Possibility neutrosophic soft sets with applications in decision making and similarity measure

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Abstract

In this paper, concept of possibility neutrosophic soft set and its operations are defined, and their properties are studied. An application of this theory in decision making is investigated. Also a similarity measure of two possibility neutrosophic soft sets is introduced and discussed. Finally an application of this similarity measure in personal selection for a firm.

Keywords: Soft set, neutrosophic soft set, possibility neutrosophic soft set, similarity measure, decision making

1. Introduction

In this physical world problems in engineering, medical sciences, economics and social sciences the information involved are uncertainty in nature. To cope with these problems, researchers proposed some theories such as the theory of fuzzy set [35], the theory of intuitionistic fuzzy set [4], the theory of rough set [26], the theory of vague set [18]. In 1999, Molodtsov [20] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties as different from these theories. A wide range of applications of soft sets have been developed in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Maji et al. [21, 22] applied soft set theory to decision making problem and in 2003, they introduced some new operations of soft sets. After Maji’s work,
works on soft set theory and its applications have been progressing rapidly. see [1, 2, 9, 10, 11, 15, 16, 24, 25, 29, 33, 34, 36].

Neutrosophy has been introduced by Smarandache [30, 31] as a new branch of philosophy and generalization of fuzzy logic, intuitionistic fuzzy logic, paraconsistent logic. Fuzzy sets and intuitionistic fuzzy sets are characterized by membership functions, membership and non-membership functions, respectively. In some real life problems for proper description of an object in uncertain and ambiguous environment, we need to handle the indeterminate and incomplete information. But fuzzy sets and intuitionistic fuzzy sets don’t handle the indeterminant and inconsistent information. Thus neutrosophic set is defined by Samarandache [31], as a new mathematical tool for dealing with problems involving incomplete, indeterminacy, inconsistent knowledge.

Maji [23] introduced concept of neutrosophic soft set and some operations of neutrosophic soft sets. Karaaslan [19] redefined concept and operations of neutrosophic soft sets as different from Maji’s neutrosophic soft set definition and operations. Recently, the properties and applications on the neutrosophic soft sets have been studied increasingly [6, 7, 8, 12, 13, 28].

Alkhazaleh et al [3] were firstly introduced concept of possibility fuzzy soft set and its operations. They gave applications of this theory in solving a decision making problem and they also introduced a similarity measure of two possibility fuzzy soft sets and discussed their application in a medical diagnosis problem. In 2012, Bashir et al. [5] introduced concept of possibility intuitionistic fuzzy soft set and its operations and discussed similarity measure of two possibility intuitionistic fuzzy sets. They also gave an application of this similarity measure.

This paper is organized as follow: in Section 2, some basic definitions and operations are given regarding neutrosophic soft set required in this paper. In Section 3, possibility neutrosophic soft set is defined as a generalization of possibility fuzzy soft set and possibility intuitionistic fuzzy soft set introduced by Alkhazaleh [3] and Bashir [5], respectively. In Section 4, a decision making method is given using the possibility neutrosophic soft sets. In Section 5, similarity measure is introduced between two possibility neutrosophic soft sets and in Section 6, an application related personal selection for a firm is given as regarding this similarity measure method.
2. Preliminary

In this paper, we recall some definitions, operation and their properties related to the neutrosophic soft set [19, 23] required in this paper.

**Definition 1.** A neutrosophic soft set (or namely ns-set) $f$ over $U$ is a neutrosophic set valued function from $E$ to $\mathcal{N}(U)$. It can be written as

$$f = \left\{ (e, \langle u, t_{f(e)}(u), i_{f(e)}(u), f_{f(e)}(u) \rangle : u \in U) : e \in E \right\}$$

where, $\mathcal{N}(U)$ denotes set of all neutrosophic sets over $U$. Note that if $f(e) = \{ \langle u, 0, 1, 1 \rangle : u \in U \}$, the element $(e, f(e))$ is not appeared in the neutrosophic soft set $f$. Set of all ns-sets over $U$ is denoted by $\mathbb{NS}$.

**Definition 2.** [19] Let $f, g \in \mathbb{NS}$. $f$ is said to be neutrosophic soft subset of $g$, if $t_{f(e)}(u) \leq t_{g(e)}(u)$, $i_{f(e)}(u) \geq i_{g(e)}(u)$, $f_{f(e)}(u) \geq f_{g(e)}(u)$, $\forall e \in E$, $\forall u \in U$. We denote it by $f \subseteq g$. $f$ is said to be neutrosophic soft super set of $g$ if $g$ is a neutrosophic soft subset of $f$. We denote it by $f \supseteq g$.

If $f$ is neutrosophic soft subset of $g$ and $g$ is neutrosophic soft subset of $f$. We denote it $f = g$.

**Definition 3.** [19] Let $f \in \mathbb{NS}$. If $t_{f(e)}(u) = 0$ and $i_{f(e)}(u) = f_{f(e)}(u) = 1$ for all $e \in E$ and for all $u \in U$, then $f$ is called null ns-set and denoted by $\emptyset$.

**Definition 4.** [19] Let $f \in \mathbb{NS}$. If $t_{f(e)}(u) = 1$ and $i_{f(e)}(u) = f_{f(e)}(u) = 0$ for all $e \in E$ and for all $u \in U$, then $f$ is called universal ns-set and denoted by $\bar{U}$.

**Definition 5.** [19] Let $f, g \in \mathbb{NS}$. Then union and intersection of ns-sets $f$ and $g$ denoted by $f \sqcup g$ and $f \sqcap g$ respectively, are defined by as follow

$$f \sqcup g = \left\{ (e, \langle u, t_{f(e)}(u) \lor t_{g(e)}(u), i_{f(e)}(u) \land i_{g(e)}(u), f_{f(e)}(u) \land f_{g(e)}(u) \rangle : u \in U) : e \in E \right\},$$

and ns-intersection of $f$ and $g$ is defined as

$$f \sqcap g = \left\{ (e, \langle u, t_{f(e)}(u) \land t_{g(e)}(u), i_{f(e)}(u) \lor i_{g(e)}(u), f_{f(e)}(u) \lor f_{g(e)}(u) \rangle : u \in U) : e \in E \right\}.$$
Definition 6. [19] Let \( f, g \in \text{NS} \). Then complement of ns-set \( f \), denoted by \( f^\check{\text{c}} \), is defined as follow
\[
f^\check{\text{c}} = \left\{ (e, \{ (u, f_{f(e)}(u), 1 - i_{f(e)}(u), t_{f(e)}(u)) : u \in U \}) : e \in E \right\}.
\]

Proposition 1. [19] Let \( f, g, h \in \text{NS} \). Then,
\[
i. \bar{\check{\Phi}} \subseteq f
\]
\[
ii. f \subseteq \check{U}
\]
\[
iii. f \subseteq f
\]
\[
iv. f \subseteq g \text{ and } g \subseteq h \Rightarrow f \subseteq h
\]

Proposition 2. [19] Let \( f \in \text{NS} \). Then
\[
i. \bar{\check{\Phi}} = \check{U}
\]
\[
ii. \check{U}^\check{\text{c}} = \bar{\check{\Phi}}
\]
\[
iii. (f^\check{\text{c}})^\check{\text{c}} = f.
\]

Proposition 3. [19] Let \( f, g, h \in \text{NS} \). Then,
\[
i. f \cap f = f \text{ and } f \cup f = f
\]
\[
ii. f \cap g = g \cap f \text{ and } f \cup g = g \cup f
\]
\[
iii. f \cap \bar{\check{\Phi}} = \bar{\check{\Phi}} \text{ and } f \cap \bar{\check{U}} = f
\]
\[
iv. f \cup \bar{\check{\Phi}} = f \text{ and } f \cup \bar{\check{U}} = \bar{\check{U}}
\]
\[
v. f \cap (g \cap h) = (f \cap g) \cap h \text{ and } f \cup (g \cup h) = (f \cup g) \cup h
\]
\[
vi. f \cap (g \cup h) = (f \cap g) \cup (f \cap h) \text{ and } f \cup (g \cap h) = (f \cup g) \cap (f \cup h).
\]

Proof. The proof is clear from definition and operations of neutrosophic soft sets.

Theorem 1. [19] Let \( f, g \in \text{NS} \). Then, De Morgan’s law is valid.
\[
i. (f \cup g)^\check{\text{c}} = f^\check{\text{c}} \cap g^\check{\text{c}}
\]
\[ (f \sqcup g)^\circ = f^\circ \cap g^\circ \]

**Definition 7.** [10] Let \( f, g \in \text{NS} \). Then 'OR' product of ns-sets \( f \) and \( g \) denoted by \( f \sqcup g \), is defined as follow

\[
f \sqcup g = \{ ((e, e'), \{ (u, t_f(e)(u)) \land t_g(e)(u), i_f(e)(u) \lor i_g(e)(u), \}
\]

\[
f_f(e)(u) \land f_g(e)(u) : u \in U \} : (e, e') \in E \times E \}.
\]

**Definition 8.** [10] Let \( f, g \in \text{NS} \). Then 'AND' product of ns-sets \( f \) and \( g \) denoted by \( f \sqcap g \), is defined as follow

\[
f \sqcap g = \{ ((e, e'), \{ (u, t_f(e)(u)) \lor t_g(e)(u), i_f(e)(u) \land i_g(e)(u), \}
\]

\[
f_f(e)(u) \lor f_g(e)(u) : u \in U \} : (e, e') \in E \times E \}.
\]

**Proposition 4.** [10] Let \( f, g \in \text{NS} \). Then,

1. \((f \sqcup g)^\circ = f^\circ \sqcap g^\circ\)
2. \((f \sqcap g)^\circ = f^\circ \sqcup g^\circ\)

**PROOF.** The proof is clear from Definition 7 and 8.

**Definition 9.** [3] Let \( U = \{ u_1, u_2, ..., u_n \} \) be the universal set of elements and \( E = \{ e_1, e_2, ..., e_m \} \) be the universal set of parameters. The pair \((U, E)\) will be called a soft universe. Let \( F : E \rightarrow I^U \) and \( \mu \) be a fuzzy subset of \( E \), that is \( \mu : E \rightarrow I^U \), where \( I^U \) is the collection of all fuzzy subsets of \( U \). Let \( F_\mu : E \rightarrow I^U \times I^U \) be a function defined as follows:

\[
F_\mu(e) = (F(e)(u), \mu(e)(u)), \forall u \in U.
\]

Then \( F_\mu \) is called a possibility fuzzy soft set (PFSS in short) over the soft universe \((U, E)\). For each parameter \( e_i \), \( F_\mu(e_i) = (F(e_i)(u), \mu(e_i)(u)) \) indicates not only the degree of belongingness of the elements of \( U \) in \( F(e_i) \), but also the degree of possibility of belongingness of the elements of \( U \) in \( F(e_i) \), which is represented by \( \mu(e_i) \).

**Definition 10.** [3] Let \( U = \{ u_1, u_2, ..., u_n \} \) be the universal set of elements and \( E = \{ e_1, e_2, ..., e_m \} \) be the universal set of parameters. The pair \((U, E)\) will be called a soft universe. Let \( F : E \rightarrow (I \times I)^U \times I^U \) where \((I \times I)^U \) is
the collection of all intuitionistic fuzzy subsets of $U$ and $I^U$ is the collection of all intuitionistic fuzzy subsets of $U$. Let $p$ be a fuzzy subset of $E$, that is, $p : E \to I^U$ and let $F_p : E \to (I \times I)^U \times I^U$ be a function defined as follows:

$$F_p(e) = (F(e)(u), p(e)(u)), F(e)(u) = (\mu(u), \nu(u)) \forall u \in U.$$  

Then $F_p$ is called a possibility intuitionistic fuzzy soft set (PIFSS in short) over the soft universe $(U, E)$. For each parameter $e_i$, $F_p(e_i) = (F(e_i)(u), p(e_i)(u))$ indicates not only the degree of belongingness of the elements of $U$ in $F(e_i)$, but also the degree of possibility of belongingness of the elements of $U$ in $F(e_i)$, which is represented by $p(e_i)$.

3. Possibility neutrosophic soft sets

In this section, we introduced the concept of possibility neutrosophic soft set, possibility neutrosophic soft subset, possibility neutrosophic soft null set, possibility neutrosophic soft universal set and possibility neutrosophic soft set operations such as union, intersection and complement.

Throughout paper $U$ is an initial universe, $E$ is a set of parameters and $\Lambda$ is an index set.

**Definition 11.** Let $U$ be an initial universe, $E$ be a parameter set, $\mathcal{N}(U)$ be the collection of all neutrosophic sets of $U$ and $I^U$ is collection of all fuzzy subset of $U$. A possibility neutrosophic soft set (PNS-set) $f_{\mu}$ over $U$ is defined by the set of ordered pairs

$$f_{\mu} = \{ (e_i, \{(\frac{u_j}{f(e_i)(u_j)}, \mu(e_i)(u_j)) : u_j \in U\}) : e_i \in E \}$$

where, $i, j \in \Lambda$, $f$ is a mapping given by $f : E \to \mathcal{N}(U)$ and $\mu(e_i)$ is a fuzzy set such that $\mu : E \to I^U$. Here, $f_{\mu}$ is a mapping defined by $f_{\mu} : E \to \mathcal{N}(U) \times I^U$.

For each parameter $e_i \in E$, $f(e_i) = \{(u_j, t_{f(e_i)}(u_j), i_{f(e_i)}(u_j), f_{f(e_i)}(u_j)) : u_j \in U\}$ indicates neutrosophic value set of parameter $e_i$ and where $t, i, f : U \to [0, 1]$ are the membership functions of truth, indeterminacy and falsity respectively of the element $u_j \in U$. For each $u_j \in U$ and $e_i \in E$, $0 \leq t_{f(e_i)}(u_j) + i_{f(e_i)}(u_j) + f_{f(e_i)}(u_j) \leq 3$. Also $\mu(e_i)$, degrees of possibility of belongingness of elements of $U$ in $f(e_i)$. So we can write

$$f_{\mu}(e_i) = \{ (\frac{u_1}{f(e_i)(u_1)}, \mu(e_i)(u_1)), (\frac{u_2}{f(e_i)(u_2)}, \mu(e_i)(u_2)), \ldots, (\frac{u_n}{f(e_i)(u_n)}, \mu(e_i)(u_n)) \}$$
From now on, we will show set of all possibility neutrosophic soft sets over $U$ with $\mathcal{PS}(U, E)$ such that $E$ is parameter set.

**Example 1.** Let $U = \{u_1, u_2, u_3\}$ be a set of three cars. Let $E = \{e_1, e_2, e_3\}$ be a set of qualities where $e_1 = \text{cheap}$, $e_2 = \text{equipment}$, $e_3 = \text{fuel consumption}$ and let $\mu : E \rightarrow I^U$. We define a function $f_\mu : E \rightarrow \mathcal{N}(U) \times I^U$ as follows:

$$f_\mu = \begin{cases} f_\mu(e_1) = \left\{ \left( \frac{u_1}{(0.5,0.2,0.6), (0.7,0.3,0.5)}, 0.8 \right), \left( \frac{u_2}{(0.4,0.5,0.8)}, 0.7 \right) \right\} \\ f_\mu(e_2) = \left\{ \left( \frac{u_1}{(0.8,0.4,0.5)}, 0.6 \right), \left( \frac{u_2}{(0.5,0.7,0.2)}, 0.8 \right), \left( \frac{u_3}{(0.7,0.3,0.9)}, 0.4 \right) \right\} \\ f_\mu(e_3) = \left\{ \left( \frac{u_1}{(0.6,0.7,0.5)}, 0.2 \right), \left( \frac{u_2}{(0.5,0.3,0.7)}, 0.6 \right), \left( \frac{u_3}{(0.6,0.5,0.4)}, 0.5 \right) \right\} \end{cases}$$

also we can define a function $g_\nu : E \rightarrow \mathcal{N}(U) \times I^U$ as follows:

$$g_\nu = \begin{cases} g_\nu(e_1) = \left\{ \left( \frac{u_1}{(0.6,0.3,0.8)}, 0.4 \right), \left( \frac{u_2}{(0.6,0.5,0.9)}, 0.7 \right), \left( \frac{u_3}{(0.2,0.6,0.4)}, 0.8 \right) \right\} \\ g_\nu(e_2) = \left\{ \left( \frac{u_1}{(0.5,0.4,0.3)}, 0.3 \right), \left( \frac{u_2}{(0.4,0.6,0.5)}, 0.6 \right), \left( \frac{u_3}{(0.7,0.2,0.5)}, 0.8 \right) \right\} \\ g_\nu(e_3) = \left\{ \left( \frac{u_1}{(0.7,0.5,0.3)}, 0.8 \right), \left( \frac{u_2}{(0.4,0.4,0.6)}, 0.5 \right), \left( \frac{u_3}{(0.8,0.5,0.3)}, 0.6 \right) \right\} \end{cases}$$

For the purpose of storing a possibility neutrosophic soft set in a computer, we can use matrix notation of possibility neutrosophic soft set $f_\mu$. For example, matrix notation of possibility neutrosophic soft set $f_\mu$ can be written as follows: for $m, n \in \Lambda$,

$$f_\mu = \begin{pmatrix} ((0.5,0.2,0.6),0.8) & ((0.7,0.3,0.5),0.4) & ((0.4,0.5,0.8),0.7) \\ ((0.8,0.4,0.5),0.6) & ((0.5,0.7,0.2),0.8) & ((0.7,0.3,0.9),0.4) \\ ((0.6,0.7,0.5),0.2) & ((0.5,0.3,0.7),0.6) & ((0.6,0.5,0.4),0.5) \end{pmatrix}$$

where the $m$–th row vector shows $f(e_m)$ and $n$–th column vector shows $u_n$.

**Definition 12.** Let $f_\mu, g_\nu \in \mathcal{PS}(U, E)$. Then, $f_\mu$ is said to be a possibility neutrosophic soft subset ($PNS$-subset) of $g_\nu$, and denoted by $f_\mu \subseteq g_\nu$, if

i. $\mu(e)$ is a fuzzy subset of $\nu(e)$, for all $e \in E$

ii. $f$ is a neutrosophic subset of $g$,

**Example 2.** Let $U = \{u_1, u_2, u_3\}$ be a set of tree houses, and let $E = \{e_1, e_2, e_3\}$ be a set of parameters where $e_1 = \text{modern}$, $e_2 = \text{big}$ and $e_3 = \text{cheap}$. Let $f_\mu$ be a $PNS$-set defined as follows:
\[ f_\mu = \begin{cases} f_\mu(e_1) = \left\{ \left( \frac{u_1}{0.5,0.2,0.6}, 0.8 \right), \left( \frac{u_2}{0.7,0.3,0.5}, 0.4 \right), \left( \frac{u_3}{0.4,0.5,0.9}, 0.7 \right) \right\} \\
    f_\mu(e_2) = \left\{ \left( \frac{u_1}{0.8,0.4,0.5}, 0.6 \right), \left( \frac{u_2}{0.5,0.7,0.2}, 0.8 \right), \left( \frac{u_3}{0.7,0.3,0.9}, 0.4 \right) \right\} \\
    f_\mu(e_3) = \left\{ \left( \frac{u_1}{0.6,0.7,0.5}, 0.2 \right), \left( \frac{u_2}{0.5,0.3,0.8}, 0.6 \right), \left( \frac{u_3}{0.6,0.5,0.4}, 0.5 \right) \right\} \end{cases} \]

\[ g_\nu : E \to \mathcal{N}(U) \times I^U \text{ be another PNS-set defined as follows:} \]

\[ g_\nu = \begin{cases} g_\nu(e_1) = \left\{ \left( \frac{u_1}{0.6,0,1,0.5}, 0.9 \right), \left( \frac{u_2}{0.8,0,2,0.3}, 0.6 \right), \left( \frac{u_3}{0.7,0,5,0.8}, 0.8 \right) \right\} \\
    g_\nu(e_2) = \left\{ \left( \frac{u_1}{0.9,0,2,0.4}, 0.7 \right), \left( \frac{u_2}{0.9,0,5,0.1}, 0.9 \right), \left( \frac{u_3}{0.8,0,1,0.9}, 0.5 \right) \right\} \\
    g_\nu(e_3) = \left\{ \left( \frac{u_1}{0.6,0,5,0.4}, 0.4 \right), \left( \frac{u_2}{0.7,0,1,0.7}, 0.9 \right), \left( \frac{u_3}{0.8,0,2,0.4}, 0.7 \right) \right\} \end{cases} \]

it is clear that \( f_\mu \) is PNS - subset of \( g_\nu \)

**Definition 13.** Let \( f_\mu, g_\nu \in \mathcal{PN}(U, E) \). Then, \( f_\mu \) and \( g_\nu \) are called possibility neutrosophic soft equal set and denoted by \( f_\mu = g_\nu \), if \( f_\mu \subseteq g_\nu \) and \( f_\mu \supseteq g_\nu \).

**Definition 14.** Let \( f_\mu \in \mathcal{PN}(U, E) \). Then, \( f_\mu \) is said to be possibility neutrosophic soft null set denoted by \( \phi_\mu \), if \( \forall e \in E, \phi_\mu : E \to \mathcal{N}(U) \times I^U \) such that \( \phi_\mu(e) = \{(u, \mu_e(u)) : u \in U\} \), where \( \phi(e) = \{(u, 0, 1, 1) : u \in U\} \) and \( \mu(e) = \{(u, 0) : u \in U\} \).

**Definition 15.** Let \( f_\mu \in \mathcal{PN}(U, E) \). Then, \( f_\mu \) is said to be possibility neutrosophic soft universal set denoted by \( U_\mu \), if \( \forall e \in E, U_\mu : E \to \mathcal{N}(U) \times I^U \) such that \( U_\mu(e) = \{(u, \mu_e(u)) : u \in U\} \), where \( U(e) = \{(u, 1, 0, 0) : u \in U\} \) and \( \mu(e) = \{(u, 1) : u \in U\} \).

**Proposition 5.** Let \( f_\mu, g_\nu \) and \( h_\delta \in \mathcal{PN}(U, E) \). Then,

\[ i. \phi_\mu \subseteq f_\mu \]
\[ ii. f_\mu \subseteq U_\mu \]
\[ iii. f_\mu \subseteq f_\mu \]
\[ iv. f_\mu \subseteq g_\nu \text{ and } g_\nu \subseteq h_\delta \Rightarrow f_\mu \subseteq h_\delta \]

**Proof.** It is clear from Definition 13, 14 and 15.

**Definition 16.** Let \( f_\mu \in \mathcal{PN}(U) \), where \( f_\mu(e_i) = \{(f(e_i)(u_j), \mu_e(u_j)) : e_i \in E, u_j \in U\} \) and \( f(e_i) = \{(u, f(e_i)(u_j)) : e_i \in E, u_j \in U\} \) for all \( e_i \in E, u_j \in U \). Then for \( e_i \in E \) and \( u_j \in U \),
We can write a possibility neutrosophic soft set in form:

\[
t_{\mu} = \{(f^t_{ij}(e_i), \mu_{ij}(e_i))\} \text{ and } f^i_{ij}(e_i) = \{(u_j, t_{f(e_i)}(u_j))\}, \mu_{ij}(e_i) = \{(u_j, \mu(e_i)(u_j))\}
\]

Definition 17. [27] A binary operation \(\otimes: [0, 1] \times [0, 1] \rightarrow [0, 1]\) is continuous \(t\)-norm if \(\otimes\) satisfies the following conditions:

i. \(\otimes\) is commutative and associative,

ii. \(\otimes\) is continuous,

iii. \(a \otimes 1 = a, \forall a \in [0, 1]\),

iv. \(a \otimes b \leq c \otimes d\) whenever \(a \leq c, b \leq d\) and \(a, b, c, d \in [0, 1]\).

Definition 18. [27] A binary operation \(\oplus: [0, 1] \times [0, 1] \rightarrow [0, 1]\) is continuous \(t\)-conorm (s-norm) if \(\oplus\) satisfies the following conditions:

i. \(\oplus\) is commutative and associative,

ii. \(\oplus\) is continuous,
iii. \( a \oplus 0 = a, \forall a \in [0, 1], \)

iv. \( a \oplus b \leq c \oplus d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0, 1]. \)

**Definition 19.** Let \( I^3 = [0, 1] \times [0, 1] \times [0, 1] \) and \( N(I^3) = \{ (a, b, c) : a, b, c \in [0, 1] \}. \) Then \((N(I^3), \oplus, \otimes)\) be a lattices together with partial ordered relation \( \leq, \) where order relation \( \leq \) on \( N(I^3) \) can be defined by for \((a, b, c), (d, e, f) \in N(I^3)\)

\[
(a, b, c) \leq (e, f, g) \iff a \leq e, b \geq f, c \geq g
\]

**Definition 20.** A binary operation

\[
\tilde{\otimes} : \left( [0, 1] \times [0, 1] \times [0, 1] \right)^2 \rightarrow [0, 1] \times [0, 1] \times [0, 1]
\]

is continuous \( n-\)norm if \( \tilde{\otimes} \) satisfies the following conditions

i. \( \tilde{\otimes} \) is commutative and associative,

ii. \( \tilde{\otimes} \) is continuous,

iii. \( a \tilde{\otimes} 0 = 0, a \tilde{\otimes} 1 = a, \forall a \in [0, 1] \times [0, 1] \times [0, 1], \) \((1 = (1, 0, 0)) \) and \((0 = (0, 1, 1))\)

iv. \( a \tilde{\otimes} b \leq c \tilde{\otimes} d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0, 1] \times [0, 1] \times [0, 1]. \)

Here,

\[
a \tilde{\otimes} b = \tilde{\otimes}(\langle t(a), i(a), f(a) \rangle, \langle t(b), i(b), f(b) \rangle) = \langle t(a) \tilde{\otimes} t(b), i(a) \oplus i(b), f(a) \oplus f(b) \rangle
\]

**Definition 21.** A binary operation

\[
\tilde{\oplus} : \left( [0, 1] \times [0, 1] \times [0, 1] \right)^2 \rightarrow [0, 1] \times [0, 1] \times [0, 1]
\]

is continuous \( n-\)conorm if \( \tilde{\oplus} \) satisfies the following conditions

i. \( \tilde{\oplus} \) is commutative and associative,

ii. \( \tilde{\oplus} \) is continuous,

iii. \( a \tilde{\oplus} 0 = a, a \tilde{\oplus} 1 = 1, \forall a \in [0, 1] \times [0, 1] \times [0, 1], \) \((1 = (1, 0, 0)) \) and \((0 = (0, 1, 1))\)
iv. \( a \oplus b \leq c \oplus d \) whenever \( a \leq c, b \leq d \) and \( a, b, c, d \in [0, 1] \times [0, 1] \times [0, 1] \).

Here,

\[
a \oplus b = \oplus((t(a), i(a), f(a)), (t(b), i(b), f(b))) = (t(a) \oplus t(b), i(a) \otimes i(b), f(a) \otimes f(b))
\]

**Definition 22.** Let \( f_\mu, g_\nu \in \mathcal{PN}(U, E) \). The union of two possibility neutrosophic soft sets \( f_\mu \) and \( g_\nu \) over \( U \), denoted by \( f_\mu \cup g_\nu \), is defined by as follow:

\[
f_\mu \cup g_\nu = \left\{ e_i : \left( \frac{u_j}{(f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i), f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i)), f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i))}, \mu_{ij}(e_i) \otimes \nu_{ij}(e_i) \right) : u_j \in U \right\} : e_i \in E
\]

**Definition 23.** Let \( f_\mu, g_\nu \in \mathcal{PN}(U, E) \). The intersection of two possibility neutrosophic soft sets \( f_\mu \) and \( g_\nu \) over \( U \), denoted by \( f_\mu \cap g_\nu \), is defined by as follow:

\[
f_\mu \cap g_\nu = \left\{ e_i : \left( \frac{u_j}{(f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i), f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i)), f_{ij}^\mu(e_i) \oplus g_{ij}^\nu(e_i))}, \mu_{ij}(e_i) \otimes \nu_{ij}(e_i) \right) : u_j \in U \right\} : e_i \in E
\]

**Example 3.** Let us consider the possibility neutrosophic soft sets \( f_\mu \) and \( g_\nu \) defined as in Example 7. Let us suppose that \( t \)-norm is defined by \( a \otimes b = \min \{a, b\} \) and the \( t \)-conorm is defined by \( a \oplus b = \max \{a, b\} \) for \( a, b \in [0, 1] \). Then,

\[
f_\mu \cup g_\nu = \begin{cases} (f_\mu \cup g_\nu)(e_1) = \begin{pmatrix} \frac{u_1}{(0.6,0.2,0.6)} \end{pmatrix}, (0.8), \begin{pmatrix} \frac{u_2}{(0.7,0.3,0.5)} \end{pmatrix}, (0.7), \begin{pmatrix} \frac{u_3}{(0.4,0.5,0.4)} \end{pmatrix}, (0.8) \\ (f_\mu \cup g_\nu)(e_2) = \begin{pmatrix} \frac{u_1}{(0.8,0.4,0.3)} \end{pmatrix}, (0.6), \begin{pmatrix} \frac{u_2}{(0.5,0.6,0.2)} \end{pmatrix}, (0.8), \begin{pmatrix} \frac{u_3}{(0.7,0.2,0.9)} \end{pmatrix}, (0.8) \\ (f_\mu \cup g_\nu)(e_3) = \begin{pmatrix} \frac{u_1}{(0.7,0.3,0.3)} \end{pmatrix}, (0.8), \begin{pmatrix} \frac{u_2}{(0.5,0.3,0.6)} \end{pmatrix}, (0.6), \begin{pmatrix} \frac{u_3}{(0.8,0.5,0.3)} \end{pmatrix}, (0.6) \end{cases}
\]

and

\[
f_\mu \cap g_\nu = \begin{cases} (f_\mu \cap g_\nu)(e_1) = \begin{pmatrix} \frac{u_1}{(0.5,0.3,0.8)} \end{pmatrix}, (0.4), \begin{pmatrix} \frac{u_2}{(0.6,0.5,0.5)} \end{pmatrix}, (0.4), \begin{pmatrix} \frac{u_3}{(0.2,0.6,0.8)} \end{pmatrix}, (0.7) \\ (f_\mu \cap g_\nu)(e_2) = \begin{pmatrix} \frac{u_1}{(0.9,0.4,0.5)} \end{pmatrix}, (0.3), \begin{pmatrix} \frac{u_2}{(0.4,0.7,0.5)} \end{pmatrix}, (0.6), \begin{pmatrix} \frac{u_3}{(0.7,0.3,0.9)} \end{pmatrix}, (0.4) \\ (f_\mu \cap g_\nu)(e_3) = \begin{pmatrix} \frac{u_1}{(0.6,0.7,0.5)} \end{pmatrix}, (0.2), \begin{pmatrix} \frac{u_2}{(0.4,0.5,0.7)} \end{pmatrix}, (0.5), \begin{pmatrix} \frac{u_3}{(0.6,0.5,0.4)} \end{pmatrix}, (0.5) \end{cases}
\]

**Proposition 6.** Let \( f_\mu, g_\nu, h_\delta \in \mathcal{PN}(U, E) \). Then,

i. \( f_\mu \cap f_\mu = f_\mu \) and \( f_\mu \cup f_\mu = f_\mu \)

ii. \( f_\mu \cap g_\nu = g_\nu \cap f_\mu \) and \( f_\mu \cup g_\nu = g_\nu \cup f_\mu \)

iii. \( f_\mu \cap \phi_\mu = \phi_\mu \) and \( f_\mu \cap U_\mu = f_\mu \)

iv. \( f_\mu \cap \phi = f_\mu \) and \( f_\mu \cup U_\mu = U_\mu \)
v. \( f_\mu \cap (g_\nu \cap h_\delta) = (f_\mu \cap g_\nu) \cap h_\delta \) and \( f_\mu \cup (g_\nu \cup h_\delta) = (f_\mu \cup g_\nu) \cup h_\delta \)

vi. \( f_\mu \cap (g_\nu \cup h_\delta) = (f_\mu \cap g_\nu) \cup (f_\mu \cap h_\delta) \) and \( f_\mu \cup (g_\nu \cap h_\delta) = (f_\mu \cup g_\nu) \cap (f_\mu \cup h_\delta) \).

**Proof.** The proof can be obtained from Definitions 22 and 23.

**Definition 24.** [17, 32] A function \( N : [0, 1] \to [0, 1] \) is called a negation if \( N(0) = 1, N(1) = 0 \) and \( N \) is non-increasing \((x \leq y \Rightarrow N(x) \geq N(y))\). A negation is called a strict negation if it is strictly decreasing \((x < y \Rightarrow N(x) > N(y))\) and continuous. A strict negation is said to be a strong negation if it is also involutive, i.e. \( N(N(x)) = x \).

**Definition 25.** [30] A function \( n_N : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1] \times [0, 1] \) is called a negation if \( n_N(0) = \top, n_N(1) = \bot \) and \( n_N \) is non-increasing \((x \preceq y \Rightarrow n_N(x) \geq n_N(y))\). A negation is called a strict negation if it is strictly decreasing \((x < y \Rightarrow N(x) > N(y))\) and continuous.

**Definition 26.** Let \( f_\mu \in \mathcal{PN}(U, E) \). Complement of possibility neutrosophic soft set \( f_\mu \), denoted by \( f_\mu^c \), is defined as follows:

\[
f_\mu^c = \left\{ \left( e, \left( \frac{u_j}{N(x)} \right), N(x) \right) : u_j \in U \right\} : e \in E
\]

where \( (n_N(f_{ij}(e_i))) = (N(f_{ij}^c(e_i)), N(f_{ij}^c(e_i)), N(f_{ij}^c(e_i))) \) for all \( i, j \in \Lambda \).

**Example 4.** Let us consider the possibility neutrosophic soft set \( f_\mu \) defined in Example 7. Suppose that the negation is defined by \( N(f_{ij}^c(e_i)) = f_{ij}^c(e_i), N(f_{ij}^c(e_i)) = f_{ij}^c(e_i), N(f_{ij}^c(e_i)) = 1 - f_{ij}^c(e_i) \) and \( N(\mu_{ij}(e_i)) = 1 - \mu_{ij}(e_i) \), respectively. Then, \( f_\mu^c \) is defined as follows:

\[
f_\mu^c = \begin{cases} f_\mu^c(e_1) = \left\{ \left( \frac{u_1}{(0.6,0.8,0.5)}, 0.2 \right), \left( \frac{u_2}{(0.5,0.7,0.7)}, 0.6 \right), \left( \frac{u_3}{(0.8,0.5,0.4)}, 0.3 \right) \right\} \\
 f_\mu^c(e_2) = \left\{ \left( \frac{u_1}{(0.5,0.6,0.8)}, 0.4 \right), \left( \frac{u_2}{(0.2,0.3,0.5)}, 0.2 \right), \left( \frac{u_3}{(0.9,0.7,0.7)}, 0.6 \right) \right\} \\
 f_\mu^c(e_3) = \left\{ \left( \frac{u_1}{(0.5,0.3,0.6)}, 0.8 \right), \left( \frac{u_2}{(0.7,0.7,0.5)}, 0.4 \right), \left( \frac{u_3}{(0.4,0.5,0.6)}, 0.5 \right) \right\}
\end{cases}
\]

**Proposition 7.** Let \( f_\mu \in \mathcal{PN}(U, E) \). Then,

i. \( \phi_\mu^c = U_\mu \)

ii. \( U_\mu^c = \phi_\mu \)

iii. \( (f_\mu^c)^c = f_\mu \).
Proof. It is clear from Definition \[26\]

Proposition 8. Let \(f_{\mu}, g_{\nu} \in \mathcal{P}\mathcal{N}(U, E)\). Then, De Morgan's law is valid.

\(i.\) \((f_{\mu} \cup g_{\nu})^c = f_{\mu}^c \cap g_{\nu}^c\)

\(ii.\) \((f_{\mu} \cap g_{\nu})^c = f_{\mu}^c \cup g_{\nu}^c\)

Proof. \(i.\) Let \(i, j \in \Lambda\)

\[
(f_{\mu} \cup g_{\nu})^c = \left\{ \begin{array}{l}
\{ (\mu_{ij}(e_i) \cup \nu_{ij}(e_i)) : u_j \in U \} : e_i \in E \end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\{ (\mu_{ij}(e_i) \cup \nu_{ij}(e_i)) : u_j \in U \} : e_i \in E \end{array} \right\}
\]

\[
\cap \left\{ \begin{array}{l}
\{ (\mu_{ij}(e_i) \cup \nu_{ij}(e_i)) : u_j \in U \} : e_i \in E \end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\{ (\mu_{ij}(e_i) \cup \nu_{ij}(e_i)) : u_j \in U \} : e_i \in E \end{array} \right\}
\]

\(\cap\) \(f_{\mu}^c \cap g_{\nu}^c\)

\(ii.\) The proof can be made with similar way.

Definition 27. Let \(f_{\mu} \in \mathcal{P}\mathcal{N}(U, E)\). Then 'AND' product of PNS-set \(f_{\mu}\) and \(g_{\nu}\), denoted by \(f_{\mu} \land g_{\nu}\), is defined as follow:

\[
f_{\mu} \land g_{\nu} = \left\{ (e_k, e_l) : (f_{k_j}(e_k) \land g_{ij}(e_l), f_{k_j}(e_k) \lor g_{ij}(e_l)) \right\}
\]

\(\mu_{k_j}(e_k) \land \nu_{ij}(e_l)\) : \((e_k, e_l) \in E \times E, j, k, l \in \Lambda\)
Definition 28. Let \( f_\mu \) and \( g_\nu \in \mathcal{PN}(U, E) \). Then 'OR' product of PNS-set \( f_\mu \) and \( g_\nu \) denoted by \( f_\mu \lor g_\nu \), is defined as follow:

\[
f_\mu \lor g_\nu = \left\{ (e_k, e_l), (f^i_{kj}(e_k) \lor g^i_{lj}(e_l), f^f_{kj}(e_k) \land g^f_{lj}(e_l)) + \mu_{kj}(e_k) \lor \nu_{lj}(e_l)) : (e_k, e_l) \in E \times E, j, k, l \in \Lambda \right\}
\]

4. Decision making method

In this section we will construct a decision making method over the possibility neutrosophic soft set. Firstly, we will define some notions that necessary to construct algorithm of decision making method. And then we will present an application of possibility neutrosophic soft set theory in a decision making problem.

Definition 29. Let \( f_\mu \in \mathcal{PN}(U, E) \), \( f^t_\mu \), \( f^i_\mu \) and \( f^f_\mu \) be the truth, indeterminacy and falsity matrices of \( \land \)-product matrix, respectively. Then, weighted matrices of \( f^t_\mu \), \( f^i_\mu \) and \( f^f_\mu \) are denoted by \( \land^t \), \( \land^i \) and \( \land^f \), respectively and computed by as follows:

\[
\land^t(e_{ij}, u_k) = t(f_\mu \land g_\nu)(e_{ij})(u_k) + (\mu_{ik}(e_i) \land \nu_{jk}(e_j)) - t(f_\mu \land g_\nu)(e_{ij})(u_k) \times (\mu_{ik}(e_i) \land \nu_{jk}(e_j))
\]

\[
\land^i(e_{ij}, u_k) = i(f_\mu \land g_\nu)(e_{ij})(u_k) \times (\mu_{ik}(e_i) \land \nu_{jk}(e_j))
\]

\[
\land^f(e_{ij}, u_k) = f(f_\mu \land g_\nu)(e_{ij})(u_k) \times (\mu_{ik}(e_i) \land \nu_{jk}(e_j))
\]

for \( i, j, k \in \Lambda \)

Definition 30. Let \( f_\mu \in \mathcal{PN}(U, E) \), \( \land^t \), \( \land^i \) and \( \land^f \) be the weighed matrices of \( f^t_\mu \), \( f^i_\mu \) and \( f^f_\mu \), respectively. Then, for all \( u_t \in U \) such that \( t \in \Lambda \), scores of \( u_t \) is in the weighed matrices \( \land^t \), \( \land^i \) and \( \land^f \) are denoted by \( s^t(u_k) \), \( s^i(u_k) \) and \( s^f(u_k) \), and computed by as follow, respectively

\[
s^t(u_t) = \sum_{i,j \in \Lambda} \delta^t_{ij}(u_t)
\]

\[
s^i(u_t) = \sum_{i,j \in \Lambda} \delta^i_{ij}(u_t)
\]
\[ s^f(u_t) = \sum_{i,j \in \Lambda} \delta^f_{ij}(u_t) \]

Here

\[ \delta^s_{ij}(u_t) = \begin{cases} \wedge^s(e_{ij}, u_t), & \wedge^s(e_{ij}, u_t) = \max\{\wedge^s(e_{ij}, u_k) : u_k \in U\} \\ 0, & \text{otherwise} \end{cases} \]

\[ \delta^i_{ij}(u_t) = \begin{cases} \wedge^i(e_{ij}, u_t), & \wedge^i(e_{ij}, u_t) = \max\{\wedge^i(e_{ij}, u_k) : u_k \in U\} \\ 0, & \text{otherwise} \end{cases} \]

\[ \delta^f_{ij}(u_t) = \begin{cases} \wedge^f(e_{ij}, u_t), & \wedge^f(e_{ij}, u_t) = \max\{\wedge^f(e_{ij}, u_k) : u_k \in U\} \\ 0, & \text{otherwise} \end{cases} \]

**Definition 31.** Let \( s^t(u_t), s^i(u_t) \) and \( s^f(u_t) \) be scores of \( u_t \in U \) in the weighted matrices \( \wedge^t \), \( \wedge^i \) and \( \wedge^f \). Then, decision score of \( u_t \in U \), denoted by \( ds(u_t) \), is computed as follow:

\[ ds(u_t) = s^t(u_t) - s^i(u_t) - s^f(u_t) \]

Now, we construct a possibility neutrosophic soft set decision making method by the following algorithm;

**Algorithm**

**Step 1:** Input the possibility neutrosophic soft set \( f_\mu \).

**Step 2:** Construct the matrix \( \wedge \)-product

**Step 3:** Construct the matrices \( f^t_\mu, f^i_\mu \) and \( f^f_\mu \)

**Step 4:** Construct the weighted matrices \( \wedge^t \), \( \wedge^i \) and \( \wedge^f \).

**Step 5:** Compute score of \( u_t \in U \), for each of the weighted matrices,

**Step 6:** Compute decision score, for all \( u_t \in U \),

**Step 7:** The optimal decision is to select \( u_t = \max ds(u_t) \).
Example 5. Assume that $U = \{u_1, u_2, u_3\}$ is a set of houses and $E = \{e_1, e_2, e_3\} = \{\text{cheap}, \text{large}, \text{moderate}\}$ is a set of parameters which is attractiveness of houses. Suppose that Mr. X want to buy one the most suitable house according to to himself depending on three of the parameters only.

Step 1: Based on the choice parameters of Mr. X, let there be two observations $f_\mu$ and $g_\nu$ by two experts defined as follows:

$$
\begin{align*}
  f_\mu(e_1) &= \left\{ \left( \frac{u_1}{0.5,0.3,0.7}, 0.6 \right), \left( \frac{u_2}{0.6,0.2,0.5}, 0.2 \right), \left( \frac{u_3}{0.7,0.6,0.5}, 0.4 \right) \right\} \\
  f_\mu(e_2) &= \left\{ \left( \frac{u_1}{0.35,0.2,0.6}, 0.4 \right), \left( \frac{u_2}{0.7,0.8,0.3}, 0.5 \right), \left( \frac{u_3}{0.2,0.4,0.4}, 0.6 \right) \right\} \\
  f_\mu(e_3) &= \left\{ \left( \frac{u_1}{0.7,0.2,0.5}, 0.5 \right), \left( \frac{u_2}{0.4,0.5,0.2}, 0.3 \right), \left( \frac{u_3}{0.5,0.3,0.6}, 0.2 \right) \right\}
\end{align*}
$$

$$
\begin{align*}
  g_\nu(e_1) &= \left\{ \left( \frac{u_1}{0.3,0.4,0.5}, 0.2 \right), \left( \frac{u_2}{0.7,0.3,0.4}, 0.5 \right), \left( \frac{u_3}{0.4,0.5,0.2}, 0.3 \right) \right\} \\
  g_\nu(e_2) &= \left\{ \left( \frac{u_1}{0.4,0.6,0.2}, 0.3 \right), \left( \frac{u_2}{0.2,0.5,0.3}, 0.7 \right), \left( \frac{u_3}{0.4,0.5,0.2}, 0.8 \right) \right\} \\
  g_\nu(e_3) &= \left\{ \left( \frac{u_1}{0.2,0.1,0.6}, 0.7 \right), \left( \frac{u_2}{0.8,0.4,0.5}, 0.4 \right), \left( \frac{u_3}{0.6,0.4,0.3}, 0.4 \right) \right\}
\end{align*}
$$

Step 2: Let us consider possibility neutrosophic soft set $\land$-product which is the mapping $\land: E \times E \rightarrow \mathcal{N}(U) \times I^U$ given as follows:

<table>
<thead>
<tr>
<th>$\land$</th>
<th>$u_1, \mu$</th>
<th>$u_2, \mu$</th>
<th>$u_3, \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{11}$</td>
<td>$((0.3,0.4,0.7),0.2)$</td>
<td>$((0.6,0.3,0.5),0.2)$</td>
<td>$((0.4,0.6,0.5),0.3)$</td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>$((0.4,0.6,0.7),0.3)$</td>
<td>$((0.2,0.5,0.5),0.2)$</td>
<td>$((0.4,0.6,0.5),0.4)$</td>
</tr>
<tr>
<td>$e_{13}$</td>
<td>$((0.2,0.3,0.7),0.6)$</td>
<td>$((0.6,0.4,0.5),0.2)$</td>
<td>$((0.6,0.6,0.5),0.4)$</td>
</tr>
<tr>
<td>$e_{21}$</td>
<td>$((0.3,0.4,0.6),0.2)$</td>
<td>$((0.7,0.8,0.4),0.5)$</td>
<td>$((0.2,0.5,0.5),0.3)$</td>
</tr>
<tr>
<td>$e_{22}$</td>
<td>$((0.35,0.6,0.6),0.3)$</td>
<td>$((0.2,0.8,0.3),0.5)$</td>
<td>$((0.2,0.6,0.5),0.6)$</td>
</tr>
<tr>
<td>$e_{23}$</td>
<td>$((0.2,0.2,0.6),0.4)$</td>
<td>$((0.7,0.8,0.5),0.4)$</td>
<td>$((0.2,0.4,0.5),0.4)$</td>
</tr>
<tr>
<td>$e_{31}$</td>
<td>$((0.3,0.4,0.5),0.2)$</td>
<td>$((0.4,0.5,0.4),0.3)$</td>
<td>$((0.4,0.5,0.6),0.2)$</td>
</tr>
<tr>
<td>$e_{32}$</td>
<td>$((0.4,0.6,0.5),0.3)$</td>
<td>$((0.2,0.5,0.3),0.3)$</td>
<td>$((0.4,0.6,0.6),0.2)$</td>
</tr>
<tr>
<td>$e_{33}$</td>
<td>$((0.2,0.2,0.6),0.5)$</td>
<td>$((0.4,0.5,0.5),0.3)$</td>
<td>$((0.5,0.4,0.6),0.2)$</td>
</tr>
</tbody>
</table>

Matrix representation of $\land$-product
Step 3: We construct matrices $f^i_\mu$, $f^i_\mu$ and $f^f_\mu$ as follows:

$$\begin{align*}
\begin{array}{c|ccc}
\wedge & u_1, \mu & u_2, \mu & u_3, \mu \\
\hline
e_{11} & (0.3, 0.2) & (0.6, 0.2) & (0.4, 0.3) \\
e_{12} & (0.4, 0.3) & (0.2, 0.2) & (0.4, 0.4) \\
e_{13} & (0.2, 0.6) & (0.6, 0.2) & (0.6, 0.4) \\
e_{21} & (0.3, 0.2) & (0.7, 0.5) & (0.2, 0.3) \\
e_{22} & (0.35, 0.3) & (0.2, 0.5) & (0.2, 0.6) \\
e_{23} & (0.2, 0.4) & (0.7, 0.4) & (0.2, 0.4) \\
e_{31} & (0.3, 0.2) & (0.4, 0.3) & (0.4, 0.2) \\
e_{32} & (0.4, 0.3) & (0.2, 0.3) & (0.4, 0.2) \\
e_{33} & (0.2, 0.5) & (0.4, 0.3) & (0.5, 0.2) \\
\end{array}
\end{align*}$$

Matrix $f^i_\mu$ of $\wedge$-product

$$\begin{align*}
\begin{array}{c|ccc}
\wedge & u_1, \mu & u_2, \mu & u_3, \mu \\
\hline
e_{11} & (0.4, 0.2) & (0.3, 0.2) & (0.6, 0.3) \\
e_{12} & (0.6, 0.3) & (0.5, 0.2) & (0.6, 0.4) \\
e_{13} & (0.3, 0.6) & (0.4, 0.2) & (0.6, 0.4) \\
e_{21} & (0.4, 0.2) & (0.8, 0.5) & (0.5, 0.3) \\
e_{22} & (0.6, 0.3) & (0.8, 0.5) & (0.6, 0.6) \\
e_{23} & (0.2, 0.4) & (0.8, 0.4) & (0.4, 0.4) \\
e_{31} & (0.4, 0.2) & (0.5, 0.3) & (0.5, 0.2) \\
e_{32} & (0.6, 0.3) & (0.5, 0.3) & (0.6, 0.2) \\
e_{33} & (0.2, 0.5) & (0.5, 0.3) & (0.4, 0.2) \\
\end{array}
\end{align*}$$

Matrix $f^f_\mu$ of $\wedge$-product

$$\begin{align*}
\begin{array}{c|ccc}
\wedge & u_1, \mu & u_2, \mu & u_3, \mu \\
\hline
e_{11} & (0.7, 0.2) & (0.5, 0.2) & (0.5, 0.3) \\
e_{12} & (0.7, 0.3) & (0.5, 0.2) & (0.5, 0.4) \\
e_{13} & (0.7, 0.6) & (0.5, 0.2) & (0.5, 0.4) \\
e_{21} & (0.6, 0.2) & (0.4, 0.5) & (0.5, 0.3) \\
e_{22} & (0.6, 0.3) & (0.3, 0.5) & (0.5, 0.6) \\
e_{23} & (0.6, 0.4) & (0.5, 0.4) & (0.5, 0.4) \\
e_{31} & (0.5, 0.2) & (0.4, 0.3) & (0.6, 0.2) \\
e_{32} & (0.5, 0.3) & (0.3, 0.3) & (0.6, 0.2) \\
e_{33} & (0.6, 0.5) & (0.5, 0.3) & (0.6, 0.2) \\
\end{array}
\end{align*}$$
Matrix \( f^\wedge_e \) of \( \wedge \)-product

**Step 4:** We obtain weighted matrices \( \wedge^e, \wedge^i \) and \( \wedge^f \) using Definition 24 as follows:

\[
\begin{pmatrix}
\wedge^e & u_1 & u_2 & u_3 \\
e_{11} & 0.44 & 0.64 & 0.58 \\
e_{12} & 0.58 & 0.36 & 0.64 \\
e_{13} & 0.68 & 0.68 & 0.76 \\
e_{21} & 0.44 & 0.85 & 0.44 \\
e_{22} & 0.55 & 0.60 & 0.68 \\
e_{23} & 0.52 & 0.82 & 0.48 \\
e_{31} & 0.44 & 0.58 & 0.52 \\
e_{32} & 0.58 & 0.44 & 0.52 \\
e_{33} & 0.60 & 0.58 & 0.60 \\
\end{pmatrix}
\begin{pmatrix}
\wedge^i & u_1 & u_2 & u_3 \\
e_{11} & 0.08 & 0.16 & 0.18 \\
e_{12} & 0.18 & 0.10 & 0.24 \\
e_{13} & 0.18 & 0.08 & 0.24 \\
e_{21} & 0.08 & 0.40 & 0.15 \\
e_{22} & 0.18 & 0.40 & 0.36 \\
e_{23} & 0.08 & 0.32 & 0.16 \\
e_{31} & 0.08 & 0.15 & 0.10 \\
e_{32} & 0.18 & 0.15 & 0.12 \\
e_{33} & 0.10 & 0.15 & 0.08 \\
\end{pmatrix}
\begin{pmatrix}
\wedge^f & u_1 & u_2 & u_3 \\
e_{11} & 0.14 & 0.10 & 0.15 \\
e_{12} & 0.21 & 0.10 & 0.20 \\
e_{13} & 0.42 & 0.10 & 0.20 \\
e_{21} & 0.21 & 0.18 & 0.15 \\
e_{22} & 0.24 & 0.20 & 0.15 \\
e_{23} & 0.24 & 0.20 & 0.15 \\
e_{31} & 0.10 & 0.12 & 0.12 \\
e_{32} & 0.15 & 0.09 & 0.12 \\
e_{33} & 0.30 & 0.15 & 0.12 \\
\end{pmatrix}
\]

Weighed matrices of \( f^i_\mu, f^e_\mu \) and \( f^f_\mu \) from left to right, respectively.

**Step 5:** For all \( u \in U \), we find scores using Definition 30 as follows:

\[
\begin{align*}
\text{Step 5: } & s^f(u_1) = 1, 18, \quad s^f(u_2) = 2, 89, \quad s^f(u_3) = 2, 68 \\
& s^i(u_1) = 0, 18 \quad s^i(u_2) = 1, 42 \quad s^i(u_3) = 0, 66 \\
& s^e(u_1) = 1, 32 \quad s^e(u_2) = 0, 32 \quad s^e(u_3) = 0, 57 \\
\end{align*}
\]

*Step 5:* For all \( u \in U \), we find scores using Definition 30 as follows:

\[
\begin{align*}
\text{Step 5: } & ds(u_1) = 1, 18 - 0, 18 - 1, 32 = -0, 32 \\
& ds(u_2) = 2, 89 - 1, 42 - 0, 32 = 0, 90 \\
& ds(u_3) = 2, 68 - 0, 66 - 0, 57 = 1, 45 \\
\end{align*}
\]

*Step 5:* Then the optimal selection for Mr.X is \( u_3 \).

5. Similarity measure of possibility neutrosophic soft sets

In this section, we introduce a measure of similarity between two PNS-sets.
Definition 32. Similarity between two PNS-sets $f_\mu$ and $g_\nu$, denoted by $S(f_\mu, g_\nu)$, is defined as follows:

$$S(f_\mu, g_\nu) = M(f(e), g(e)).M(\mu(e), \nu(e))$$

such that

$$M(f(e), g(e)) = \frac{1}{n} M_i(f(e), g(e)), M(\mu, \nu) = \frac{1}{n} \sum_{i=1}^{n} M(\mu(e_i), \nu(e_i)),$$

where

$$M_i(f(e), g(e)) = 1 - \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} |\phi_{f_\mu}(e_i)(u_j) - \phi_{g_\nu}(u_j)|^p, 1 \leq p \leq \infty$$

such that and

$$\phi_{f_\mu}(e_i)(u_j) = \frac{f_{ij}^1(e_i) + f_{ij}^2(e_i) + f_{ij}^3(e_i)}{3}, \phi_{g_\nu}(e_i)(u_j) = \frac{g_{ij}^1(e_i) + g_{ij}^2(e_i) + g_{ij}^3(e_i)}{3},$$

$$M(\mu(e), \nu(e)) = 1 - \frac{\sum_{j=1}^{n} |\mu_{ij}(e) - \nu_{ij}(e)|}{\sum_{j=1}^{n} |\mu_{ij}(e) + \nu_{ij}(e)|}$$

Definition 33. Let $f_\mu$ and $g_\nu$ be two PNS-sets over $U$. We say that $f_\mu$ and $g_\nu$ are significantly similar if $S(f_\mu, g_\nu) \geq \frac{1}{2}$

Proposition 9. Let $f_\mu, g_\nu \in \mathcal{PN}(U, E)$. Then,

i. $S(f_\mu, g_\nu) = S(g_\nu, f_\mu)$

ii. $0 \leq S(f_\mu, g_\nu) \leq 1$

iii. $f_\mu = g_\nu \Rightarrow S(f_\mu, g_\nu) = 1$

Proof. The proof is straightforward and follows from Definition 32.

Example 6. Let us consider PNS-sets $f_\mu$ and $g_\nu$ in Example 1 given as follows:
\[
\begin{align*}
\begin{cases}
  f_\mu(e_1) &= \left\{ \left( \frac{u_1}{(0.5,0.2,0.6)}, 0.8 \right), \left( \frac{u_2}{(0.7,0.3,0.5)}, 0.4 \right), \left( \frac{u_3}{(0.4,0.3,0.8)}, 0.7 \right) \right\} \\
  f_\mu(e_2) &= \left\{ \left( \frac{u_1}{(0.8,0.4,0.5)}, 0.6 \right), \left( \frac{u_2}{(0.5,0.7,0.2)}, 0.8 \right), \left( \frac{u_3}{(0.7,0.3,0.9)}, 0.4 \right) \right\} \\
  f_\mu(e_3) &= \left\{ \left( \frac{u_1}{(0.6,0.7,0.5)}, 0.2 \right), \left( \frac{u_2}{(0.5,0.3,0.7)}, 0.6 \right), \left( \frac{u_3}{(0.6,0.5,0.4)}, 0.5 \right) \right\}
\end{cases}
\end{align*}
\]

and
\[
\begin{align*}
\begin{cases}
  g_\nu(e_1) &= \left\{ \left( \frac{u_1}{(0.6,0.3,0.8)}, 0.4 \right), \left( \frac{u_2}{(0.6,0.5,0.5)}, 0.7 \right), \left( \frac{u_3}{(0.2,0.6,0.4)}, 0.8 \right) \right\} \\
  g_\nu(e_2) &= \left\{ \left( \frac{u_1}{(0.5,0.4,0.3)}, 0.3 \right), \left( \frac{u_2}{(0.4,0.6,0.5)}, 0.6 \right), \left( \frac{u_3}{(0.7,0.2,0.5)}, 0.8 \right) \right\} \\
  g_\nu(e_3) &= \left\{ \left( \frac{u_1}{(0.7,0.5,0.3)}, 0.8 \right), \left( \frac{u_2}{(0.4,0.4,0.6)}, 0.5 \right), \left( \frac{u_3}{(0.8,0.5,0.3)}, 0.6 \right) \right\}
\end{cases}
\end{align*}
\]

then,

\[
M(\mu(e_1), \nu(e_1)) = 1 - \frac{\sum_{j=1}^{3} |\mu_{1j}(e_1) - \nu_{1j}(e_1)|}{\sum_{j=1}^{3} |\mu_{1j}(e_1) + \nu_{1j}(e_1)|}
= 1 - \frac{|0.8 - 0.4| + |0.4 - 0.7| + |0.7 - 0.8|}{|0.8 + 0.4| + |0.4 + 0.7| + |0.7 + 0.8|} = 0.79
\]

Similarly we get \(M(\mu(e_2), \nu(e_2)) = 0.74\) and \(M(\mu(e_3), \nu(e_3)) = 0.75\), then

\[
M(\mu, \nu) = \frac{1}{3}(0.79 + 0.75 + 0.74) = 0.76
\]

\[
M_1(f(e), g(e)) = 1 - \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} (\phi_{f_\mu(e_i)}(u_j) - \phi_{g_\nu(e_i)}(u_j))^p}
= 1 - \frac{1}{\sqrt{3}} \sqrt{(0.43 - 0.57)^2 + (0.50 - 0.53)^2 + (0.57 - 0.40)^2} = 0.73
\]

\[
M_2(f(e), g(e)) = 0.86
\]

\[
M_3(f(e), g(e)) = 0.94
\]

\[
M(f(e), g(e)) = \frac{1}{3}(0.73 + 0.86 + 0.94) = 0.84
\]

and

\[
S(f_\mu, g_\nu) = 0.84 \times 0.76 = 0.64
\]

20
6. Decision-making method based on the similarity measure

In this section, we give a decision making problem involving possibility neutrosophic soft sets by means of the similarity measure between the possibility neutrosophic soft sets.

Let our universal set contain only two elements "yes" and "no", that is $U = \{y, n\}$. Assume that $P = \{p_1, p_2, p_3, p_4, p_5\}$ are five candidates who fill in a form in order to apply formally for the position. There is a decision maker committee. They want to interview the candidates by model possibility neutrosophic soft set determined by committee. So they want to test similarity of each of candidate to model possibility neutrosophic soft set.

Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the set of parameters, where $e_1$=experience, $e_2$=computer knowledge, $e_3$=training, $e_4$=young age, $e_5$=higher education, $e_6$=marriage status and $e_7$=good health.

Our model possibility neutrosophic soft set determined by committee for suitable candidates properties $f_\mu$ is given in Table 1.

<table>
<thead>
<tr>
<th>$f_\mu$</th>
<th>$e_1, \mu$</th>
<th>$e_2, \mu$</th>
<th>$e_3, \mu$</th>
<th>$e_4, \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$((1,0,0),1)$</td>
<td>$((1,0,0),1)$</td>
<td>$((0,1,1),1)$</td>
<td>$((1,1,1),1)$</td>
</tr>
<tr>
<td>$n$</td>
<td>$((0,1,1),1)$</td>
<td>$((1,0,0),1)$</td>
<td>$((0,1,1),1)$</td>
<td>$((1,0,0),1)$</td>
</tr>
</tbody>
</table>

Table 1: The tabular representation of model possibility neutrosophic soft set

<table>
<thead>
<tr>
<th>$g_\nu$</th>
<th>$e_1, \nu$</th>
<th>$e_2, \nu$</th>
<th>$e_3, \nu$</th>
<th>$e_4, \nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$((0.7,0.2,0.5),0.4)$</td>
<td>$((0.5,0.4,0.6),0.2)$</td>
<td>$((0.2,0.3,0.4),0.5)$</td>
<td>$((0.8,0.4,0.6),0.3)$</td>
</tr>
<tr>
<td>$n$</td>
<td>$((0.3,0.7,0.1),0.3)$</td>
<td>$((0.7,0.3,0.5),0.4)$</td>
<td>$((0.6,0.5,0.3),0.2)$</td>
<td>$((0.2,0.1,0.5),0.4)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$g_\nu$</th>
<th>$e_5, \nu$</th>
<th>$e_6, \nu$</th>
<th>$e_7, \nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$((0.2,0.4,0.3),0.5)$</td>
<td>$((0.1,1),0.3)$</td>
<td>$((0.1,0.4,0.7),0.2)$</td>
</tr>
<tr>
<td>$n$</td>
<td>$((0.1,0.5,0.2),0.6)$</td>
<td>$((1,0,0),0.5)$</td>
<td>$((0.3,0.5,0.1),0.4)$</td>
</tr>
</tbody>
</table>

Table 2: The tabular representation of possibility neutrosophic soft set for $p_1$
Table 3: The tabular representation of possibility neutrosophic soft set for $p_2$

<table>
<thead>
<tr>
<th>$r_\theta$</th>
<th>$e_1,\theta$</th>
<th>$e_2,\theta$</th>
<th>$e_3,\theta$</th>
<th>$e_4,\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>(0.3,0.2,0.5),0.4</td>
<td>(0.7,0.1,0.5),0.6</td>
<td>(0.6,0.5,0.3),0.2</td>
<td>(0.3,0.1,0.4),0.5</td>
</tr>
<tr>
<td>$n$</td>
<td>(0.1,0.7,0.6),0.3</td>
<td>(0.4,0.2,0.3),0.7</td>
<td>(0.7,0.4,0.3),0.5</td>
<td>(0.7,0.1,0.2),0.1</td>
</tr>
</tbody>
</table>

Table 4: The tabular representation of possibility neutrosophic soft set for $p_3$

<table>
<thead>
<tr>
<th>$s_\alpha$</th>
<th>$e_1,\alpha$</th>
<th>$e_2,\alpha$</th>
<th>$e_3,\alpha$</th>
<th>$e_4,\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>(0.2,0.1,0.4),0.5</td>
<td>(0.7,0.5,0.4),0.8</td>
<td>(0.8,0.1,0.2),0.4</td>
<td>(0.5,0.4,0.5),0.4</td>
</tr>
<tr>
<td>$n$</td>
<td>(0.6,0.5,0.1),0.1</td>
<td>(0.3,0.7,0.2),0.2</td>
<td>(0.7,0.5,0.1),0.7</td>
<td>(0.1,0.3,0.7),0.5</td>
</tr>
</tbody>
</table>

Table 5: The tabular representation of possibility neutrosophic soft set for $p_4$

<table>
<thead>
<tr>
<th>$m_\gamma$</th>
<th>$e_1,\gamma$</th>
<th>$e_2,\gamma$</th>
<th>$e_3,\gamma$</th>
<th>$e_4,\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>(0.1,0.2,0.1),0.3</td>
<td>(0.2,0.3,0.5),0.8</td>
<td>(0.4,0.1,0.3),0.9</td>
<td>(0.7,0.3,0.2),0.3</td>
</tr>
<tr>
<td>$n$</td>
<td>(0.4,0.5,0.3),0.2</td>
<td>(0.7,0.6,0.1),0.3</td>
<td>(0.2,0.3,0.4),0.5</td>
<td>(0.5,0.2,0.3),0.6</td>
</tr>
</tbody>
</table>

Table 6: The tabular representation of possibility neutrosophic soft set for $p_5$

Now we find the similarity between the model possibility neutrosophic soft set and possibility neutrosophic soft set of each person as follow

$S(f_\mu,g_\nu) \cong 0,49 < \frac{1}{2}$, $S(f_\mu,h_\delta) \cong 0,47 < \frac{1}{2}$, $S(f_\mu,r_\theta) \cong 0,51 > \frac{1}{2}$, $S(f_\mu,s_\alpha) \cong 0,54 > \frac{1}{2}$, $S(f_\mu,m_\gamma) \cong 0,57 > \frac{1}{2}$.

Consequently, $p_5$ is should be selected by the committee.
7. Conclusion

In this paper we have introduced the concept of possibility neutrosophic soft set and studied some of the related properties. Applications of this theory have been given to solve a decision making problem. We also presented a new method to find out the similarity measure of two possibility neutrosophic soft sets and we applied to a decision making problem. In future, these seem to have natural applications and algebraic structure.

References


