \le j \le n-1;\ \ j\ \ even$

v_j^i **of clique** i **(the *j*th vertex in clique *i*) joins to** v_j^{i+1} **(the *j*th vertex in clique** $(i+1)$**),**
$j = n-1;\ \ n\ \ even,\ \ j\ \ odd,\ \ i\ \ odd$ **:**

$a_{1+\lambda n+j,1+(\lambda+1)n+j} = 1;\ 0 \le \lambda \le n-1;\ \ j = n-1,\ \ \lambda\ even$

$a_{1+(\lambda+1)n+j,1+\lambda n+j} = 1;\ 0 \le \lambda \le n-1;\ \ j = n-1,\ \ \lambda\ even$

If for $a_{i,j}$ $i > (1+n^2)$ *then* $i = i - (1+n^2)$ and if for $a_{i,j}$ $j > (1+n^2)$ *then* $j = j - (1+n^2)$

$a_{i,j} = 0;\ 1 \le i \le (1+n^2),\ 1 \le j \le (1+n^2)$ otherwise.

3. Eigenvalues of *q*-cliqued graphs and their eigen-bi-balanced property

In thissection, we focus on the *q*-cliqued graphs as constructed in section 1. We show that the *q*-cliqued graphs have eigenvalue *q* and conjugate eigen-pair:

$$\lambda = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}.$$

The determination of the conjugate eigen-pair is equivalent to showing that the cubic

$$\lambda^3 - \lambda^2(q-1) - \lambda q - \lambda(q-1) + q(q-1) \quad = (\lambda - q)(\lambda^2 + \lambda - (q-1))$$

is a factor of the characteristic equation determined by $A(G_k^{\ *})\underline{x} = \lambda \underline{x}$ where $A(G_k^{\ *})$ is the adjacency matrix of the*q*-cliqued graph.The proof requires a number of specific definitions of vertices within the *q*-clique graph, and we use the connectivity between the first clique, the second to last clique, and the last clique in the proof of the conjugate eigen-pair. The central vertex also plays a key role in this proof, as each sub-clique of K_q is connected to the central vertex. The proof of determining the conjugate eigen-pair and the associated eigenvectors, is first determined explicitly for the cases *q*=3,4, and then generalized for the *q*-cliqued graph.

Once we have found the conjugate eigen-pair of the *q*-cliqued graph, we then determine the eigen-bi-balanced properties of the class of *q*-cliqued graphs associated with this eigen-pair in section 4. The values of all the newly defined eigen-bi-balanced properties, as defined in Winter and Jessop [7], are easily determined for this class of graphs

Theorem 3.1

The *q*-cliqued graphs, as constructed in section 1,have eigenvalues $\lambda = q$ (and the *q*-cliqued graph is *q*-regular) and conjugate eigen-pair:

$$\lambda = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}.$$

The conjugate pairarise out of the "tightness" of the connection between the central vertex and the cliques, and between two adjacent cliques – for convention we chose the second last and last clique.

Proof of Theorem 3.1

We will illustrate Theorem 3.1 for $q = 4$ and 5 and then give the proof for all $q \geq 6$. Proof of cases $q = 2,3$ can be found in Jessop [5]. First, we need the following definitions.

3.1.1Vertex notation convention

Several vertices will be important in the proof, and hence we will give them special labels as follows:

1. First vertex (central vertex), x_1;
2. Second vertex, x_2;
3. Third vertex, x_3;
4. Vertices in first clique = $\{x_{2,} x_3, ..., x_q, x_{q+1}\}$;
5. Anchor vertexof each clique = vertex in each clique which is joined to the first vertex x_1
6. Anchor vertex of the last clique, $x_a = x_{2+q(q-1)}$;
7. Switching pair of vertices, $x_{q^2-1} = x_{l-2}$ (third last vertex) and $x_{q^2} = x_{l-1}$ (second last vertex);
8. Last vertex, $x_l = x_{q^2+1}$.

3.1.2 The generating set

The following definitions are also required for this proof:

Let $T = \{x_1, x_2\}$

and T' = {the set of vertices of the second last clique which are adjacent to vertices in the last clique}.

Then $T' = \{x_{k_1}, x_{k_2}, \ldots, x_{k_t}\}$, where $t = \frac{q-1}{2}$, q odd, or $t = \frac{q}{2}$, q even, and

S = the generating setof vertices

$= T \cup T'$

Also, if $S = \{x_1, x_2, ..., x_k\}$, then we define $\sum S = \sum_{i=1}^{k} x_i$.

3.1.3The two main equations that generate the conjugate eigen-pairs

We will use the relationship $A\underline{x} = \lambda \underline{x}$ to determine the two main equations that generate the conjugate eigen-pairs:

$$\sum S = \lambda^2 x_l - q x_l \qquad (1)$$

and

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l$$

$$\Rightarrow \sum S = \frac{(q-1)x_l}{(\lambda - (q-1))} \qquad (2)$$

Substituting (2), into (1) we get:

$$\frac{(q-1)\lambda x_l}{\lambda - (q-1)} = \lambda^2 x_l - q x_l; \quad \lambda \neq q-1$$

so that:

$$(q-1)\lambda = \lambda^2(\lambda - (q-1)) - q(\lambda - (q-1))$$
$$\Rightarrow \lambda^3 - \lambda^2(q-1) - q\lambda + q(q-1) - \lambda(q-1) = 0$$
$$\Rightarrow (\lambda - q)(\lambda^2 + \lambda - (q-1)) = 0$$

This gives us three eigenvalues:

- $\lambda = q$; and
- the conjugate eigen-pair $\lambda = \dfrac{-1 \pm \sqrt{1 + 4(q-1)}}{2}$.

The proof of cases q=2 and 3 can be found in Jessop [ ]’

3,1.4The case $q = 4$

Refer to Jessop [5].

Step 1: Write down first equation using last vertex: equation part of $A\underline{x} = \lambda \underline{x}:\ x_3 + x_{14} + x_{15} + x_{16} = \lambda x_{17}$

Expand left hand side with their neighborsto get vertices belonging to set S:

$$(x_2 + x_4 + x_5 + x_{17}) + (x_1 + x_{15} + x_{16} + x_{17}) +$$
$$(x_{13} + x_{14} + x_{16} + x_{17}) + (x_{12} + x_{14} + x_{15} + x_{17})$$
$$= \lambda(x_3 + x_{14} + x_{15} + x_{16})$$

$$x_1 + x_2 + x_4 + x_5 + 2(x_{14} + x_{15} + x_{16}) + x_{12} + x_{13} + 4x_{17} = \lambda(\lambda x_{17})$$

Step 2: Put $x_{16} = -x_{15}$ (second and third largest have opposite signs and are called the switching pair) – this guarantees $x_{15}, x_{16} \notin S$.

Set $T = \{x_1, x_2\}$ and $T' = \{$all vertices in S that belong to the second last clique, and which are neighbours of the lastclique $\} = \{x_{12}, x_{13}\}$. Then the generating set $S = T \cup T' = \{x_1, x_2\} \cup \{x_{12}, x_{13}\} = \{x_1, x_2, x_{12}, x_{13}\}$.

Then we have

$$x_1 + x_2 + x_4 + x_5 + 2(x_{14}) + x_{12} + x_{13} + 4x_{17} = \lambda(\lambda x_{17})$$

Step 3: Set $x_4 = x_5 = x_{14} = 0$;

$$x_1 + x_2 + x_{12} + x_{13} + 4x_{17} = \lambda^2 x_{17}$$
$$\Rightarrow x_1 + x_2 + x_{12} + x_{13} = \lambda^2 x_{17} - 4x_{17}$$

$$\Rightarrow \sum S = \lambda^2 x_{17} - 4x_{17} \qquad (1)$$

This verifies equation (1) of Section 5.1.2 for the case $q = 4$.

Step 4: Taking the neighbours of the vertices in $S = \{x_1, x_2, x_{12}, x_{13}\}$ we get

$$(x_2 + x_6 + x_{10} + x_{14}) + (x_1 + x_3 + x_4 + x_5) +$$
$$(x_{10} + x_{11} + x_{13} + x_{16}) + (x_{10} + x_{11} + x_{12} + x_{15}) = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

From above, $x_4 = x_5 = x_{14} = 0; x_{15} = -x_{16}$

$$x_2 + x_6 + x_{10} + x_1 + x_3 + x_{10} + x_{11} + x_{13} + x_{10} + x_{11} + x_{12} = 0$$

$$x_1 + x_2 + x_3 + x_6 + 3x_{10} + 2x_{11} + x_{12} + x_{13} = 0$$

$Set \; x_{10} = \lambda x_{17};$

$Set \; x_3 = 2x_1$

$Set \; x_{11} = x_2$

$x_{12} = 0$

$Set \; 2x_{13} = x_6$

$$x_1 + x_2 + 2x_1 + 3\lambda x_{17} + 2x_2 + 3x_{12} + 3x_{13} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

$$3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

$$\Rightarrow \quad x_1 + x_2 + x_{12} + x_{13} = \frac{3\lambda x_{17}}{\lambda - 3}$$

$$\Rightarrow \quad \sum S = \frac{3\lambda x_{17}}{\lambda - 3} \qquad (2)$$

This verifies equation (2) of Section 5.1.2 for the case $q = 3$.

Step 5: Substitute (2) into (1) to we get

$$\frac{3\lambda x_{17}}{\lambda - 3} = \lambda^2 x_{17} - 4x_{17}$$

$$\Rightarrow \quad \lambda^2(\lambda - 3)x_{17} - 4(\lambda - 3)x_{17} = 3\lambda x_{17}$$

$$\Rightarrow \quad \lambda^3 x_{17} - 3\lambda^2 x_{17} - 4\lambda x_{26} + 12x_{17} - 3\lambda x_{17} = 0$$

$$\Rightarrow \quad \lambda^3 x_{17} - 3\lambda^2 x_{17} - 7\lambda x_{26} + 12x_{17} = 0$$

$$\Rightarrow \quad (\lambda - 4)(\lambda^2 + \lambda - 3)x_{17} = 0$$

$$\Rightarrow \quad \lambda = 4 \text{ or } \lambda = \frac{-1 \pm \sqrt{1 - (4)(-3)}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = 4$, (which is the same as the degree of the vertices in the *4*-cliqued graph), and the conjugate eigen-pairs $\lambda = \dfrac{-1 \pm \sqrt{13}}{2}$.

Let $\underline{x} = [x_1, x_2, \ldots, x_{17}]^T$. Then $A(G_{K_4}{}^*)\underline{x} = \lambda \underline{x}$ gives

$$\begin{bmatrix} x_2+x_6+x_{10}+x_{14} \\ x_3+x_4+x_5+x_1 \\ x_2+x_4+x_5+x_{17} \\ x_3+x_5+x_2+x_8 \\ x_2+x_3+x_4+x_7 \\ x_1+x_7+x_8+x_9 \\ x_5+x_6+x_8+x_9 \\ x_4+x_6+x_7+x_9 \\ x_6+x_7+x_8+x_{11} \\ x_1+x_{11}+x_{12}+x_{13} \\ x_9+x_{10}+x_{12}+x_{13} \\ x_{10}+x_{11}+x_{13}+x_{16} \\ x_{10}+x_{11}+x_{12}+x_{15} \\ x_1+x_{15}+x_{16}+x_{17} \\ x_{13}+x_{14}+x_{16}+x_{17} \\ x_{12}+x_{14}+x_{15}+x_{17} \\ x_3+x_{14}+x_{15}+x_{16} \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \end{bmatrix}$$

$$= \begin{bmatrix} x_2+2x_{13}+\lambda x_{17}+0 \\ 2x_1+0+0+x_1 \\ x_2+0+0+x_{17} \\ 2x_1+0+x_2+x_8 \\ x_2+2x_1+0+x_7 \\ x_1+x_7+x_8+x_9 \\ 0+2x_{13}+x_8+x_9 \\ 0+2x_{13}+x_7+x_9 \\ 2x_{13}+x_7+x_8+x_2 \\ x_1+x_2+x_{13} \\ x_9+\lambda x_{17}+x_{13} \\ \lambda x_{17}+x_2+x_{13}+x_{16} \\ \lambda x_{17}+x_2+0-x_{16} \\ x_1+0+x_{17} \\ x_{13}+0+x_{16}+x_{17} \\ -x_{16}+x_{17} \\ 2x_1+0+0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 \\ 0 \\ 0 \\ 2x_{13} \\ x_7 \\ x_8 \\ x_9 \\ \lambda x_{17} \\ x_2 \\ 0 \\ x_{13} \\ 0 \\ -x_{16} \\ x_{16} \\ x_{17} \end{bmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{pmatrix}$$

Step 6: We will now verify equations (1) and (2) in this section, using the definition of the eigenvector above.

We use the generating set $S = \{x_1, x_2, x_{12}, x_{13}\}$ with its sum $\sum S = x_1 + x_2 + x_{12} + x_{13}$. Now, using equations (4), (5) and (15) and(16) in the above, and noting that the variable x_2 is 0:

$$\begin{aligned}
\lambda\sum S \quad &= \lambda(x_1 + x_2 + x_{12} + x_{13}) \\
&= (x_2 + 2x_{13} + \lambda x_{17}) + (2x_1 + x_1) + (\lambda x_{17} + x_2 + x_{13} + x_{16}) + (\lambda x_{17} + x_2 - x_{16}) \\
&= 3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} \\
&= 3\sum S + 2\lambda x_{17}
\end{aligned}$$

$$\Rightarrow \quad \sum S = \frac{3\lambda x_{17}}{\lambda - 3} \tag{21}$$

This is the same result as equation (2) above.

Step 7: We now verify equation (1) in this section, using the definition of the eigenvector above.

Next: equation 20 and 6 gives:

$$\lambda\lambda x_{17} = \lambda(2x_1) = x_2 + x_{17} + 0 + 0 + 0 + 0$$

Equation 17 gives $x_1 + x_{17} = 0$ and 18 +19 gives: $x_{13} + 2x_{17} = 0$ and $x_{12} = 0$, so that:

$$\lambda^2 x_{17} = x_1 + x_2 + x_{12} + x_{13} + 4x_{17}$$

$$\Rightarrow \sum S = \lambda^2 x_{17} - 4x_{17} \tag{22}$$

This is the same result as equation (1) above.

So we have verified both equations (1) and (2) by using the definition of the eigenvector.

3.1.5 The case $q = 5$

Step 1: Write down first equation using last vertexpart of the equation

$A\underline{x} = \lambda \underline{x}$

$$x_3 + x_{22} + x_{23} + x_{24} + x_{25} = \lambda x_{26}$$

Expand left hand side with their neighborsto get vertices belonging to set S:

$$\left(x_2+x_4+x_5+x_6+x_{26}\right)+\left(x_1+x_{23}+x_{24}+x_{25}+x_{26}\right)$$
$$+\left(x_{21}+x_{22}+x_{24}+x_{25}+x_{26}\right)+\left(x_{20}+x_{22}+x_{23}+x_{25}+x_{26}\right)$$
$$+\left(x_4+x_{22}+x_{23}+x_{24}+x_{26}\right)$$
$$=\lambda\left(x_3+x_{22}+x_{23}+x_{24}+x_{25}\right)$$

$$x_1+x_2+2x_4+x_5+x_6+x_{20}+x_{21}+3x_{22}+3x_{23}+3x_{24}+3x_{25}+5x_{26}$$
$$=\lambda\left(\lambda x_{26}\right)$$

Step 2: Put $x_{25}=-x_{24}$ (second and third largest have opposite signs and are called the switching pair) – this guarantees no $x_{24}, x_{25} \notin S$.

Set $T=\{x_1, x_2\}$ and $T'=\{$all vertices in S that belong to the second last clique, and which are neighbours of the lastclique $\}=\{x_{20}, x_{21}\}$. Then the generating set $S=T \cup T'=\{x_1, x_2\} \cup \{x_{20}, x_{21}\}=\{x_1, x_2, x_{20}, x_{21}\}$

Then we have

$$x_1+x_2+2x_4+x_5+x_6+x_{20}+x_{21}+3x_{22}+3x_{23}+5x_{26}=\lambda^2 x_{26}$$

Step 3: Put

$$x_4=x_5=0; x_6=-3x_{22}; x_{23}=0$$
$$x_1+x_2+x_{20}+x_{21}+5x_{26}=\lambda^2 x_{26}$$
$$\Rightarrow x_1+x_2++x_{20}+x_{21}=\lambda^2 x_{26}-5x_{26}$$
$$\Rightarrow \sum S=\lambda^2 x_{26}-5x_{26} \qquad (1)$$

This verifies equation (1) of Section 5.1.2 for the case $q=4$.

Step 4: Taking the neighbours of the vertices in $S=\{x_1, x_2, x_{20}, x_{21}\}$ we get

$$\left(x_2+x_7+x_{12}+x_{17}+x_{22}\right)+\left(x_1+x_3+x_4+x_5+x_6\right)$$
$$+\left(x_{17}+x_{18}+x_{19}+x_{21}+x_{24}\right)+\left(x_{17}+x_{18}+x_{19}+x_{20}+x_{23}\right)$$
$$=\lambda(x_1+x_2+x_{20}+x_{21})$$

Switching pair: $x_{24}=-x_{25}$ and set $x_4=x_5=0; x_6=-3x_{22}; x_{23}=0$

$$3x_1 = x_7; x_{12} = 4\lambda x_{26}$$

$$x_3 = 3x_2$$

$$x_{18} = \frac{3}{2}x_{20}; \quad x_{19} = \frac{3}{2}x_{21}$$

$$x_4 = x_5 = x_{17} = 0$$

$$x_{25} = -2x_{22} = -x_{24}$$

$$\left(x_2 + 3x_1 + 4x_{26} + 0 + x_{22}\right) + \left(x_1 + 3x_2 + 0 + 0 - 3x_{22}\right)$$
$$+\left(0 + \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + x_{24}\right) + \left(0 + \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0\right)$$
$$= 4(x_1 + x_2 + x_{20} + x_{21}) + 4x_{26}$$
$$= \lambda(x_1 + x_2 + x_{20} + x_{21})$$

Therefore

$$\Rightarrow x_1 + x_2 + x_{20} + x_{21} = \frac{4x_{26}}{\lambda - 4} \qquad \Rightarrow$$

$$\sum S = \frac{4x_{26}}{\lambda - 4} \qquad (2)$$

This verifies equation (2) of Section 5.1.2 for the case $q = 4$.

Step 5: Substitute (2) into (1) to we get

$$\frac{4\lambda x_{26}}{\lambda - 4} = \lambda^2 x_{26} - 5x_{26}$$
$$\Rightarrow \quad \lambda^2(\lambda - 4)x_{26} - 5(\lambda - 4)x_{26} = 4\lambda x_{26}$$
$$\Rightarrow \quad \lambda^3 x_{26} - 4\lambda^2 x_{26} - 5\lambda x_{26} + 20x_{26} - 4\lambda x_{26} = 0$$
$$\Rightarrow \quad \lambda^3 x_{26} - 4\lambda^2 x_{26} - 9\lambda x_{26} + 20x_{26} = 0$$
$$\Rightarrow \quad (\lambda - 5)(\lambda^2 + \lambda - 4)x_{26} = 0$$
$$\Rightarrow \quad \lambda = 5 \text{ or } \lambda = \frac{-1 \pm \sqrt{1 - (4.-4)}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = 5$, (which is the same as the degree of the vertices in the 5-cliqued graph), and the conjugate eigen-pairs $\lambda = \dfrac{-1 \pm \sqrt{17}}{2}$.

Let $\underline{x} = [x_1, x_2, \ldots, x_{26}]^T$. Then $A\left(G_{K_5}{}^{*}\right)\underline{x} = \lambda \underline{x}$ gives (see Jessop [ ]:

$$\begin{bmatrix} x_2 + 3x_1 + 4\lambda x_{26} + x_{22} \\ x_1 + 3x_2 - 3x_{22} \\ x_2 - 3x_{22} + x_{26} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ x_{15} + 0 + \frac{3}{2}x_{20} + x_{20} + x_{21} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + 2x_{22} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0 \\ x_1 + 0 + x_{26} \\ x_{21} + x_{22} + 0 + x_{26} \\ x_{20} + x_{22} + 2x_{22} + x_{26} \\ 0 + x_{22} - 2x_{22} + x_{26} \\ x_3 + x_{22} + 0 + 0 + 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ 0 \\ -x_{25} \\ x_{25} \\ x_{26} \end{bmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25 \\ 26 \\ 27 \\ 28 \\ 29 \end{pmatrix}$$

Step 6: We will now verify equation (2) in this section, using the definition of the eigenvector above.

We use the generating set $S = \{x_1, x_2, x_{21}, x_{22}\}$ with its sum $\sum S = x_1 + x_2 + x_{21} + x_{22}$. Now, using equations (4),(5),(23) and (24) in the above, and noting that the variable $x_2 = 0$:

$$\lambda \sum S = \lambda(x_1 + x_2 + x_{21} + x_{22})$$

$$= (x_2 + 3x_1 + 4\lambda x_{26} + x_{22}) + (x_1 + 3x_2 - 3x_{22}) + \left(\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + 2x_{22}\right)$$

$$+ \left(\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0\right)$$

$$= 4(x_1 + x_2 + x_{20} + x_{21}) + 4\lambda x_{26}$$

$$= 4\sum S + 4\lambda x_{26}$$

$$\Rightarrow \quad \sum S = \frac{4\lambda x_{26}}{\lambda - 4} \qquad (30)$$

This is the same result as equation (2) above.

Step 7: We now verify equation (1) in this section, using the definition of the eigenvector above.

(24) +(25) yield:

$$x_{20} + 2x_{22} + 2x_{26} = 0 \qquad (31)$$

From (29) we get

$$x_3 + x_{22} = \lambda\ x_{26}$$

$$\Rightarrow \lambda^2 x_{26} = \lambda x_3 + \lambda x_{22}$$

Substituting (6) and (25), we get

$$\lambda^2 x_{26} \quad = (x_2 - 3x_{22} + x_{26}) + (x_1 + 0 + x_{26}) \qquad (32)$$

Adding (31) and (26) to (32) we get

$$\lambda^2 x_{26} \quad = (x_2 - 3x_{22} + x_{26}) + (x_1 + 0 + x_{26}) + (x_{20} + 2x_{22} + 2x_{26}) + (x_{21} + x_{22} + 0 + x_{26})$$

$$= (x_1 + x_2 + x_{20} + x_{21}) + 5x_{26}$$

$$\Rightarrow \sum S = \lambda^2 x_{26} - 5x_{26} \qquad (33)$$

This is the same result as equation (1) above.

So we have verified both equations (1) and (2) by using the definition of the eigenvector.

3.1.6 Eigenvalues of general case

Refer to Section 3.1 for the vertex notation and definitions. We require the following additional definitions to clarify the proof for the general case, where $q \geq 6$.

1. x_1 is the first vertex (central vertex);
2. x_2 is the second vertex;
3. x_3 is the third vertex;
4. Vertices in first clique = $\{x_{2,} x_3,..., x_q, x_{q+1}\}$;
5. Vertices in last clique = $\{x_{a,} x_{a+1},..., x_{l-3}, x_{l-2}, x_{l-1}, x_l\}$;
6. Anchor vertexof clique is the vertex in each clique which is joined to the first vertex x_1;
7. Anchor vertex of the last clique, $x_a = x_{2+q(q-1)}$;
8. Switching pair of vertices are $x_{q^2-1} = x_{l-2}$ (third last vertex) and $x_{q^2} = x_{l-1}$ (second last vertex);
9. $x_l = x_{q^2+1}$ is the last vertex;
10. λx_l is the sum of the neighbours of x_l i.e.

    $\lambda x_l = x_3 + x_a + x_{a+1} + x_{a+2} + ..x_{l-3} + x_{l-2} + x_{l-2}$
11. Q is the set of vertices in the last clique which give ''0'' equations, i.e., $Q = \{x_{a+1}, x_{a+2},..., x_{l-3}\}$ and $\{x_{l-2}, x_{l-1}, x_l\} \notin Q$
12. Neighbours of $x_a = \{x_1, x_{l-2}, x_{l-1}, x_l\}$ and all other neighbours of x_a from from Q (which are 0)
13. Neighbours of $x_l = N(x_l)$

$$= \{x_l^1, x_l^2, \ldots, x_l^q\}$$

$$= \{x_3, x_a, x_{a+1}, x_{a+2}, \ldots, x_{l-3}, x_{l-2}, x_{l-1}\}$$

14. The sum of the neigbours of $x_l^i;\ 1 \le i \le q$ is $\lambda(\lambda x_l)$

15. The set T' consists of vertices from $\lambda(\lambda x_l)$ which belong to the second last $((q-1)$ th) clique, which are neighbours of the vertices from the last (q th) clique

i.e., $T' = \{x_{k_1}, x_{k_2}, \ldots, x_{k_t}\}$ where $t = \begin{cases} \frac{q-1}{2}; & q \ odd \\ \frac{q}{2}; & q \ even \end{cases}$.

16. $T = \{x_1, x_2\}$

17. Let $S =$ the generating set of vertices then $S = T \cup T'$.

18. P = the set of vertices in the second last clique, excluding the anchor vertex, which are not neighbours of the last clique, and are therefore not in T' as defined above

i.e., $P = \{x_{p_1}, x_{p_2}, \ldots, x_{p_{q-1-t}}\}$ where $t = \begin{cases} \frac{q-1}{2}; & q \ odd \\ \frac{q}{2}; & q \ even \end{cases}$.

19. Q' isa subset of Q, whose vertices join backwards to vertices of T'. All vertices in Q' are in the last clique.

20. If $S = \{x_1, x_2, ..., x_k\}$, then we define $\sum S = \sum_{i=1}^{k} x_i$.

Step 1- write down the first equation using the last vertex:

$$\begin{aligned} \lambda^2 x_l &= \lambda(\lambda x_l) \\ &= x_1 && \text{central vertex} \\ &\quad + x_2 + x_4 + x_5 + ... + x_q + x_{q+1} && \text{all vertices in first clique} \\ &\quad + (q-2)x_a + (q-2)x_{a+1}, (q-2)x_{a+2} + ... + (q-2)x_{a+t} + ... + (q-2)x_{l-3} \\ &\quad + (q-2)x_{l-2} + (q-2)x_{l-1} + qx_l + (x_{k_1} + x_{k_2} + \ldots + x_{k_t}) \\ &\quad + (x_{p_1} + x_{p_2} + \ldots + x_{p_{q-1-t}}) \end{aligned}$$

Step 2: Set $x_{l-1} = -x_{l-2}$ switching vertices

Step 3: Put $x_4 = x_5 = ... = x_q = 0;\quad Q = \{0,0,...,0\};$

$x_{q+1} = -(q-2)x_a;$

Then,

$$\begin{aligned}\lambda^2 x_l = \;& x_1 + x_2 + 0 + 0 + \ldots + 0 \\ & + 0 + 0 + \ldots + 0 + 0 \\ & + (q-2)x_{l-2} - (q-2)x_{l-2} + qx_l \\ & + (x_{k_1} + x_{k_2} + \ldots + x_{k_t}) \\ & + (0 + 0 + \ldots + 0)\end{aligned}$$

$$\Rightarrow \lambda^2 x_l = x_1 + x_2 + qx_l + (x_{k_1} + x_{k_2} + \ldots + x_{k_t})$$

$$\Rightarrow \lambda^2 x_l - qx_l = x_1 + x_2 + \left(x_{k_1} + x_{k_2} + \ldots + x_{k_t}\right)$$

$$\Rightarrow \lambda^2 x_l - qx_l = \sum S$$

$$\Rightarrow \sum S = \lambda^2 x_l - qx_l \qquad (1)$$

Step 4:

Now we look at the neighbors of the generating set *S*:

$$S = T \cup T' = \{x_1, x_2\} \cup \{x_{k_1}, x_{k_2}, \ldots, x_{k_t}\}$$

$$\text{where } t = \frac{q-1}{2},\ q \text{ odd, and } t = \frac{q}{2},\ q \text{ even.}$$

Neighbours of $x_1 : x_2, x_{2+q}, x_{2+2q}, \ldots, x_{2+q(q-1)} = x_a$

Neighbours of $x_2 : x_1, x_3, x_{4,}, \ldots, x_{q+1}$

Sum of neighbours of $T' = (t-1)\sum T' + t\sum P + \sum Q'$

Then the sum of the neighbors of the elements of *S*:

$$\begin{aligned}\lambda \sum S \;& = (x_2 + x_{2+q} + x_{2+2q} + \ldots + x_{2+q(q-1)}) + (x_1 + x_3 + x_4 + \ldots + x_q + x_{q+1}) \\ & \quad + (t-1)\sum T' + t\sum P + \sum Q'\end{aligned}$$

From before:

Put $x_4 = x_5 = \ldots = x_q = 0;\ \ Q = \{0,0,\ldots,0\};\ \ x_{q+1} = -(q-2)x_a;\, x_{l-1} = -x_{l-2},$

$$\begin{aligned}\lambda \sum S \;& = (x_2 + x_{2+q} + x_{2+2q} + \ldots + x_{2+q(q-1)}) \\ & \quad + (x_1 + x_3 + 0 + \ldots + 0 - (q-2)x_a) \\ & \quad + (t-1)\sum T' + t\sum P + x_{l-2} \\ & = x_1 + x_2 + x_3 - (q-2)x_a + x_{2+q} + x_{2+2q} + \ldots + x_{2+q(q-1)} \\ & \quad + (t-1)\sum T' + t\sum P + x_{l-2}\end{aligned}$$

Set

$$x_3 = (q-2)x_2;$$
$$x_{2+q} = (q-2)x_1$$
$$x_{2+2q} = (q-1)\lambda x_l$$
$$x_{a-q} = x_{2+q(q-2)} = 0$$
$$x_{l-2} = (q-3)x_a = -x_{l-1}$$

$$\begin{aligned}\lambda\sum S &= x_1 + x_2 + (q-2)x_2 - (q-2)x_a + (q-2)x_1 + (q-1)\lambda x_l + ... \\ &\quad + 0 + x_a + (t-1)\sum T' + t\sum P + (q-3)x_a \\ &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (t-1)\sum T' + t\sum P\end{aligned}$$

Set

$$x_{p_1} = \frac{q-t}{t}x_{k_1};$$
$$x_{p_2} = \frac{q-t}{t}x_{k_2};$$
$$\ldots$$
$$x_{p_t} = \frac{q-t}{t}x_{k_t}$$

and

$x_{p(t+1)} = 0$ if q is even, as P has one more vertex than T' when q is even.

Then,

$$\begin{aligned}\lambda\sum S &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l \\ &\quad + (t-1)(x_{k_1} + x_{k_{2+}} + ... + x_{k_t}) + t\left[\frac{q-t}{t}(x_{k_1} + x_{k_2} + ... + x_{k_t})\right] \\ &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (q-1)(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t}) \\ &= (q-1)(x_1 + x_2 + x_{k_1} + x_{k_2} + \ldots + x_{k_t}) + (q-1)\lambda x_l \\ &= (q-1)\sum S + (q-1)\lambda x_l\end{aligned}$$

$$\Rightarrow (\lambda - (q-1))\sum S = (q-1)\lambda x_l$$

$$\Rightarrow \sum S = \frac{(q-1)\lambda x_l}{\lambda - (q-1)} \qquad (2)$$

Substituting (2) into (1), we get

$$\frac{(q-1)\lambda x_l}{\lambda-(q-1)} = \lambda^2 x_l - qx_l$$

$$\Rightarrow \quad \lambda^2(\lambda-(q-1))x_l - q(\lambda-(q-1))x_l = (q-1)\lambda x_l$$

$$\Rightarrow \quad \lambda^3 x_l - (q-1)\lambda^2 x_l - q\lambda x_l + q(q-1)x_l - (q-1)\lambda x_l = 0$$

$$\Rightarrow \quad \lambda^3 x_l - (q-1)\lambda^2 x_l - (2q-1)\lambda x_l + q(q-1)x_l = 0$$

$$\Rightarrow \quad (\lambda-q)(\lambda^2+\lambda-(q-1))x_l = 0$$

$$\Rightarrow \quad \lambda = q \text{ or } \lambda = \frac{-1\pm\sqrt{1-(-4(q-1))}}{2} = \frac{-1\pm\sqrt{1+4(q-1)}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = q$, (which is the same as the degree of the vertices in the *q*-cliqued graph), and the conjugate eigen-pairs $\lambda = \frac{-1\pm\sqrt{1+4(q-1)}}{2}$.

3.1.7General eigenvector

Let $\underline{x} = \left[x_1, x_2, ...x_{q^2+1}\right]^T$ be an eigenvector of $G_{K_q}{}^*$.Then, from applying the construction of the *q*-cliqued graphs and the anaylsis in the preceding sections, we have:

$$x_1 = x_2 + x_{2+q} + x_{2+2q} + ... + x_{a-q} + x_a$$

$$x_2 = x_1 + x_3 + x_4 + \ldots + x_{q+1}$$

$$x_3 = x_2 + x_4 + x_5 + x_6 + \ldots + x_{q+1} + x_l$$

$$= (q-1)x_2$$

$$x_4, x_5, \ldots, x_q = 0$$

$$x_{q+1} = -(q-2)x_a$$

$$x_{2+q} = (q-1)x_1$$

$$x_{2+2q} = \lambda(q-1)x_l$$

$$x_a = x_{l-(q-1)} = x_1 + x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + x_l$$

$$x_{p_1} = \frac{q-t}{t} x_{k_1};$$
$$x_{p_2} = \frac{q-t}{t} x_{k_2};$$
$$\ldots$$
$$x_{p_t} = \frac{q-t}{t} x_{k_t}$$
and

$x_{p(t+1)} = 0$ if q is even, as P has one more vertex than T' when q is even.

$$x_{q^2-1} = x_{l-2}$$
$$= x_a + (q-3)x_a + x_\alpha + x_l = \lambda x_{l-2},$$
where $x_\alpha \in T'$ and is connected to switching vertex x_{l-2}
$$x_{q^2} = x_{l-1}$$
$$= x_a - (q-3)x_a + x_l$$
$$x_{l-2} = (q-3)x_a = -x_{l-1}$$
$$x_{l-1} = -x_{l-2}$$
$$x_{q^2+1} = x_l$$
$$= x_3 + x_a$$

The general eigenvector will have $q-4-(t-1)$ entries which contain $x_a + x_l + x_{l-1} + x_{l-2}$.

Zero equations (obtained from all vertices in the last clique, which connect backwards to the $(q-1)$ clique, i.e., to the vertices of $T' \backslash \{x_\alpha\}$. $(t-1)$ of these such equations

$x_a + x_{\beta_1} + (x_{l-1} + x_{l-2}) + x_l = x_a + x_{k_\beta} + 0 + x_l$ $(t-1)$ of these such equationswhere $1 \leq \beta \leq t$, and $x_{k_\beta} \neq x_\alpha$.

Sum of generating set T' without x_α : $(t-2)T' \backslash \{x_{k_i}\} + (t-1)P + (t-1)x_{a-q} = \lambda x_{k_i}$;

Equation for x_α in generating set: $T' \backslash \{x_\alpha\} + P + x_{a-q} + x_{l-2}$

3.1.8The final general equations

As in the specific cases for $q = 4$ and 5, we need to verify the following two equations using the values of the entries in the eigenvector:

$$\sum S = \lambda^2 x_l - q x_l. \qquad (1)$$

and

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l \qquad (2)$$

We shall now prove that equation (1) holds for values of the eigenvector:

The last equation in $A(G_{K_n}{}^*)\underline{x} = \lambda \underline{x}$ yields

$$x_3 + x_a = \lambda x_l$$

$$\Rightarrow \lambda x_3 + \lambda x_a = \lambda^2 x_l$$

Substituting *a*th and *3*rdequations of $A(G_{K_n}{}^*)\underline{x} = \lambda \underline{x}$ we get

$$\begin{aligned} \lambda^2 x_l &= \lambda x_3 + \lambda x_a \\ &= (x_2 + x_4 + x_5 + x_6 + ... + x_{q+1} + x_l) \\ &\qquad + (x_1 + x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + x_l) \\ &= x_1 + x_2 + (x_4 + x_5 + x_6 + ... + x_{q+1}) \\ &\qquad + (x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + 2x_l) \end{aligned}$$

Setting $x_4 = x_5 = ... = x_q = 0,$ and $x_{q+1} = -(q-2)x_a,$ we get

$$\lambda^2 x_l = x_1 + x_2 - (q-2)x_a + (x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + 2x_l)$$

Now, adding the switching vertices, we get

$$x_{l-1} + x_{l-2} = (x_a - (q-3)x_a + x) + (x_a + (q-3)x_a + x_\alpha + x_l) = 0$$

$$\Rightarrow 2x_a + x_\alpha + 2x_l = 0$$

Adding the 0 equations yields: $(t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)}$

Adding the other 0 equations yield: $q - 4 - (t-1)$ of $x_a + x_l$

This all yields:

$$\begin{aligned} \lambda^2 x_l &= x_1 + x_2 - (q-2)x_a \quad + (x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + 2x_l) \\ &\quad + 2x_a + x_\alpha + 2x_l \end{aligned}$$

$$+(t-1)x_a+(t-1)x_l+x_{a-1}+x_{a-2}+..+x_{a-(t-1)}$$

$$+(q-4-(t-1))[x_a+x_l]$$

$$=x_1+x_2-(q-2)x_a+2x_l+2x_a+x_\alpha+2x_l$$

$$+(t-1)x_a+(t-1)x_l+x_{a-1}+x_{a-2}+..+x_{a-(t-1)}$$

$$+(q-4-(t-1))[x_a+x_l]$$

= sum of elements from generating set $+qx_l$

Therefore,
$\sum S=\lambda^2 x_l - qx_l$, which is equation (1) above.

Using the vector values as per 5.1.8, and referring to section 5.1.7, we have verified that

$$\Rightarrow \sum S=\frac{(q-1)\lambda x_l}{\lambda-(q-1)} \tag{2}$$

So we have verified both equations (1) and (2) by using the general definition of the eigenvector. Substituting (2) into (1), we solve for the conjugate eigen-pair. □

This concludes the proof of the conjugate eigen-pair of the adjacency matrix associated with the *q*-cliqued graphs, as constructed in section 2. It is interesting to note that the conjugate eigen-pair are a function of the clique number of the graph.

In the next section, we determine the eigen-bi-balanced properties of q-cliqued graphs associated with the conjugate eigen-pair $\lambda=\frac{-1\pm\sqrt{1+4(q-1)}}{2}$.

4 Eigen-bi-balanced properties of *q*-cliqued graphs

Now that we have determined the conjugate eigen-pair for the class of *q*-cliqued graphs, we can determine the eigen-bi-blanced properties as defined in Winter and Jessop [7], for this newly defined class of graphs. We recall from Section 3 that the conjugate eigen-pair is $(a,b)=\frac{-1\pm\sqrt{1+4(q-1)}}{2}$ for all *q*-cliqued graphs as defined in Section 2. We will determine the eigen-bi-balanced properties of the class of *q*-cliqued graphs, associated with this conjugate eigen-pair. We note the importance of the central vertex, which is connected to the anchor vertex of each of the q sub-cliques in the q-cliqued graphs. The proof of the following results can easily be verified.

Theorem 4.1

For the class of q-cliqued graphs and the conjugate eigen-pair

$$(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$$

1. The class of q-cliqued graphs is sum*(-1)*eigen-pair balanced with respect to the conjugate eigen-pair $(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$;

2. The class of q-cliqued graphs is product* $(1-q)$*eigen-pair balanced with respect to the conjugate eigen-pair $(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$;

3. The class of q-cliqued graphs has eigen-bi-balanced ratio

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2} G_{K_q}{}^{*} \frac{-1-\sqrt{1+4(q-1)}}{2}\right) = \frac{1}{(q-1)} ,$$

witheigen-bi-balanced ratio asymptote

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2} G_{K_q}{}^{*} \frac{-1-\sqrt{1+4(q-1)}}{2}\right)^{\infty} = 0, \text{ and}$$

density $\Omega_r(G_{K_q}{}^{*}) = \left| asymp\left(r\left(\frac{-1+\sqrt{1+4(q-1)}}{2} G_{K_q}{}^{*} \frac{-1-\sqrt{1+4(q-1)}}{2}\right)\right)\right| = 0;$

4. The class of q-cliqued graphs has eigen-bi-balanced ratio area

$$Ar\left(G_{K_q}{}^{*}\right)_{\frac{-1+\sqrt{1+4(q-1)}}{2}, \frac{-1-\sqrt{1+4(q-1)}}{2}} = \sqrt{n-1}\left(4\sqrt{n-1} + 4\ln\left|\sqrt{n-1}-1\right|\right); \text{ and}$$

5. The class of q-cliqued graphs has $|a+b|+|ab|=q$ with respect to the conjugateeigen-pair $(a,b)=\frac{-1\pm\sqrt{1+4(q-1)}}{2}$.

Proof

1. The sum of the conjugate eigen-pair (a,b) is

$$\text{sum}\left(\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$
$$=\frac{-1+\sqrt{1+4(q-1)}}{2}+\frac{-1-\sqrt{1+4(q-1)}}{2}$$
$$=-1$$

Therefore, the class of q-cliqued graphs isexact sum*(-1)*eigen-pair balanced. It is interesting that it is the conjugate pair of eigenvalues that satisfy the sum*(-1)*eigen-pair balanced criteria.

2. The product of the conjugate eigen-pair (a,b) is

$$\text{product}\left(\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$

$$=\frac{(-1)^2-(1+4(q-1))}{4}$$
$$=-(q-1)$$

We have shown that the product of the conjugate eigen-pair is an *integral function of q i.e.*, $f(q)=-(q-1)$ where $q-1$ is also the degree of the vertices in a complete graph of order$q$. These eigenvalues are therefore non-exact product*$(1-q)$*eigen-pair balanced.

3. The eigen-bi-balanced ratio is

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$
$$=\frac{-1}{-(q-1)}$$
$$=\frac{1}{(q-1)}$$

Note that the eigen-bi-balanced ratio is equal to the negative of the reciprocal of the product of the conjugate pairs. The asymptote of this ratio is 0, as the value of q increases. So

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)^{\infty}=0\text{, and}$$

$$\Omega_r(G_{K_q}{}^{*})=\left|asymp\left(r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)\right)\right|=0;$$

4. The eigen-bi-balanced ratio area is

$$Ar\left(G_{K_q}{}^{*}\right)^{\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}}=\frac{2m}{n}\left|\int\frac{a+b}{ab}dn\right|$$
$$=\frac{q(q^2+1)}{q^2+1}\left|\int\frac{-1}{-(q-1)}dn\right|$$
$$=2q\left|\int\frac{1}{\sqrt{n-1}-1}dn\right|$$
$$=4q\left|\int\frac{udu}{u-1}\right|$$
$$=4q\left|\int\frac{u-1}{u-1}+\frac{1}{u-1}du\right|$$
$$=\sqrt{n-1}\left(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|\right)+c$$

When $n=1$ we have $Ar=0$ so that $c=0$.

So

$$Ar\left(G_{K_q}{}^*\right)^{\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}} = \sqrt{n-1}(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|)$$

5. $|a+b|+|ab|$

$$=\left|\frac{-1+\sqrt{1+4(q-1)}}{2}+\frac{-1-\sqrt{1+4(q-1)}}{2}\right|+\left|\frac{-1+\sqrt{1+4(q-1)}}{2}*\frac{-1-\sqrt{1+4(q-1)}}{2}\right|$$

$$=\left|\frac{-2}{2}\right|+\left|\frac{1-\left(1+4(q-1)\right)}{4}\right|$$

$$=1+(q-1)$$

$$=q$$

5. Design graphs and an entomological experiment

The study of the interaction between insects and host-specific plants is important in bio-control situations and is well documented - see Jans and Nylin [4]. Many such experiments use block designs (see, for example, Coll [2]) and optimal scheduling would be advantageous when there is the occurrence of large number of treatments and blocks.

5.1 Design graphs

We can associate designs with the q-cliqued graphs as follows: the vertices are the treatments and the blocks are the neighbours of each vertex (see Jessop [5]). Since we have a q-cliques graph which is a block design graph, any application of graph theory to our graphs can be applied to its associated design, and in particular to experiments where block designs can be used to study the interaction of insects and plants. One of the important studies in graph theory is vertex colourings of graphs. It can be shown that a graph's chromatic number is greater or equal to the order of its largest clique, since a complete graph on *n* vertices requires *n* colours for a proper colouring.

Thus for our *q*-cliqued block graphs, their chromatic number is greater than or equal to *q*. Jessop [5]showed that $\chi(G_{K_q}{}^*)=q$. We now apply a 3-colouring to the design associated with the 3-cliqued block graph relating to an entomological experiment.

5.2 Experiment

We investigate the effect of 3 different species of insects on 10 different types of leaves (plants). We will have 10 cages containing the leaves and the insects, and they will be labelled as Cage 1, …, Cage 10.

We have 3 sets of leaves, each containing 10 different leaves. These leaves are to be divided (arbitrarily) into 10 cages, each cage labelled Cage 1, Cage 2,..., Cage 10. Thus each type of leaf must appear 3 times in the experiment so that we need 3 sets of the 10 leaves.

The effect of three species of insects (using 10 insects per species) on the leaves in each cage will be studied. The insects will be labelled The application of the 3 different insects to the mini-groups (cages) must be done in the smallest number of time sessions, such that the following conditions hold:

A1. Each mini-group of triple leaves must be exposed to 3 different insects.

A2. An arbitrary mini-group of leaves will be called the *central-trial set* or *central cage*, and denoted by v_1.

A3. There must be 3 groups of 3-cliques P, Q and R of cages not containing the central trial set.

A4. Each cage in a clique cannot receive insects at the same time.

A5. Exactly one member from each different clique must receive a 3-set of insects at the same time, as well as not at the same time as the central cage receives its 3-set of insects.

A6. Exactly one member of each different clique, different from the cages in A5, must not receive a 3-set of insects at the same time.

A7. The three clique groups receiving the insects must be interchangeable (permutable) so that each clique can be exposed to all 3 insects other than the control.

These requirementscan be depicted in a *3*-cliqued graph, where its central vertex is the central-trial set. The 10 vertices (labeled 1 to 10) represent the 10 cages each containing a set of 3 leaves, the 3 leaves in each cage (vertex) having their labels from the neighbour of the vertex (this is the block of the associated design).

The edges (adjacent cages) of the 3-cliqued graph represent tubes connected to the cages (vertices) with the condition that the tube cannot be open at both ends at the same time - forcing the insect into only one cage incident with the edge at a time.

The 3-cliqued graph has 15 edges, each vertex incident with 3 edges so that three different insect sets of 10 insects will be used. The proper colouring of the graph will refer to the time sessions when the insects can be released subject to conditions A1 – A7.

The chromatic number 3 refers to the condition where we require the smallest number of time sessions so that conditions A1 – A7hold.

The 10 blocks containing 3 different leaves from the 10 different leaves will be:

1. {2,5,8};
2. {1,3,4};
3. {2,4,10};
4. {2,3,6};
5. {1,6,7};
6. {4,5,7};
7. {5,6,9};
8. {1,9,10};
9. {7,8,10};
10. {3,8,9}.

The colouring is as follows (see figure 5.1):

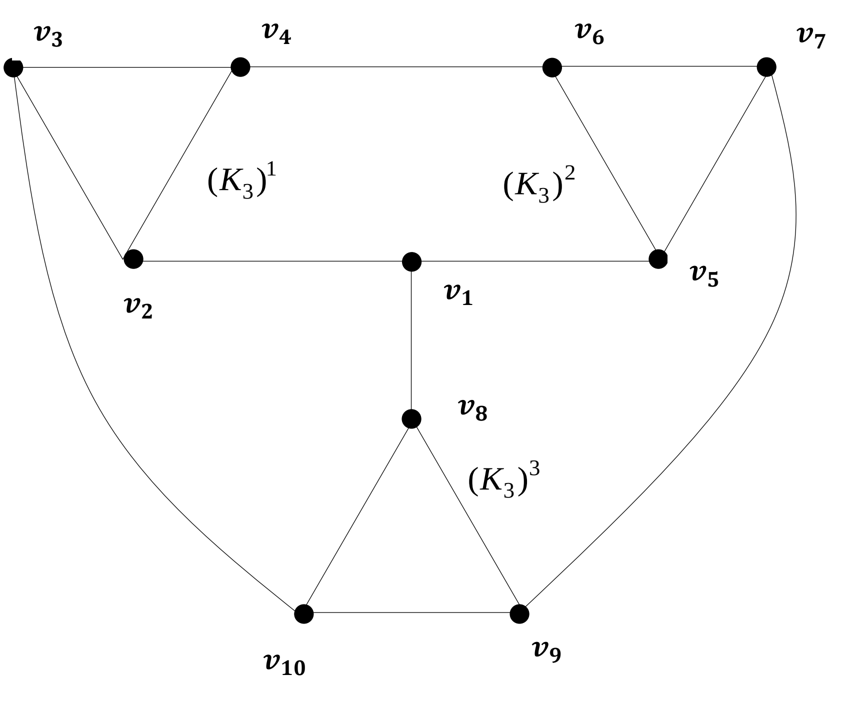

Figure 5.1 The graph of $G_{K_3}{}^{*}$

Put 3 colours red, green and blue – vertex 1 coloured blue, vertices 2,5,8colouredgreen, vertices 4,7,10 coloured blue, vertices 3,6,9 coloured red.

Label the insects *i*(1), *i*(2),…,*i*(30), where i(1+; 3k); i(2+3k) and i(3+3k), k=0,1,2,…9,represent the three different species (10 each), and allocate them as follows:

1. The trial-set is the (arbitrary) block 1={2,5,8} – this block contains leaves 2,5 and 8 and is colouredblue. The other blocks which are colouredblue are: block 4={2,3,6}; block 7={5,6,9}; block 10={3,8,9}. We release insects *i*(1), *i*(2), i(3) into cage 1, *i*(4), *i*(5), *i*(6) into cage 4, *i*(7), *i*(8), *i*(9) into cage 7 and *i*(10), *i*(11), *i*(12) into cage 10 (we only open the side incident with these vertices).
2. For the vertices 2={1,4,3}; 5={1,6,7}; 8={1,9,10} colouredgreen we release the next 9 insects (3 per vertex): *i*(13) to *i*(21).
3. For the remaining 3 vertices 3={2,4,10}; 6={4,5,7}; 9={7,8,10} colouredred,we release the remaining 9 insects (3 for each vertex): *i*(22) to *i*(30).

With this assignment of colours in $G_{K_3}{}^{*}$, we will now show that the 7 conditions are satisfied.

We have now released all the insects in the least number of time sessions of 3, each cage being exposed to 3 different insects, satisfying A1.

The central cage receives insects at a different time from a block from each clique, and these respective blocks receive insects at the same time, satisfying A5.

The 3 cliques P,Q and R each do not have their 3 blocks receiving insects at the same time (all blocks are adjacent in each clique) and do not contain the central cage, satisfying conditions A3 and A4.

The edges between the cliques allow condition A6 to be satisfied.

Three 5-cycles through the central cage are each coloured with 3 colours representing the central cage not receiving insects at the same time as a cage from each block required in A5. 2 cages from 2 separate cliques do receive insects at the same time and 2 cages from the same separate cliques do not.

Once we have applied the insects with 3 different time sessions, we keep the central vertices fixed and rotate the vertices (cages) of each clique once keeping the edges (tubes) fixed releasing 27 (fresh insects other than those released into vertex 1). For example, the block represented by vertex 2 with colourgreen, has edges (insects) *i*(13),*i*(14),and *i*(15). These insects remain connected to the tubes when we rotate, but vertex 4 will replace vertex 2 or vertex 3 will replace vertex 2. This rotation allows each block of the clique to receive each of the 3 (edges of the triangle) of the clique. Keeping the edges fixed of each clique and rotating the vertices of each clique (not the colours of the vertices), and doing this for two sessions on 3-time intervals, each block of each clique will then have been exposed to the 9 insects connected to each clique.

After the first two time sessions, we fix the edges (tubes) and we move the whole cliques (as vertices) around without changing the vertex colouring, so that conditions A1, A2 still hold, and each blockother than the trial block, is exposed to all 27 insects involved in the 3 cliques. Thus condition A7 holds without violating any other condition.

6. References

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