On filter ($\alpha$)-convergence and exhaustiveness of function nets in lattice groups and applications

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Abstract: We consider (strong uniform) continuity of the limit of a pointwise convergent net of lattice group-valued functions, (strong weak) exhaustiveness and (strong) ($\alpha$)-convergence with respect to a pair of filters, which in the setting of nets are more natural than the corresponding notions formulated with respect to a single filter. Some comparison results are given between such concepts, in connection with suitable properties of filters. Moreover, some modes of filter (strong uniform) continuity for lattice group-valued functions are investigated, giving some characterization. As an application, we get some Ascoli-type theorem in an abstract setting.

A bornology on a topological space $X$ is a family $\mathcal{B}$ of nonempty subsets of $X$ which covers $X$, stable with respect to finite unions and with $B \in \mathcal{B}$ for each nonempty subset $B$ of any element $B \in \mathcal{B}$.

If $X$ is a topological space and $x \in X$, then we say that a function $f : X \to R$ is continuous at $x$ iff there is an $(O)$-sequence $(\sigma_p)_p$ (depending on $X$) with the property that for each $p \in \mathbb{N}$ and $x \in X$ there exists a neighborhood $U_x$ of $x$ with $|f(x) - f(z)| \leq \sigma_p$ whenever $z \in U_x$. We say that $f \in R^X$ is globally continuous on $X$ iff it is continuous at every point $x \in X$ with respect to a single $(O)$-sequence, which can be taken independently of $x$.

Let $X = (X, \mathcal{D})$ be a uniform space. The elements of $\mathcal{D}$ are often called entourages. If $\emptyset \neq B \subset X$, then a function $f : X \to R$ is strongly uniformly continuous on $B$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there is an entourage $D \in \mathcal{D}$ with $|f(\beta) - f(x)| \leq \sigma_p$ whenever $x \in X$, $\beta \in B$ and $(x, \beta) \in D$. If $B$ is a bornology on $X$, then we say that $f : X \to R$ is strongly uniformly continuous on $B$ if it is uniformly continuous on $B$ for every $B \in B$, with respect to an $(O)$-sequence independent of $B$.

We denote by $\mathcal{F}_{\text{cofin}}$ the filter of all subsets of $\mathbb{N}$ whose complement is finite, by $\mathcal{I}_{\text{fin}}$ its dual ideal, namely the family of all finite subsets of $\mathbb{N}$, by $\mathcal{F}_{\text{st}}$ the filter of all subsets of $\mathbb{N}$ having asymptotic density 1, and by $\mathcal{I}_{\text{st}}$ its dual ideal, that is the family of all subsets of $\mathbb{N}$ with asymptotic density 0, by $\mathcal{F}_{\Lambda}$ the class $\{A \subset \Lambda$ and $A \supset M_\lambda : \lambda \in \Lambda\}$ and by $\mathcal{I}_{\Lambda}$ the dual ideal of $\mathcal{F}_{\Lambda}$. Observe that $\mathcal{F}_{\text{fin}} = \mathcal{F}_{\text{cofin}}$ and $\mathcal{I}_{\text{fin}} = \mathcal{I}_{\text{cofin}}$.

We will sometimes consider free filters $\mathcal{F}$ of $\mathbb{N}$, with the property that there is a partition of the type $\mathbb{N} = \bigcup_{k=1}^{\infty} \Delta_k$, such that

$$\mathcal{I} = \{A \subset \mathbb{N} : A \text{ intersects at most a finite number of } \Delta_k \text{'s}\},$$

where $\mathcal{I}$ denotes the dual ideal of $\mathcal{F}$ (see also [6]).

Remarks 1 (a) Observe that the ideal $\mathcal{I}_{\text{fin}}$ satisfies condition (1): indeed it is enough to take $\Delta_k = \{k\}$ for each $k \in \mathbb{N}$.
(b) If $\mathcal{I}$ is as in (1), and $(A_i)_i$ is any sequence of subsets of $\mathbb{N}$, with $A_i \notin \mathcal{I}$ for all $j \in \mathbb{N}$, then there exists a disjoint sequence $(B_i)_i$ in $\mathcal{I}$, with $B_i \subset A_j$ for every $j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} B_i \notin \mathcal{I}$.

(c) The ideal $\mathcal{I}_{st}$ does not fulfil condition (1). To this aim, it is enough to show that for every partition $(\Delta_k)_k$ of $\mathbb{N}$ there is a set belonging to $\mathcal{I}_{st}$ which intersects infinitely many $\Delta_k$'s. Set $q(1)=1$: there is $k_1 \in \mathbb{N}$ with $1 \in \Delta_{k_1}$. At the second step, take a natural number $q(2)$ greater than $q(1)+3$ and belonging to $\Delta_{k_2}$, where $k_2$ is a suitable integer strictly greater than $k_1$. At the $n+1$-th step, if we have chosen $q(n)$, let $q(n+1)$ be an integer greater than $q(n)+2n+1$ and belonging to $\Delta_{k_{n+1}}$, where $k_1 < k_2 < \ldots < k_n < k_{n+1}$. It is not difficult to check that the set $A = \{q(n): n \in \mathbb{N}\}$ has asymptotic density smaller or equal than that of the set of squares, that is $0$, and thus $A \in \mathcal{I}_{st}$. Moreover, by construction, $A$ intersects infinitely many $\Delta_k$'s.

Let $X$ be any Hausdorff topological space, $x \in X$ and $\mathcal{F}$ be a $(\Lambda)$-free filter of $\Lambda$. A net $(x_\lambda)_{\lambda \in \Lambda}$ be $\mathcal{F}$-converges to $x$ (shortly, $(\mathcal{F})\lim x_\lambda = x$) iff $\{\lambda \in \Lambda: x_\lambda \in U\} \in \mathcal{F}$ for each neighborhood $U$ of $x$.

We say that a net $(x_\lambda)_{\lambda \in \Lambda}$ in $R$ $(\mathcal{O} \mathcal{F})$-converges to $x \in R$ (briefly, $(\mathcal{O} \mathcal{F})\lim x_\lambda = x$) iff there exists an $(\mathcal{O})$-sequence $(\sigma_p)_p$ with $\{\lambda \in \Lambda: x_\lambda - x \leq \sigma_p\} \in \mathcal{F}$ for each $p \in \mathbb{N}$.

Let $\Xi$ be any nonempty set. A family $\{(x_\zeta, \xi)_{\zeta \in \Xi}\}$ $(\mathcal{R} \mathcal{O} \mathcal{F})$-converges to $x_\xi \in R$ iff there is an $(\mathcal{O})$-sequence $(\sigma_p)_p$ in $\mathcal{R}$ such that for each $p \in \mathbb{N}$ and $\xi \in \Xi$ we get $\{\lambda \in \Lambda: x_\lambda - x_\xi \leq \sigma_p\} \in \mathcal{F}$. We will denote by $(\mathcal{R} \mathcal{O})$-convergence the $(\mathcal{R} \mathcal{O} \mathcal{F})$-convergence. Observe that $(\mathcal{R} \mathcal{O})$-convergence coincides with the usual pointwise $(\mathcal{O})$-convergence of a family with respect to a single $(\mathcal{O})$-sequence and, when $R = \mathbb{R}$, $(\mathcal{R} \mathcal{O} \mathcal{F})$-convergence coincides with filter convergence in the ordinary sense.

Let $(X, D)$ be a uniform space, $\emptyset \neq B \subset X$, $\Xi = (\Xi, \geq)$ be a directed set and $S$ be a $(\Xi)$-free filter of $\Xi$. We say that the pair of nets $(z_\xi)_{\xi \in \Xi}, (x_\xi)_{\xi \in \Xi}$, satisfies condition H1) with respect to $S$ iff $x_\xi, z_\xi \in B$ for each $\xi \in \Xi$, and for every $D \in D$ there is a set $F \in S$ with $(x_\xi, z_\xi) \in D$ whenever $\xi \in F$.

Let $S$ and $\mathcal{F}$ be any two fixed $(\Xi)$-free filters of $\Xi$. A function $f: X \to R$ is said to be strongly $(S, \mathcal{F})$-uniformly continuous on $B$ iff there is an $(\mathcal{O})$-sequence $(\sigma_p)_p$ in $\mathcal{R}$ such that for every pair of nets $(z_\xi)_{\xi \in \Xi}, (x_\xi)_{\xi \in \Xi}$, satisfying condition H1) with respect to $S$, we have:

for each $p \in \mathbb{N}$ there is $F^* \in \mathcal{F}$ with $f(x_\xi) - f(z_\xi) \leq \sigma_p$ for each $\xi \in F^*$.

If $B$ is a bornology on $X$, we say that $f: X \to R$ is strongly $(S, \mathcal{F})$-uniformly continuous on $B$ iff it is strongly $(S, \mathcal{F})$-uniformly continuous on every $B \in B$ with respect to a single $(\mathcal{O})$-sequence, independent of $B$.

Let $X$ be any Hausdorff topological space. We say that $f: X \to R$ is $(S, \mathcal{F})$-continuous at $x \in X$ iff there is an $(\mathcal{O})$-sequence $(\sigma_p)_p$ in $\mathcal{R}$ such that, for every net $(x_\lambda)_{\lambda \in \Lambda}$ $S$-convergent to $x$, the net $(f(x_\lambda))_{\lambda \in \Lambda} (\mathcal{O} \mathcal{F})$-converges to $f(x)$ with respect to $(\sigma_p)_p$. We say that $f: X \to R$ is $(S, \mathcal{F})$-continuous on $X$ iff $f$ is $(S, \mathcal{F})$-continuous at every $x \in X$ with respect to a single $(\mathcal{O})$-sequence,
independent of $x \in X$ and of $(x_\xi)_\xi$.

Let $(X, D)$ be a uniform space and $\emptyset \neq B \subset X$. A net of functions $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be **strongly $\mathcal{F}$-exhaustive on $B$** iff there is an $(O)$-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there exist an entourage $D \in D$ and a set $A \in \mathcal{F}$ such that for each $\lambda \in A$ and $x \in X$, $\beta \in B$ with $(x, \beta) \in D$ we have $|f_\lambda(x) - f_\lambda(\beta)| \leq \sigma_p$.

We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **strongly weakly $\mathcal{F}$-exhaustive on $B$** iff there is an $(O)$-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is an entourage $D \in D$ such that, for every $x \in X$ and $\beta \in B$ with $(x, \beta) \in D$, there is $A \in \mathcal{F}$ (depending on $x$ and $\beta$) with $|f_\lambda(x) - f_\lambda(\beta)| \leq \sigma_p$ whenever $\lambda \in A$.

Given a bornology $\mathcal{B}$ on $X$, we say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be **strongly (weakly) $\mathcal{F}$-exhaustive on $B$** iff it is strongly (weakly) $\mathcal{F}$-exhaustive on every $B \in \mathcal{B}$ with respect to a single $(O)$-sequence, independent of $B$.

Let $x \in X$. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **$\mathcal{F}$-exhaustive at $x$** iff there is an $(O)$-sequence $(\sigma_p)_p$ with the property that for any $p \in \mathbb{N}$ there exist a neighborhood $U$ of $x$ and a set $A \in \mathcal{F}$ such that for each $\lambda \in A$ and $z \in U$ we have $|f_\lambda(z) - f_\lambda(x)| \leq \sigma_p$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **weakly $\mathcal{F}$-exhaustive at $x$** iff there is an $(O)$-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is a neighborhood $U$ of $x$ with the property that for any $z \in U$ there is $A \in \mathcal{F}$ with $|f_\lambda(z) - f_\lambda(x)| \leq \sigma_p$ whenever $\lambda \in A$.

We say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **(weakly) $\mathcal{F}$-exhaustive on $X$** iff it is (weakly) $\mathcal{F}$-exhaustive at every $x \in X$ with respect to a single $(O)$-sequence, independent of $x \in X$.

Let $S$ and $\mathcal{F}$ be as above, $(X, D)$ be a uniform space and $\emptyset \neq B \subset X$. A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be **strongly $(S, \mathcal{F})$-exhaustive on $B$** iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that, for every pair of nets $(x_\xi)_{\xi \in \mathbb{Z}}$, $(z_\xi)_{\xi \in \mathbb{Z}}$, satisfying H1) with respect to $S$ and for any $p \in \mathbb{N}$ there are $S \in S$, $F \in \mathcal{F}$ with $|f_\lambda(x_\xi) - f_\lambda(z_\xi)| \leq \sigma_p$ for every $\xi \in S$ and $\lambda \in F$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **strongly weakly $(S, \mathcal{F})$-exhaustive on $B$** iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that, for each pair of nets $(x_\xi)_{\xi \in \mathbb{Z}}$, $(z_\xi)_{\xi \in \mathbb{Z}}$, satisfying H1) with respect to $S$ and for every $p \in \mathbb{N}$ there is a set $S \in S$ such that for each $\xi \in S$ there exists $F_\xi \in \mathcal{F}$ with $|f_\lambda(x_\xi) - f_\lambda(z_\xi)| \leq \sigma_p$ for every $\lambda \in F_\xi$.

Given a bornology $\mathcal{B}$ on $X$, we say that the net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **strongly (weakly) $(S, \mathcal{F})$-exhaustive on $B$** iff it is strongly (weakly) $(S, \mathcal{F})$-exhaustive on every $B \in \mathcal{B}$, with respect to a single $(O)$-sequence, independent of $B$.

Let $X$ be any Hausdorff topological space and $x \in X$. A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be **$(S, \mathcal{F})$-exhaustive at $x \in X$** iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that, for every net $(x_\xi)_{\xi \in \mathbb{Z}}$ $S$-convergent to $x$ and for any $p \in \mathbb{N}$ there are $S \in S$, $F \in \mathcal{F}$ with $|f_\lambda(x_\xi) - f_\lambda(x)| \leq \sigma_p$ for every $\xi \in S$ and $\lambda \in F$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is **weakly $(S, \mathcal{F})$-exhaustive at $x \in X$** iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that, for each net $(x_\xi)_{\xi \in \mathbb{Z}}$ $S$-convergent to $x$ and for every $p \in \mathbb{N}$ there is
a set $S \in S$ such that for each $\xi \in S$ there exists $F_\xi \in \mathcal{F}$ with $|f_\lambda(x_\xi) - f_\lambda(x)| \leq \sigma_p$ whenever $\lambda \in F_\xi$.

The net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is (weakly) $(S, \mathcal{F})$-exhaustive on $X$ iff it is (weakly) $(S, \mathcal{F})$-exhaustive at every $x \in X$ with respect to a single $(O)$-sequence, independent of $x$.

Note that the analogous concepts of (strong weak) filter exhaustiveness can be formulated analogously for sequences of functions, by taking $\Lambda = \mathbb{N}$ with the usual order.

Let $\mathcal{B}$ be a bornology on $X$. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{F}\mathcal{B})$-converges to $f : X \to R$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ such that $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$-convergent to $f$ with respect to $(\sigma_p)_p$, and for every $B \in \mathcal{B}$ and $p \in \mathbb{N}$ there is $F \in \mathcal{F}$ with $|f_\lambda(x) - f(x)| \leq \sigma_p$ for each $x \in B$ and $\lambda \in F$.

We now consider $(\alpha)$-convergence of nets $(f_\lambda)_{\lambda \in \Lambda}$ of $(\ell)$-group-valued functions, defined on a Hausdorff topological space $X$. When we deal with nets, in general it is not always advisable to follow an approach similar as that used for sequences, since the cardinality of $\Lambda$ can be larger than the one of $X$, and we want to consider possibly nets in $X$ of the type $x_\xi$, $\xi \in \mathcal{Z}$, whose points are all distinct. We say that $f_n : X \to R$, $n \in \mathbb{N}$, $c$-converges (continuously converges) to $f : X \to R$ at $x \in X$ iff there is an $(O)$-sequence $(\sigma_p)_p$ such that for each sequence $(x_n)_n$ in $X$ with $(O)\lim_n x_n = x$ we get $(O)\lim_n f_n(x_n) = f(x)$ (with respect to the $(O)$-sequence $(\sigma_p)_p$).

The sequence $(f_n)_n$ $c$-converges to $f : X \to R$ on $X$ iff it $c$-converges to $f$ at every $x \in X$ with respect to a single $(O)$-sequence, independent of $x \in X$.

Let $\mathcal{F}$ be a fixed free filter of $\mathbb{N}$. We say that a sequence $(f_n)_n$ in $R^X$ $(\mathcal{F}c)$-converges (filter continuously converges) to $f \in R^X$ at $x \in X$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ such that for each sequence $(x_n)_n$ in $X$ with $(\mathcal{F})\lim_n x_n = x$ we get $(O\mathcal{F})\lim_n f_n(x_n) = f(x)$ with respect to $(\sigma_p)_p$.

The sequence $(f_n)_n$ is $(\mathcal{F}c)$-convergent to $f : X \to R$ on $X$ iff it $(\mathcal{F}c)$-converges to $f$ at every $x \in X$ with respect to a single $(O)$-sequence $(\sigma_p)_p$, independent of the choice of $x$.

Note that $(f_n)_n$ is $c$-convergent to $f$ if and only if $(f_n)_n$ is $(\mathcal{F}cofin)$-convergent to $f$.

Let now $\Lambda$ and $\mathcal{Z}$ be two directed sets, $\mathcal{S}$ and $\mathcal{F}$ be a $(\mathcal{Z})$-free filter of $\mathcal{Z}$ and a $(\Lambda)$-free filter of $\Lambda$ respectively. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{S}_\mathcal{F}\alpha)$-converges to $f : X \to R$ at $x \in X$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ in $R$ such that, for every net $(x_\xi)_\xi \in \mathcal{S}$, $S$-convergent to $x$ and for each $p \in \mathbb{N}$ there are $S \in \mathcal{S}$, $F \in \mathcal{F}$ with $|f_\lambda(x_\xi) - f(x)| \leq \sigma_p$ whenever $\xi \in S$ and $\lambda \in F$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{S}_\mathcal{F}\alpha)$-converges to $f : X \to R$ on $x \in X$ iff it $(\mathcal{S}_\mathcal{F}\alpha)$-converges to $f : X \to R$ at every $x \in X$ with respect to a single $(O)$-sequence, independent of the choice of $x$.

Let $(X, \mathcal{D})$ be a uniform space and $\emptyset \neq B \subset X$. We say that a sequence $(f_n)_n$ in $R^X$ strongly $(\mathcal{F}c)$-converges to $f \in R^X$ on $B$ iff there exists an $(O)$-sequence $(\sigma_p)_p$ such that for each pair of sequences $(x_n)_n$, $(z_n)_n$ in $X$ satisfying condition H1) with respect to $\mathcal{F}$ we get $(O\mathcal{F})\lim_n f_n(x_n) = f(x)$ with respect to $(\sigma_p)_p$.

Given a bornology $\mathcal{B}$ on $X$, we say that a sequence $(f_n)_n$ in $R^X$ is strongly $(\mathcal{F}c)$-convergent to $f : X \to R$ on $B$ iff it strongly $(\mathcal{F}c)$-converges to $f$ on every $B \in \mathcal{B}$ with respect to an $(O)$-
sequence \((\sigma_p)_p\), independent of \(B\).

A net \(f_\lambda : X \to R\), \(\lambda \in \Lambda\), is said to be {\bf strongly \((S,F\alpha)\)-convergent to \(f : X \to R\) on \(B\)} iff there exists an \((O)\)-sequence \((\sigma_p)_p\) in \(R\) such that, for every pair of nets \((x_\xi)_\xi \in S\), \((z_\xi)_\xi \in S\) satisfying condition H1) with respect to \(S\) and for each \(p \in \mathbb{N}\) there are \(S \in S\), \(F \in \mathcal{F}\) with \(|f_\lambda(z_\xi) - f(x_\xi)| \leq \sigma_p\) whenever \(\xi \in S\) and \(\lambda \in F\).

A net \(f_\lambda : X \to R\), \(\lambda \in \Lambda\), {\bf strongly \((S,F\alpha)\)-converges to \(f : X \to R\) on \(B\)} iff it strongly \((S,F\alpha)\)-converges to \(f : X \to R\) on every \(B \in B\) with respect to a single \((O)\)-sequence, independently of \(B\).

A net \(f_\lambda : X \to R\), \(\lambda \in \Lambda\), is said to be {\bf \((F\alpha)\)-convergent to \(f : X \to R\) at \(x \in X\) and on \(X\) (resp. on \(B\) and on \(B\))} iff it is {\bf \((S,F\alpha)\)-convergent to \(f : X \to R\) at \(x \in X\) and on \(X\) (resp. on \(B\) and on \(B\))} for every directed set \(\Xi = (\Xi, \tau)\) and for each \((\Xi)\)-free filter \(S\) of \(\Xi\).

Remarks

2. (a) Note that, even when \(R = \mathbb{R}\) and \(F = \mathcal{F}_\Lambda\), to use arbitrary nets of the type \((x_\xi)_\xi \in S\) instead of arbitrary sequences \((x_n)_n\) is essential. Indeed there are nets \((f_\lambda)_\lambda\) and functions \(f\) in \(\mathbb{R}^X\) such that \((f_\lambda)_\lambda\) does not \((\mathcal{F}_\Lambda\alpha)\)-convergent to \(f\), but for every \(\varepsilon > 0\), \(x \in X\) and for each sequence \((x_n)_n\) convergent to \(x\) in the usual sense there are \(n_0 \in \mathbb{N}\) and \(\lambda_0 \in \Lambda\), with \(|f_\lambda(x_n) - f(x)| \leq \varepsilon\) whenever \(n \geq n_0\) and \(\lambda \geq \lambda_0\).

(b) Observe that, in general, \((S,F\alpha)\)-convergence is strictly weaker than strong \((S,F\alpha)\)-convergence. Indeed, let \(\Lambda = \mathbb{N}\) be with the usual order, \(X = [0,1]\) with the usual metric, \(R = \mathbb{R}\), \(f_n(x) = x^n\) for each \(n \in \mathbb{N}\) and \(x \in [0,1]\), \(f(1) = 1\) and \(f(x) = 0\) for every \(x \in [0,1]\). We prove that for every \((\Xi)\)-free filter \(S\) of \(\Xi\) and for each free filter \(\mathcal{F}\) of \(\mathbb{N}\) the sequence \((f_n)_n\) \((S,F\alpha)\)-converges to \(f\), but does not strongly \((S,F\alpha)\)-convergent to \(f\) on \(B = [0,1]\). Indeed, if \((x_\xi)_\xi \in S\) is any net in \(B\), \(S\)-convergent to \(x_0 \in B\), then \(f_n(x_\xi) = x_\xi^n\) and \(f(x_0) = 0\), and there exist \(S \in S\) and \(x_0 < y < 1\) with \(x_\xi < y\) whenever \(\xi \in S\). Choose arbitrarily \(\varepsilon > 0\). Since \(0 < y < 1\), there exists \(n \in \mathbb{N}\) with \(y^n < \varepsilon\) for each \(n \geq n\). Hence for such \(n\)’s and \(\xi \in S\) we get \(0 < x_\xi^n < y^n < \varepsilon\). Thus \((f_n)_n\) \((S,F\alpha)\)-converges to \(f\) on \(B\). On the other hand, pick any pair of nets \((x_\xi)_\xi\), \((z_\xi)_\xi \in B\), with \((S)\text{lim}_\xi x_\xi = (S)\text{lim}_\xi z_\xi = 1\). Then we get \(f(x_\xi) = 0\) for each \(\xi \in \Xi\). Now we claim that

for every \(S \in S\) and \(F \in \mathcal{F}\) there are \(\xi_0 \in S, n_0 \in F\) with \(z_\xi^n > 1 - \frac{1}{3}\).

Choose arbitrarily \(S \in S\) and \(F \in \mathcal{F}\). As \(\lim_n \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}\), there is \(n_0 \in \mathbb{N}\) with

\[
\left(1 - \frac{1}{n}\right)^n > \frac{1}{3}
\]

for every \(n \geq n_0\).

Let \(n\) be any integer greater than \(n_0\) and belonging to \(F\). Since \((S)\text{lim}_\xi z_\xi = 1\), in correspondence with \(n\) there is \(S_n \in S\) with \(z_\xi^n > 1 - \frac{1}{n}\) whenever \(\xi \in S_n\). Let \(\xi_0 \in S \cap S_n\). Since \(z_\xi^n > 1 - \frac{1}{n}\), taking
into account (3) we get
\[ z_n^x > \left(1 - \frac{1}{n}\right)^n > \frac{1}{3}, \]
that is (2). Thus, the sequence \((f_n)_n\) does not strongly \((S, F)\)-converge to \(f\) on \(B\).

**Theorem 3** Let \(X\) be a Hausdorff topological space, \(x\) be a fixed element of \(X\), \(f_n : X \to R\), \(n \in \mathbb{N}\), be a sequence, \((ROF)\)-convergent to \(f : X \to R\). If \((f_n)_n\) is \(F\)-exhaustive at \(x\), then \((f_n)_n\) \((F)\)-converges to \(f\) at \(x\).

Conversely, if \(F\) satisfies condition (1) and \((f_n)_n\) is \((F)\)-convergent to \(f\) at \(x\), then \((f_n)_n\) is \(F\)-exhaustive at \(x\).

**Theorem 4** Let \((X, D)\) be any uniform space, \(\emptyset \neq B \subset X\), \(S\) and \(F\) be two free filters of \(\mathbb{N}\) with \(S \supset F\), and \(f_n : X \to R\), \(n \in \mathbb{N}\), be a function sequence, strongly \((S, F)\)-convergent to \(f \in R^X\) on \(B\). Then \((f_n)_n\) is strongly \((Sc)\)-convergent to \(f\) on \(B\).

**Theorem 5** Let \(X\) be a Hausdorff topological space, \(x \in X\), \(S\) and \(F\) be as in Theorem 3.2 and \(f_n : X \to R\), \(n \in \mathbb{N}\), be a sequence, \((S, F)\)-convergent to \(f \in R^X\) at \(x\). Then \((f_n)_n\) is \((S)\)-convergent to \(f\) at \(x\).

**Theorem 6** Let \((X, D)\) be a uniform space with a decreasing base \((U_k)_k\) of entourages, \(\emptyset \neq B \subset X\), \(S\) be a \((\Xi)\)-free filter of \(\Xi\), \(F\) be a free filter of \(\mathbb{N}\), satisfying condition (1), and \(f_n : X \to R\), \(n \in \mathbb{N}\), be a sequence, strongly \((F)\)-convergent to \(f \in R^X\) on \(B\). Then \((f_n)_n\) is strongly \((S, F)\)-convergent to \(f\) on \(B\).

**Theorem 7** Let \(X\) be a Hausdorff topological space, \(x \in X\), \((U_k)_k\) be a decreasing base of neighborhoods of \(x\), \(S\) be a \((\Xi)\)-free filter of \(\Xi\), \(F\) be a free filter of \(\mathbb{N}\), satisfying condition (1), and \(f_n : X \to R\), \(n \in \mathbb{N}\), be a sequence, \((F)\)-convergent to \(f \in R^X\) at \(x\). Then \((f_n)_n\) is \((S, F)\)-convergent to \(f\) at \(x\).

**Theorem 8** Let \((X, D)\) be any uniform space, \(\emptyset \neq X \subset B\), \(\Lambda\) and \(\Xi\) be two directed sets, \(S\) and \(F\) be a \((\Xi)\)-free filter of \(\Xi\) and a \((\Lambda)\)-free filter of \(\Lambda\) respectively, and \(f_\lambda : X \to R\), \(\lambda \in \Lambda\), be a function net, strongly \(F\)-exhaustive on \(B\). Then \((f_\lambda)_\lambda\) is strongly \((S, F)\)-exhaustive on \(B\).

Conversely, if \((D_\xi)_{\xi \in \Xi}\) is a decreasing net in \(D\), such that

\[
\text{for each } U \in D \text{ there exists } \xi \in \Xi \text{ with } D_\xi \subset U \quad (4)
\]

and \((f_\lambda)_\lambda\) is strongly \((S, F)\)-exhaustive on \(B\), then \((f_\lambda)_\lambda\) is strongly \(F\)-exhaustive on \(B\).

**Remark 9** It is easy to check that the set \(\Xi = D\), endowed with the order \(D_1 \geq D_2\) if and only if \(D_1 \subset D_2\), is a directed set, and that (4) is satisfied.

**Theorem 10** Let \(X\) be a Hausdorff topological space, \(x \in X\) be fixed, \((T_x, \subset)\) be the set of all neighborhoods of \(x\); \(\Lambda\), \(\Xi\), \(S\), \(F\) be as in Theorem 8, and \(f_\lambda : X \to R\), \(\lambda \in \Lambda\), be a net, \(F\)-exhaustive at \(x\). Then \((f_\lambda)_\lambda\) is \((S, F)\)-exhaustive at \(x\).

Conversely, if \((D_\xi)_{\xi \in \Xi}\) is a decreasing net in \(T_x\) satisfying (5), and \((f_\lambda)_\lambda\) is \((S, F)\)-exhaustive at \(x\), then \((f_\lambda)_\lambda\) is \(F\)-exhaustive at \(x\).

**Theorem 11** Let \((X, D)\) be any uniform space, \(B\) be a bornology on \(X\), \(\Xi\) and \(\Lambda\) be as
above, \( S \) and \( F \) be any two \((\Xi)\)-free and \((\Lambda)\)-free filters of \( \Xi \) and \( \Lambda \) respectively, and \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a net of functions, \((FB)\)-convergent to \( f : X \to R \). Let \( B \in B \) be fixed. Then \((f_\lambda)_\lambda \) is strongly \((S,F\alpha)\)-convergent to \( f \) on \( B \) if and only if \((f_\lambda)_\lambda \) is strongly \((S,F)\)-exhaustive on \( B \).

**Theorem 12** Let \( X \) be a Hausdorff topological space, \( x \in X \), \( S \) and \( F \) be as in Theorem 11, and \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a net of functions, \((ROF)\)-convergent to \( f : X \to R \). Then \((f_\lambda)_\lambda \) is \((S,F\alpha)\)-convergent to \( f \) at \( x \) if and only if \((f_\lambda)_\lambda \) is \((S,F)\)-exhaustive at \( x \).

**Theorem 13** Let \( X \), \( R \), \( \Lambda \), \( \Xi \), \( F \), \( S \), \( B \) be as in Theorem 11, \( B \in B \) be fixed, and \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a function net, \((ROF)\)-convergent to \( f \in R^X \).

Then \((f_\lambda)_\lambda \) is strongly weakly \((S,F)\)-exhaustive on \( B \) if and only if \( f \) is strongly \((S,S)\)-uniformly continuous on \( B \).

**Theorem 14** Let \( X \), \( R \), \( \Lambda \), \( \Xi \), \( F \), \( S \), \( B \) be as in Theorem 12, \( x \in X \) be fixed, and \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a net, \((ROF)\)-convergent to \( f \in R^X \).

Then \((f_\lambda)_\lambda \) is weakly \((S,F)\)-convergent to \( f \) at \( x \) if and only if \( f \) is \((S,S)\)-continuous at \( x \).

**Theorem 15** Let \( (X,D) \) be a uniform space, \((\Xi,\geq)\) be a directed set, \( f : X \to R \), \( \emptyset \neq B \subset X \), \( S_1 \) and \( S_2 \) be any two fixed \((\Xi)\)-free filters of \( \Xi \). Suppose that for every point \( x \in X \) there is a net \((y_\xi)_\xi \in \Xi \) in \( X \), with

\[
(S_\xi) \lim_{\xi} y_\xi = x. \tag{5}
\]

Then the following results hold.

(a) If \( S_1 \setminus S_2 \neq \emptyset \), then \( f \) is strongly \((S_1,S_2)\)-uniformly continuous on \( B \) if and only if \( f \) is constant.

(b) If \((D_\xi)_\xi \in \Xi\) is a decreasing net in \( D \), satisfying (5), and \( S_1 \subset S_2 \), then \( f \) is strongly \((S_1,S_2)\)-uniformly continuous on \( B \) if and only if \( f \) is strongly uniformly continuous on \( B \).

**Theorem 16** Let \( X \) be a Hausdorff topological space, \( f : X \to R \), \((\Xi,\geq)\) be a directed set, \( S_1 \) and \( S_2 \) be two fixed \((\Xi)\)-free filters of \( \Xi \), \( x \in X \) be such that there is a net \((y_\xi)_\xi \in \Xi \) in \( X \), fulfilling (5). Then the following results hold.

(a) If \( S_1 \setminus S_2 \neq \emptyset \), then \( f \) is \((S_1,S_2)\)-continuous at \( x \) if and only if \( f \) is constant.

(b) If \((T_x,\subset)\) is the set of all neighborhoods of \( x \), \((D_\xi)_\xi \in \Xi\) is a decreasing net in \( T_x \), fulfilling (5) and \( S_1 \subset S_2 \), then \( f \) is \((S_1,S_2)\)-continuous at \( x \) if and only if it is continuous at \( x \).

A consequence of the previous theorems is that, if \( X \) is a Hausdorff topological space (resp. a uniform space) and a net \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), is (strongly) \((F\alpha)\)-convergent to \( f : X \to R \) on \( X \) (resp. on \( B \)), then it \((ROF)\)-converges to \( f \), and \( f \) is (strongly uniform) continuous on \( X \) (resp. on \( B \)).

As an application, we give an Ascoli-type theorem. Given a topological space \( X \), a nonempty set \( \Phi \subset R^X \) and a convergence \((\sigma)\) on \( \Phi \), we say that \( \Phi \) is \((\sigma)\)-compact iff every net \((f_\lambda)_\lambda \in \Lambda \) in \( \Phi \) admits a subnet \((f_{\lambda_k})_{k \in \Lambda} \), \((\sigma)\)-convergent to an element \( f \in \Phi \), and that \( \Phi \) is \((\sigma)\)-closed iff \( f \in \Phi \) whenever \((f_{\lambda_k})_{k \in \Lambda} \) is a net in \( \Phi \), \((\sigma)\)-convergent to \( f \in R^X \). The \((\sigma)\)-closure of \( \Phi \) is the set of the functions \( f \in R^X \), having a net \((f_{\lambda_k})_{k \in \Lambda} \) in \( \Phi \) \((\sigma)\)-convergent to \( f \). A set \( \Phi \) is \((\sigma)\)-closed if and only if it coincides with its \((\sigma)\)-closure.

**Theorem 17** Let \( X \) be a Hausdorff topological space, \( S \) and \( F \) be as above.
If \( \Phi \subset \Psi \subset \mathbb{R}^k \), where \( \Phi \) is \((S, F\alpha)\)-closed and \( \Psi \) is \((ROF)\)-compact, and every net \((f_\lambda)_{\lambda \in \Lambda} \), \((ROF)\)-convergent in \(\Phi\), has a subnet \((f_{\lambda\kappa})_{\kappa \in \Lambda} \), \((ROF)\)-convergent in \(\mathbb{R}^k \) and \((S, F)\)-exhaustive,

then \( \Phi \) is \((S, F\alpha)\)-compact.

Moreover, if \( \Phi \) is \((S, F\alpha)\)-compact, then \( \Phi \) satisfies condition \( H' \).

In our setting, a related question, which arises naturally, is to find some necessary and/or sufficient conditions under which the limit function of a suitably pointwise convergent net is constant, or \((\alpha)\)-continuous. In this framework, it is advisable to consider the following extensions of the concepts of \((S, F)\)-exhaustiveness and corresponding \((\alpha)\)-convergence.

Let \( X \) be any Hausdorff topological space, \( x \in X \), \( R \) be any Dedekind complete lattice group, \((\mathfrak{Z}, \geq)\) and \((\Lambda, \geq)\) be two directed sets, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two \((\mathfrak{Z})\)-free filters of \( \mathfrak{Z} \), \( \mathcal{F}_3 \) be a \((\Lambda)\)-free filter of \( \Lambda \), and \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a function net. We say that \((f_\lambda)_{\lambda} \) is \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-exhaustive at \( x \in X \) iff there exists an \((O)\)-sequence \((\sigma_p)_{\sigma} \) in \( R \) such that, for every net \((x_\xi)_{\xi \in \mathfrak{Z}} \) \( \mathcal{F}_1 \)-convergent to \( x \) and for any \( p \in \mathbb{N} \) there are \( F_2 \in \mathcal{F}_2 \) and \( F_3 \in \mathcal{F}_3 \) with \( |f_\lambda(x_\xi) - f_\lambda(x)| \leq \sigma_p \) for every \( \xi \in F_2 \) and \( \lambda \in F_3 \). The net \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), is weakly \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-exhaustive at \( x \in X \) iff there is an \((O)\)-sequence \((\sigma_p)_{\sigma} \) in \( R \) such that, for each net \((x_\xi)_{\xi \in \mathfrak{Z}} \) \( \mathcal{F}_1 \)-convergent to \( x \) and for every \( p \in \mathbb{N} \) there is a set \( F_2 \in \mathcal{F}_2 \) such that for each \( \xi \in F_2 \) there exists \( F_3 \in \mathcal{F}_3 \) with \( |f_\lambda(x_\xi) - f_\lambda(x)| \leq \sigma_p \) whenever \( \lambda \in F_3 \). The net \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-converges to \( f : X \to R \) at \( x \in X \) iff there exists an \((O)\)-sequence \((\sigma_p)_{\sigma} \) in \( R \) such that, for every net \((x_\xi)_{\xi \in \mathfrak{Z}} \) \( \mathcal{F}_1 \)-convergent to \( x \) and for each \( p \in \mathbb{N} \) there are \( F_2 \in \mathcal{S} \), \( F_3 \in \mathcal{F} \) with \( |f_\lambda(x_\xi) - f(x)| \leq \sigma_p \) whenever \( \xi \in F_2 \) and \( \lambda \in F_3 \).

**Theorem 18** Let \( f_\lambda : X \to R \), \( \lambda \in \Lambda \), be a net of functions, \((ROF_3)\)-convergent to \( f : X \to R \), and \( x \in X \). Then \((f_\lambda)_{\lambda} \) is \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-convergent to \( f \) at \( x \) if and only if \((f_\lambda)_{\lambda} \) is \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-exhaustive at \( x \). Moreover, \((f_\lambda)_{\lambda} \) is weakly \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\)-exhaustive at \( x \) if and only if \( f \) is \((\mathcal{F}_1, \mathcal{F}_2)\)-continuous at \( x \).