A smarandache completely prime ideal with respect to an element of near ring

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Abstract

In this paper we introduce the notion of a smarandache completely prime ideal with respect to an element belated to a near field of a near ring N (b-s-c.p.i) of N. We study some properties of this new concept and link it with some there types of ideals of a near ring.

Keywords: Smarandache Completely Prime, Near Ring.

1. Introduction

In 1905, L.E. Drckson began the study of a near ring and later in 1930; Wieland has investigated it [1]. In 1977, G.Pilz, introduced the notion of a prime ideal of near ring [1]. In 1988, N.G. Groenewald introduced of a completely prime ideal of a near ring [5]. In 2002, W.B. Vasanth Kandasamy study samaradache near ring, (samaradache ideal, of a near ring [7]. In 2012 H.H. Abbass and M.A.Mohommed introduced the notion of a completely prime ideal with respect to an element of a near ring [3].

In this work, we introduce a Samaradache completely prime ideal with respect to an element related to a near field of near ring as we mentioned in the abstract.

2. Preliminaries

In this section, we review some basic concepts about a near ring, and some types of fields of a near rind that We need in our work.

Definition 2.1 [1]: A left near ring is a set N together with two binary operations “+” and “.” such that
1. (N,+) is a group (not necessarily abelian),
2. (N,.) is a semi group?
3. \( n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3 \), for all \( n_1, n_2, n_3 \in N \).

Definition 2.2 [2]: The left near ring is called a zero symmetric if \( 0 \cdot x = 0 \), for all \( x \in N \).

Definition 2.3[7]: Left \( (N,+,\cdot) \) be a near-ring. A normal subgroup \( I \) of \( (N,+) \) is called a left ideal of \( N \) if
1. \( N \cdot I \subseteq I \)
2. for all \( n, n_i \in N \) and for all \( i \in I \),
\( n + i.n - n_i.n \in I \)

Remark 2.4: If \( N \) is a left near ring, then \( x.0 = 0 \), for all \( x \in N \) (from the left distributire law). Also, we will refer that all near rings and ideals in this work are left.
Definition 2.5 [6]: Let \( I \) be an ideal of a near ring \( N \), then \( I \) is called a completely prime ideal of \( N \) if for all \( x, y \in N \), \( x, y \in I \) implies \( x \in I \) or \( y \in I \), denoted by c.p.I of \( N \).

The a b.c.s.p.I near ring \( N \) in example (1.3) is not.

Definition 2.6 [3]: Let \( N \) be a near ring, \( I \) be an ideal of \( N \) and let \( b \in N \), then \( I \) is called a completely ideal with respect to the element \( b \) denoted by \((b - c. p.I)\) of \( N \), if for all \( x, y \in N \), \( b.(x, y) \in I \) implies \( x \in I \) or \( y \in I \).

Definition 2.7 [7]: A near ring \( N \) is called an integral domain if \( N \) has non-zero divisors.

Definition 2.8 [7]: Let \( N \) be a near ring, \( M \) be an ideal of \( N \) and let \( f: N \rightarrow M \) then \( I \) is called a completely ideal with respect to the element \( b \) denoted by \( (b - M.I) \) of \( N \), if for all \( x, y \in N \), \( f(x, y) \in I \) implies \( x \in I \) or \( y \in I \).

Definition 2.9 [7]: An empty set \( N \) is said to be a near field if \( N \) is defined by two binary operations ‘+’ and ‘.’ such that

1. \((N, +)\) is a group
2. \((N \setminus \{0\}, .)\) is a group
3. \(a.(b + c) = ab + ac\) for all \( a, b, c \in N \).

Definition 2.10 [7]: The near ring \((N, +, .)\) is said to be a smarandache near ring denoted by \((s\text{-}near\ ring)\) if it has a proper subset \( M \) such that \((M, +, .)\) is a near field.

Definition 2.11 [7]: Let \( N \) be \( s\text{-}near\ ring\). A normal subgroup \( M \) of \( N \) is called a smarandache ideal \((s\text{-}ideal)\) if,

i. For all \( x, y \in M \) and for all \( i \in \mathbb{N}x(y + i) - xy \in 1 \),
ii. \( M \subseteq 1 \).

Remark 2.12 [7]: Let \([I_i]_{i \in I}\) be a chain of \( s\text{-}ideal\)s related to a near field \( M \) of a near ring \( N \), then \([I_i]_{i \in I}\) is a \( s\text{-}ideal\)s related to near field \( M \).

Remark 2.13 [6]: Let \((N_1, +, .)\) and \((N_2, +, .)\) be two \( s\text{-}near\ ring\)s and let \( f: N_1 \rightarrow N_2 \) be an epimorphism and \( N_1 \) has \( M_1 \) as near field. Then \( f(M_1) = f(M_2) \) is a near field of \( N_2 \).

Proposition 2.14 [4]: Let \((N_1, +, .)\) and \((N_2, +, .)\) be two \( s\text{-}near\ ring\)s and \( f: N_1 \rightarrow N_2 \) be an epimorphism and let \( I \) be a \( S\text{-}ideal\) related to a near field \( M_1 \) of a near ring \( N_1 \), and then \( f(I) \) is \( s\text{-}ideal\)s related to a near field \( f(M_1) \).

Proposition 2.15 [4]: Let \((N_1, +, .)\) be a \( s\text{-}near\ ring\) has a near field \( M_1 \), \( N_2 \) be a \( s\text{-}near\ ring\) and \( f: N_1 \rightarrow N_2 \) be an epimorphism and let \( J \) be \( s\text{-}ideal\)s related to a near field \( M_2 \) of \( N_2 \), where \( f(M_1) = M_2 \) of \( N_2 \), then \( f^{-1}(J) \) is \( s\text{-}ideal\)s related to a near field \( M_1 \) of \( N_1 \).

Definition 2.16 [7]: Let \( N \) be an \( s\text{-}near\ ring\). The \( s\text{-}ideal\) \( I \) related to a near field \( M \) is called completely prime related to a near field \( M \) of \( N \) if, for all \( x, y \in M \), \( x, y \in I \) implies \( x \in I \) or \( y \in I \) denoted by \((s\text{-}c.p.I)\) of \( N \).

3. The main results

In this section, we define the notion of smarandache completely ideal with respect to an element \( b \) \((b - s\text{-}c.p.I)\) and study some properties of this notion, we will discuss the image and pre image of \( b - s\text{-}c.p.I \) under near rings epimorphism and explain the relationships between it and \( b - s\text{-}c.p.I \) of a near ring.

Definition 3.1: A \( s\text{-}ideal\) related to a near field \( M \) of a \( s\text{-}near\ ring \( N \) is called a smarandache completely ideal with respect to an element \( b \) of \( N \) \((b - s\text{-}c.p.I)\) if \( b \in (x, y) \in I \) implies \( x \in I \) or \( y \in I \) for all \( x, y \in M \).

Example 3.2: The left \( s\text{-}near\ ring\) with addition and multiplication defined by the following tables.

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The s-ideal $I = [0,a]$ related to the near field $M = [0,c]$ is b – s.c.p.I of $N$ since 0. (c.c) = 0 $\in I, but c \not\in I$.

**Proposition 3.3:** Let $I$ be a s-ideal related to a near field $M$ of a s-near ring $N,$ then $I$ is a s.c.p.I of $N$ if and only if $I$ is 1 – s.c.p.I, where 1 is the multiplicative identity element of $M$.

**Proof:** Suppose $I$ is a s.c.p.I ideal of $N$ and let $x, y \in M$ such that $1. (x.y) \in I$.

Then we have $1. (x.y) = x.y \in I$

$\Rightarrow x \in I$ or $y \in I$ [Since $I$ is a s.c.p.I of $N$].

$\Rightarrow I$ is 1 – s.c.p.I of $N$.

**Conversely,**

Let $x, y \in M$ such that $x, y \in I$

$\Rightarrow x.y = 1. (x.y) \in I \Rightarrow x \in I$ or $y \in I$ [Since $I$ is 1. (x.y) of $N$].

**Remark 3.4:** In general an S.C.P.I related to a near field $M$ of an s-near ring $N$ may not be b-S.C.P.I related to $M$ of $N$ as in the following example.

**Example 3.5:** Consider the s-near ring of integers mod 6 ($\mathbb{Z}_6, t_6, i_6$); the s-ideal $I = \{0,2,4\}$ is S.C.P.I related to the near field $M = \{0,3\}$, but it is not 2-S.C.P.I of $N$, since 3 $\in M$ and $2.(3.3)=0 \in I$ but 3 $\not\in T$.

**Proposition 3.6:** Let $I$ be a b-C.P.I related to a near field $M$ of a s-near ring $N$. then $I$ is a b-S.C.P.I of $N$.

**Proof:** Let $x, y \in M$ such that $b. (x.y) \in I$

$\Rightarrow b. (x.y) \in I$ [Since $I$ is b-C.P.I of $N$]

$\Rightarrow b$ is a b-S.C.P.I of $N$.

**Remark (3.7):** The converse of proposition (3.6) may not be true as in the following example.

**Example 3.8:** Consider the s-near ring of integers mod 12 ($\mathbb{Z}_{12}, t_{12}, i_{12}$); s-ideal $I = \{0,2,4,6,8,10\}$ is S.C.P.I related to the near field $M = \{0,4,8\}$, but it is not 2-S.C.P.I of $N$, since $3,5 \not\in N$ and $2.(3.5)=6 \in I$, but 3 and 5 $\not\in I$.

**Proposition 3.9:** Let $N$ be a s-near ring and let $I$ be a s-ideal related to a near field $M$ of $N$, then $I$ is a b-S.C.P.I of $N$ if and only if $M$ is a subset of $I$, for all $b \in I$.

**Proof:** Suppose $I$ is a b-S.C.P.I, $b \in I$ and $X \in M$.

Now,

$X^2 = x.x \in I, 0 \in I and 0, x^2 = 0, (x.x) =0 \in I$

$x \in I$ [since $I$ is b-S.C.P.I],

$\Rightarrow M$ is a subset of $I$

**Conversely,**

Let $b \in I$ and $x, y \in M$ such that $b. (x.y) \in I$

$\Rightarrow x \text{ or } y \in I$ [since $M \subseteq I$]

$\Rightarrow I$ is b-S.C.P.I of $N$.

**Proposition 3.10:** Let $N$ be a s-near integral domain. then $I = \{0\}$ is b-S.C.P.I related to a near field $M$ of $N$, for all $n \in N \setminus \{0\}$.

**Proof:** Let $b \in N \setminus \{0\}$ and $x, y \in M$, such that $b. (x.y) \in I$

$b. (x.y) =0$

$\Rightarrow x, y =0$ [since $b \neq 0$ and $N$ is a near integral domain]

$x=0$ or $y=0 \Rightarrow x \in I$ or $y \in I$

$\Rightarrow x \in I$ or $y \in I$.

$\Rightarrow I$ is a b-S.C.P.I of $N$. 

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**Proposition 3.11:** Let $N$ be a zero symmetric $s$-near ring and let $I=[0]$. Then $I$ is not o-S.C.P.I of $N$ related to all near fields of $N$.

**Proof:** Suppose $I$ is o-S.C.P.I related to a near field $M$ of $N$.
Since $M$ is a near field $\Rightarrow M \neq [0]$
$\Rightarrow \exists X \in M$, such that $x \neq 0$.
Now, 0x = 0, (x,x) = 0 $\in I$
$\Rightarrow x \in I$ $\Rightarrow x=0$ and this contradiction [since $x \neq 0$]
$\Rightarrow I$ is not o-S.C.P.I related to $M$ of $N$.

**Proposition 2.12:** Let $N$ be a $s$-near ring and let $[li]_i\in I$ be a chain of b-S.C.P.I related to a near field $M$ of $N$, for all $i \in I$. Then $V_i \in I$ is a b-S.C.P.I related to $M$ of $N$.

**Proof:** Since $[li]_i\in I$ is a chain a b-S.C.P.I related to $M$ of $N$.
$\Rightarrow li$ is a $s$-ideal of $N$ for all $i \in I$.
$\Rightarrow V_i \in I$ is a $s$-deal of $I$ [By remark (2.12)]
Now, Let $x, y \in M$, such that $b.(x,y) \in V_i \in I_i$
$\Rightarrow$ There exists b-S.C.P.I related $M$ $I_k \in [li]_i \in I$ of $N$, such that $b.(x,y) \in I_k$
$\Rightarrow x \in I_k$ or $y \in I_k$ [since $I_k$ is a b-S.C.P.I of $N$]
$\Rightarrow x \in V_i \in I_i$ or $y \in V_i \in I_i$ $\Rightarrow V_i \in I_i$ is a b-S.C.P.I of $N$.

**Remark 3.13:** In general, if $[li]_i \in I$ is a family of b-S.C.P.I related to a near field $M$ of as near ring $N$, then $\cap_{i \in I}$ and $V_i \in I_i$ may not be b-S.C.P.I
Related to $M$ of $N$, as in the following example

**Example 3.13:** Consider the $s$-near ring of integer's mod12. (Z12,t12, 12), the $s$-ideals $I=[0,6]$ and $J=[0,4,8]$ are 3-S.C.P.I related to the near field $M= [0,4,8]$ of Z12, but the $s$-ideal $I=[0]$ is not 3-S.C.P.I related to $M$ of $Z12$, since 3.(3.8)=0 $\in I$, but and $8 \notin I$. Also, the subset $I \cup J= [0,4,6,8]$ is $s$-ideal of $Z12$ and this implies $I \cup J$ is not 3-S.C.P.I related to $M$ of $Z12$.

**Theorem 3.15:** Let $(N1, *, 0)$ and $(N2, t, 0)$ be two $s$-near rings, $F: N1 \rightarrow N2$ be an epimorphism and let $I$ be a b-S.C.P.I related to near field $M$ of $N$, then $f(I)$ is $f(b)$-S.C.P.I related to the near field $f(M)$ of $N2$.

**Proof:** By remark (2.13), we have $f(I)$ is a $s$-ideal related to a near field $f(M)$
Now Let $f(m1), f(m2) \in f(m)$, such that
$f(b) + (f(m1)) \in f(m2) \in f(I)$
$\Rightarrow f(b(m1)).f(m2) \in f(I)$
$\Rightarrow f(b(m1)).f(m2) \in f(I)$
$\Rightarrow$ either $m1 \in I$ or $m2 \in I$ or $m2$ [since $I$ is b-S.C.P.I related to $M$ of $N1$]
$\Rightarrow f(m1) \in f(I)$ or $f(m2) \in f(I)$
$\Rightarrow f(I)$ is a $f(b)$-S.C.P.I related to $f(M)$ of $N2$.

**Theorem 3.16:** Let $(N1, +, .)$ be as $s$-near ring has a near field $M1$, $(N2, t, 0)$ be $S$-near ring, $f: N1 \rightarrow N2$ be an epimorphism, and let $J$ be a b-S.C.P.I related to the near field $f(M)$ of $N2$, then $f^{-1}(I)$ is a $s$-ideal related to a near field of $M1$, where $f(b(a))$.

**Proof:** By proposition (2.15), we have $f^{-1}(J)$ is a $s$-ideal related to $M$ of $N1$. Now, Let $x, y \in M$, such that a. $(x,y) \in f^{-1}(J)$
$\Rightarrow f(x), f(y) \in f(M)$ and $f(a) + (x,y) \in J$
$\Rightarrow f(x), f(y) \in f(M)$ and $f(a) + (x,y) \in f(J)$
$\Rightarrow$ either $x \in f^{-1}(J)$ or $y \in f^{-1}(J)$ [since $J$ is b-S.C.P.I related to $f(M)$ of $N2$]
$\Rightarrow e \in f^{-1}(J)$ or $y \in f^{-1}(J)$
$\Rightarrow f^{-1}(J)$ is a b-S.C.P.I related to $f(M)$ of $N2$.

**Corollary 3.17:** Let $(N1, +, 0)$ be a $s$-near ring has a near field $M$, $(N2, +, .)$ be a $S$-near ring, $f: N1 \rightarrow N2$ be an epimorphism, and if $f(0)$ be a b-S.C.P.I related to the near field $f(M)$ of $N2$ The $\ker(f)$ is b-S.C.P.I related to a near field $M$ of $N1$, where
Ker \( f = \{ x \in N1 : f(x) = 0 \} \) and \( b=f(a) \)

**Proof:** Since \( f^{-1}([0^1]) = \ker(f) \), then where \( \text{Rer}(f) \) is a S.C.P. I related to \( M \) of \( N1 \)
[By theorem (3-16)]

**References**