Some arithmetical properties of the Smarandache series

Qianli Yang

Department of Mathematics,
Weinan Normal University, Weinan, China 714000

Abstract The Smarandache function $S(n)$ is defined as the minimal positive integer $m$ such that $n|m!$. The main purpose of this paper is to study the analyze converges questions for some series of the form $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$, i.e., we proved the series $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$ diverges for any $\delta \leq 1$, and $\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon}S(n)}$ converges for any $\epsilon > 0$.

Keywords Smarandache function, smarandache series, converges.

§1. Introduction and results

For every positive integer $n$, let $S(n)$ be the minimal positive integer $m$ such that $n|m!$, i.e.,

$$S(n) = \min\{m : m \in \mathbb{N}, n|m!\}.$$  

This function is known as Smarandache function $^{[1]}$. Easily, one has $S(1) = 1, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, S(7) = 7, S(8) = 4, S(9) = 6, S(10) = 5, \cdots$.

Use the standard factorization of $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, \ p_1 < p_2 < \cdots < p_k$, it’s trivial to have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$  

Many scholars have studied the properties of $S(n)$, for example, M. Farris and P. Mitchell $^{[2]}$ show the boundary of $S(p^\alpha)$ as

$$(p - 1)\alpha + 1 \leq S(p^\alpha) \leq (p - 1)[\alpha + 1 + \log_p \alpha] + 1.$$  

Z. Xu $^{[3]}$ noticed the following interesting relationship formula

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[ \frac{S(n)}{n} \right],$$  

by the fact that $S(p) = p$ for $p$ prime and $S(n) < n$ except for the case $n = 4$ and $n = p$, where $\pi(x)$ denotes the number of prime up to $x$, and $[x]$ the greatest integer less or equal to $x$. Those and many other interesting results on Smarandache function $S(n)$, readers may refer to $^{[2]}$-$^{[6]}$.

$^{[1]}$This work is supported by by the Science Foundation of Shanxi province (2013JM1016).
Let $p$ be a fixed prime and $n \in \mathbb{N}$, the primitive numbers of power $p$, denoted by $S_p(n)$, is defined by

$$S_p(n) = \min \{ m : m \in \mathbb{N}, p^n | m! \} = S(p^n).$$

Z. Xu \cite{3} obtained the identity between Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \sigma > 1$ and an infinite series involving $S_p(n)$ as

$$\sum_{n=1}^{\infty} \frac{1}{S_p(n)} = \frac{\zeta(s)}{p^s - 1},$$

and he also obtained some other asymptotic formulæ for $S_p(n)$. F. Luca \cite{4} proved the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n) \delta}$$

converges for all $\delta \geq 1$ and diverges for all $\delta < 1$, and the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n) \epsilon \log n}$$

converges for any $\epsilon > 0$.

In this note, we studied the analyze converges problems for the infinite series involving $S(n)$. That is, we shall prove the following conclusions:

**Theorem 1.1.** For any $\delta \leq 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\delta}}$$

diverges.

**Theorem 1.2.** For any $\epsilon > 0$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon S(n)}}$$

converges.

§2. Some lemmas

To complete the proof of theorems, we need two Lemmas.

**Lemma 2.1.** Let $p$ be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1 \\text{S}_p(n) \leq x}} \frac{1}{S_p(n)} = \frac{1}{p - 1} \left( \ln x + \gamma + \frac{p \ln p}{p - 1} \right) + O(x^{-\frac{1}{2}} + \epsilon),$$

where $\gamma$ is the Euler constant, $\epsilon$ denotes any fixed positive numbers.

**Proof.** See Theorem 2 of \cite{3}.

**Lemma 2.2.** Let $\epsilon > 0$ and $d(n)$ denotes the divisor function of positive integer $n$. Then

$$d(n) = O(n^\epsilon) \leq C_n n^\epsilon,$$

where the o-constant $C_n$ depends on $\epsilon$.

**Proof.** The proof follows \cite{7} by writing $n = \prod_{p | n} p^a$, the standard factorization of $n$. Then

$$p^{a\epsilon} \geq 2^{a\epsilon} = e^{a\epsilon \ln 2} \geq a \epsilon \ln 2 \geq \frac{1}{2} (a + 1) \epsilon \ln 2.$$
If \( p^\alpha \geq 2 \), then \( p^\alpha \geq 2^\alpha \geq \alpha + 1 \). Therefore,
\[
\frac{d(n)}{n^\epsilon} = \prod_{p|n} \frac{\alpha + 1}{p^{\epsilon p}} = \prod_{\substack{p|n \\ p^\epsilon < 2}} \frac{\alpha + 1}{p^{\epsilon}} \prod_{\substack{p|n \\ p^\epsilon \geq 2}} \frac{\alpha + 1}{p^{\epsilon}} \geq \prod_{\substack{p|n \\ p^\epsilon < 2}} \frac{\alpha + 1}{p^{\epsilon (\alpha + 1) \ln 2}} \prod_{\substack{p|n \\ p^\epsilon \geq 2}} \frac{\alpha + 1}{p^{\epsilon}}.
\]

The last item in above inequality is \( \prod_{p|n} \frac{2}{\epsilon \ln 2} \), which is less than \( \prod_{p^\epsilon < 2} \frac{2}{\epsilon \ln 2} = C_\epsilon \), say, the o-constant \( C_\epsilon \) depends on \( \epsilon \).

§3. Proof of theorems

**Proof of Theorem 1.**

We may treat the case \( \delta = 1 \) first. By Lemma 1 and the notation \( S_p(n) = S(p^n) \), we have
\[
\sum_{n=1}^{\infty} \frac{1}{S(p^n)} = \lim_{x \to +\infty} \sum_{n=1}^{\infty} \frac{1}{S_p(n)} = \infty.
\]

Obviously, for \( \delta \leq 1 \), \( \sum_{n=1}^{\infty} \frac{1}{S(n)^\delta} \) diverges follows easily by the trivial inequality:
\[
\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta} \geq \sum_{n=1}^{\infty} \frac{1}{S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{S(p^n)} ;
\]

complete the proof.

**Proof of Theorem 2.**

It certainly suffices to assume that \( \epsilon \leq 1 \). We rewrite series \( \sum_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon k^x}} \) as
\[
\sum_{k=1}^{\infty} \frac{u(k)}{k^{\epsilon x}} ,
\]
where \( u(k) = \sharp \{ n : S(n) = k \} \). For every positive integer \( n \) such that \( S(n) = k \) is a divisor of \( k! \), i.e. \( u(k) \leq d(k!) \). By Lemma 2 and the inequality bellow
\[
(k!)^2 = \prod_{j=1}^{k} j(k + 1 - j) < \prod_{j=1}^{k} \left( \frac{k + 1}{2} \right)^2 = \left( \frac{k + 1}{2} \right)^{2k} .
\]

we have
\[
u(k) \leq d(k!) \leq C_\epsilon (k!)^{\epsilon} < C_\epsilon \left( \frac{k + 1}{2} \right)^{\epsilon k} ,
\]
where \( C_\epsilon \) means that the constant depending on \( \epsilon \).

Therefore, recalling that the properties of the sequence \( (1 + \frac{1}{k})^k \), we have
\[
\sum_{k=1}^{\infty} \frac{u(k)}{k^{\epsilon x}} \leq C_\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{\epsilon x}} \left( \frac{k + 1}{2} \right)^{\epsilon k} = C_\epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k^x}} \left( \frac{k + 1}{k} \right)^{\epsilon k} < C_1 \sum_{k=1}^{\infty} \frac{1}{2^{k^x}} ,
\]
for some constant $C_1$, it follows that series $\sum_{k=1}^{\infty} \frac{n(k)}{k^{x+}}$ is bounded above by

$$C_1 \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{C_1}{2^x - 1},$$

completing the proof.

References


