

To Professor ART Solarin on his 60th Birthday Celebration

**SOME NORMAL CONGRUENCES IN QUASIGROUPS
DETERMINED BY LINEAR-BIVARIATE POLYNOMIALS OVER
THE RING \mathbb{Z}_N**

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ABSTRACT. In this work, two normal congruences are built on two quasigroups with underlining set \mathbb{Z}_n^2 relative to the linear-bivariate polynomial $P(x, y) = a + bx + cy$ that generates a quasigroup over the ring \mathbb{Z}_n . Four quasigroups are built using the normal congruences and these are shown to be homomorphic to the quasigroups with underlining set \mathbb{Z}_n^2 . Some subquasigroups of the quasigroups with underlining set \mathbb{Z}_n^2 are also found.

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1. INTRODUCTION

Let G be a non-empty set. Define a binary operation (\cdot) on G . (G, \cdot) is called a groupoid if G is closed under the binary operation (\cdot) . A groupoid (G, \cdot) is called a quasigroup if the equations $a \cdot x = b$ and $y \cdot c = d$ have unique solutions for x and y for all $a, b, c, d \in G$. A quasigroup (G, \cdot) is called a loop if there exists a unique element $e \in G$ called the identity element such that $x \cdot e = e \cdot x = x$ for all $x \in G$.

A function $f : S \times S \rightarrow S$ on a finite set S of size $n > 0$ is said to be a Latin square (of order n) if for any value $a \in S$ both functions $f(a, \cdot)$ and $f(\cdot, a)$ are permutations of S . That is, a Latin square is a square matrix with n^2 entries of n different elements, none of them occurring more than once within any row or column of the matrix.

Definition 1. A pair of Latin squares $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ is said to be orthogonal if the pairs $(f_1(x, y), f_2(x, y))$ are all distinct, as x and y vary.

Definition 2. An equivalence relation θ on a quasigroup (G, \cdot) is called normal if it satisfies the following conditions:

- (i) if $ca\theta cb$, then $a\theta b$;

(ii) if $ac\theta bc$, then $a\theta b$;

(iii) if $a\theta b$ and $c\theta d$, then $ac\theta bd$.

A normal equivalence relation is also called a normal congruence.

The basic text books on quasigroups, loops are Pflugfelder [10], Bruck [1], Chein, Pflugfelder and Smith [2], Dene and Keedwell [3], Goodaire, Jespers and Milies [6], Sabinin [12], Smith [13], Jaiyéolá [7] and Vasantha Kandasamy [15].

Definition 3. (*Bivariate Polynomial*) A bivariate polynomial is a polynomial in two variables, x and y of the form $P(x, y) = \sum_{i,j} a_{ij}x^i y^j$.

Definition 4. (*Bivariate Polynomial Representing a Latin Square*) A bivariate polynomial $P(x, y)$ over \mathbb{Z}_n is said to represent (or generate) a Latin square if $(\mathbb{Z}_n, *)$ is a quasigroup where $*$: $\mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ is defined by $x * y = P(x, y)$ for all $x, y \in \mathbb{Z}_n$.

In 2001, Rivest [11] studied permutation polynomials (PPs) over the ring $(\mathbb{Z}_n, +, \cdot)$ where n is a power of 2: $n = 2^w$. This is based on the fact that modern computers perform computations modulo 2^w efficiently (where $w = 2, 8, 16, 32$ or 64 is the word size of the machine), and so it was of interest to study PPs modulo a power of 2. Below are some important results from his work.

Theorem 1. (*Rivest [11]*) A bivariate polynomial $P(x, y) = \sum_{i,j} a_{ij}x^i y^j$ represents a Latin square modulo $n = 2^w$, where $w \geq 2$, if and only if the four univariate polynomials $P(x, 0)$, $P(x, 1)$, $P(0, y)$, and $P(1, y)$ are all permutation polynomials modulo n .

Theorem 2. (*Rivest [11]*) There are no two polynomials $P_1(x, y)$, $P_2(x, y)$ modulo 2^w for $w \geq 1$ that form a pair of orthogonal Latin squares.

In 2009, Vadiraja and Shankar [14] motivated by the work of Rivest continued the study of permutation polynomials over the ring $(\mathbb{Z}_n, +, \cdot)$ by studying Latin squares represented by linear and quadratic bivariate polynomials over \mathbb{Z}_n when $n \neq 2^w$ with the characterization of some PPs. Some of the main results they got are stated below.

Theorem 3. (*Vadiraja and Shankar [14]*) A bivariate linear polynomial $a + bx + cy$ represents a Latin square over \mathbb{Z}_n , $n \neq 2^w$ if and only if one of the following equivalent conditions is satisfied:

(i) both b and c are coprime with n ;

(ii) $a + bx$, $a + cy$, $(a + c) + bx$ and $(a + b) + cy$ are all permutation polynomials modulo n .

(iii) b and c are invertible in (\mathbb{Z}_n, \cdot) .

Theorem 4. (Vadiraja and Shankar [14]) *If $P(x, y)$ is a bivariate polynomial having no cross term, then $P(x, y)$ gives a Latin square if and only if $P(x, 0)$ and $P(0, y)$ are permutation polynomials.*

Theorem 5. (Vadiraja and Shankar [14]) *Let n be even and $P(x, y) = f(x) + g(y) + xy$ be a bivariate quadratic polynomial, where $f(x)$ and $g(y)$ are permutation polynomials modulo n . Then $P(x, y)$ does not give a Latin square.*

The authors were able to establish the fact that Rivest's result for a bivariate polynomial over \mathbb{Z}_n when $n = 2^w$ is true for a linear-bivariate polynomial over \mathbb{Z}_n when $n \neq 2^w$. Although the result of Rivest was found not to be true for quadratic-bivariate polynomials over \mathbb{Z}_n when $n \neq 2^w$ with the help of counter examples, nevertheless some of such squares can be forced to be Latin squares by deleting some equal numbers of rows and columns.

Furthermore, Vadiraja and Shanhar [14] were able to find examples of pairs of orthogonal Latin squares generated by bivariate polynomials over \mathbb{Z}_n when $n \neq 2^w$ which was found impossible by Rivest for bivariate polynomials over \mathbb{Z}_n when $n = 2^w$.

The study of linear-bivariate polynomials that generate quasigroups over the ring \mathbb{Z}_n has furthered been explored in different perspectives by Jaiyéólá, Ilojide et. al. in [4, 8, 9, 5].

Theorem 6. (Theorem I.7.4, Pflugfelder [10]) *An equivalence class $G = K_g$ with respect to a normal equivalence relation θ is a subquasigroup if and only if $g\theta g^2$.*

2. MAIN RESULTS

2.1. Normal Congruences

Theorem 7. *Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n .*

(a) *Define \odot on $\mathbb{Z}_n \times \mathbb{Z}_n$ by $(x_1, y_1) \odot (x_2, y_2) = (P(x_1, x_2), P(y_1, y_2))$. Then $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$ is a quasigroup.*

(b) *Define \odot on $\mathbb{Z}_n \times \mathbb{Z}_n$ by $(x_1, y_1) \odot (x_2, y_2) = (P(x_1, y_2), P(x_2, y_1))$. Then $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$ is a quasigroup.*

Proof. (a) **Closure** Consider $(x_1, y_1) \odot (x_2, y_2) = (P(x_1, x_2), P(y_1, y_2)) \in (\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

Left Cancellation Law Let $(x, y) \odot (x_1, y_1) = (x, y) \odot (x_2, y_2)$. This implies $(P(x, x_1), P(y, y_1)) = (P(x, x_2), P(y, y_2))$ which implies $P(x, x_1) = P(x, x_2)$ and $P(y, y_1) = P(y, y_2)$ which imply $x_1 = x_2$ and $y_1 = y_2$ which imply $(x_1, y_1) = (x_2, y_2)$.

Right Cancellation Law Let $(x_1, y_1) \odot (x, y) = (x_2, y_2) \odot (x, y)$. This implies $(P(x_1, x), P(y_1, y)) = (P(x_2, x), P(y_2, y))$ which implies $P(x_1, x) = P(x_2, x)$ and $P(y_1, y) = P(y_2, y)$ which imply $x_1 = x_2$ and $y_1 = y_2$ which imply $(x_1, y_1) = (x_2, y_2)$.

We conclude that $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$ is a quasigroup.

(b) **Closure** Consider $(x_1, y_1) \odot (x_2, y_2) = (P(x_1, x_2), P(y_1, y_2)) \in (\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

Left Cancellation Law Let $(x, y) \odot (x_1, y_1) = (x, y) \odot (x_2, y_2)$. This implies $(P(x, y_1), P(x_1, y)) = (P(x, y_2), P(x_2, y))$ which implies $P(x, y_1) = P(x, y_2)$ and $P(x_1, y) = P(x_2, y)$ which imply $x_1 = x_2$ and $y_1 = y_2$ which imply $(x_1, y_1) = (x_2, y_2)$.

Right Cancellation Law Let $(x_1, y_1) \odot (x, y) = (x_2, y_2) \odot (x, y)$. This implies $(P(x_1, y), P(x, y_1)) = (P(x_2, y), P(x, y_2))$ which implies $P(x_1, y) = P(x_2, y)$ and $P(x, y_1) = P(x, y_2)$ which imply $x_1 = x_2$ and $y_1 = y_2$ which imply $(x_1, y_1) = (x_2, y_2)$.

We conclude that $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$ is a quasigroup.

Theorem 8. Let $\widetilde{P(x, y)} = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n . Define thosen relation (*thosen*) on $\mathbb{Z}_n \times \mathbb{Z}_n$ such that $(x_1, y_1)\widetilde{thosen}(x_2, y_2)$ if and only if $P(x_1, y_2) = P(x_2, y_1)$. Then

(a) \widetilde{thosen} is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

(b) if $b = c$, then \widetilde{thosen} is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

Proof. Reflexivity Clearly, $(x_1, y_1)\widetilde{thosen}(x_1, y_1)$ since $P(x_1, y_1) = P(x_1, y_1)$.

Symmetry Suppose $(x_1, y_1)\widetilde{thosen}(x_2, y_2)$. This implies that $P(x_1, y_2) = P(x_2, y_1)$ which implies that $P(x_2, y_1) = P(x_1, y_2)$. Thus, $(x_2, y_2)\widetilde{thosen}(x_1, y_1)$.

Transitivity Suppose $(x_1, y_1)\widetilde{thosen}(x_2, y_2)$ and $(x_2, y_2)\widetilde{thosen}(x_3, y_3)$. Then we have $P(x_1, y_2) = P(x_2, y_1)$ and $P(x_2, y_3) = P(x_3, y_2)$. These imply that $bx_1 + cy_3 - cy_1 - bx_3 = 0 \implies (x_1, y_1)\widetilde{thosen}(x_3, y_3)$. Hence, transitivity holds.

$\therefore \widetilde{thosen}$ is an equivalence relation over $\mathbb{Z}_n \times \mathbb{Z}_n$.

(a) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

(i) Assume that $(x_3, y_3) \odot (x_1, y_1) \widetilde{thosen}(x_3, y_3) \odot (x_2, y_2)$. This implies

$$\begin{aligned} \left(P(x_3, x_1), P(y_3, y_1) \right) \widetilde{thosen} \left(P(x_3, x_2), P(y_3, y_2) \right) &\iff \\ bcx_1 + c^2y_2 = bcx_2 + c^2y_1 &\quad (1) \end{aligned}$$

By the way,

$$(x_1, y_1) \widetilde{thosen}(x_2, y_2) \iff bcx_1 + c^2y_2 = bcx_2 + c^2y_1 \quad (2)$$

Equation 1 and Equation 2 are the same.

$$\therefore (x_3, y_3) \odot (x_1, y_1) \widetilde{thosen}(x_3, y_3) \odot (x_2, y_2) \implies (x_1, y_1) \widetilde{thosen}(x_2, y_2).$$

(ii) Assume that $(x_1, y_1) \odot (x_3, y_3) \widetilde{thosen}(x_2, y_2) \odot (x_3, y_3)$. This implies

$$\begin{aligned} \left(P(x_1, x_3), P(y_1, y_3) \right) \widetilde{thosen} \left(P(x_2, x_3), P(y_2, y_3) \right) &\iff \\ b^2x_1 + bcy_2 = b^2x_2 + bcy_1 &\quad (3) \end{aligned}$$

By the way,

$$(x_1, y_1) \widetilde{thosen}(x_2, y_2) \iff b^2x_1 + bcy_2 = b^2x_2 + bcy_1 \quad (4)$$

Equation 3 and Equation 4 are the same.

$$\therefore (x_1, y_1) \odot (x_3, y_3) \widetilde{thosen}(x_2, y_2) \odot (x_3, y_3) \implies (x_1, y_1) \widetilde{thosen}(x_2, y_2).$$

(iii) Suppose $(x_1, y_1) \widetilde{thosen}(x_2, y_2)$ and $(x_3, y_3) \widetilde{thosen}(x_4, y_4)$. These imply

$$\begin{aligned} P(x_1, y_2) = P(x_2, y_1) \text{ and } P(x_3, y_4) = P(x_4, y_3) &\iff \\ b^2x_1 + bcx_3 + bcy_2 + c^2y_4 - b^2x_2 - bcx_4 - bcy_1 - c^2y_3 = 0. &\quad (5) \end{aligned}$$

But,

$$\begin{aligned} (x_1, y_1) \odot (x_3, y_3) \widetilde{thosen}(x_2, y_2) \odot (x_4, y_4) &\iff \\ b^2x_1 + bcx_3 + bcy_2 + c^2y_4 - b^2x_2 - bcx_4 - bcy_1 - c^2y_3 = 0 &\quad (6) \end{aligned}$$

Equation 5 and Equation 6 are the same.

$$\begin{aligned} \therefore (x_1, y_1) \widetilde{thosen}(x_2, y_2) \text{ and } (x_3, y_3) \widetilde{thosen}(x_4, y_4) &\implies \\ (x_1, y_1) \odot (x_3, y_3) \widetilde{thosen}(x_2, y_2) \odot (x_4, y_4). &\end{aligned}$$

We therefore conclude that \widetilde{thosen} is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

(b) We have already shown that \widetilde{thosen} is an equivalence relation. It remains to show that if $b = c$, then \widetilde{thosen} satisfies the three conditions of a normal congruence relative to $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$. Now, let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

- (i) $(x_1, y_1) \odot (x_2, y_2) \widetilde{thosen} (x_1, y_1) \odot (x_3, y_3) \implies (P(x_1, y_2), P(x_2, y_1)) \widetilde{thosen} (P(x_1, y_3), P(x_3, y_1)) \iff P(P(x_1, y_2), P(x_3, y_1)) = P(P(x_1, y_3), P(x_2, y_1)) \iff P(a + bx_1 + cy_2, a + bx_3 + cy_1) = P(a + bx_1 + cy_3, a + bx_2 + cy_1) \iff a + b(a + bx_1 + cy_2) + c(a + bx_3 + cy_1) = a + b(a + bx_1 + cy_3) + c(a + bx_2 + cy_1) \iff bcx_3 + bcy_2 = bcx_2 + bcy_3$.
 $(x_2, y_2) \widetilde{thosen} (x_3, y_3) \iff P(x_2, y_3) = P(x_3, y_2) \iff a + bx_2 + cy_3 = a + bx_3 + cy_2 \iff bx_2 + cy_3 = bx_3 + cy_2$. Multiplying both sides by b gives $b^2x_2 + bcy_3 = b^2x_3 + bcy_2$. So, if $b = c$, $(x_1, y_1) \odot (x_2, y_2) \widetilde{thosen} (x_1, y_1) \odot (x_3, y_3) \implies (x_2, y_2) \widetilde{thosen} (x_3, y_3)$.
- (ii) $(x_2, y_2) \odot (x_1, y_1) \widetilde{thosen} (x_3, y_3) \odot (x_1, y_1) \implies (P(x_2, y_1), P(x_1, y_2)) \widetilde{thosen} (P(x_3, y_1), P(x_1, y_3)) \iff P(P(x_2, y_1), P(x_1, y_3)) = P(P(x_3, y_1), P(x_1, y_2)) \iff P(a + bx_2 + cy_1, a + bx_1 + cy_3) = P(a + bx_3 + cy_1, a + bx_1 + cy_2) \iff a + b(a + bx_2 + cy_1) + c(a + bx_1 + cy_3) = a + b(a + bx_3 + cy_1) + c(a + bx_1 + cy_2) \iff b^2x_2 + c^2y_3 = b^2x_3 + c^2y_2$.
 $(x_2, y_2) \widetilde{thosen} (x_3, y_3) \implies P(x_2, y_3) = P(x_3, y_2) \iff a + bx_2 + cy_3 = a + bx_3 + cy_2 \iff bx_2 + cy_3 = bx_3 + cy_2$. Multiplying both sides by b gives $b^2x_2 + bcy_3 = b^2x_3 + bcy_2$. So, if $b = c$, $(x_2, y_2) \odot (x_1, y_1) \widetilde{thosen} (x_3, y_3) \odot (x_1, y_1) \implies (x_2, y_2) \widetilde{thosen} (x_3, y_3)$.
- (iii) $(x_1, y_1) \widetilde{thosen} (x_2, y_2)$ and $(x_3, y_3) \widetilde{thosen} (x_4, y_4) \implies P(x_1, y_2) = P(x_2, y_1)$ and $P(x_3, y_4) = P(x_4, y_3) \iff a + bx_1 + cy_2 = a + bx_2 + cy_1$ and $a + bx_3 + cy_4 = a + bx_4 + cy_3 \iff bx_1 + cy_2 - bx_2 - cy_1 = 0$ and $bx_3 + cy_4 - bx_4 - cy_3 = 0 \implies bx_1 + cy_2 - bx_2 - cy_1 - bx_3 - cy_4 + bx_4 + cy_3 = 0$. Multiplying both sides by b gives $b^2x_1 + bcy_2 - b^2x_2 - bcy_1 - b^2x_3 - bcy_4 + b^2x_4 + bcy_3 = 0$.
 $(x_1, y_1) \odot (x_3, y_3) \widetilde{thosen} (x_2, y_2) \odot (x_4, y_4) \implies (P(x_1, y_3), P(x_3, y_1)) \widetilde{thosen} (P(x_2, y_4), P(x_4, y_2)) \iff P(P(x_1, y_3), P(x_4, y_2)) =$

$$\begin{aligned}
 P\left(P(x_2, y_4), P(x_3, y_1)\right) &\iff P(a + bx_1 + cy_3, a + bx_4 + cy_2) = P(a + \\
 &bx_2 + cy_4, a + bx_3 + cy_1) \iff a + b(a + bx_1 + cy_3) + c(a + bx_4 + cy_2) = \\
 &a + b(a + bx_2 + cy_4) + c(a + bx_3 + cy_1) \iff b^2x_1 + bcy_3 + bcx_4 + c^2y_2 - \\
 &b^2x_2 - bcy_4 - bcx_3 - c^2y_1 = 0. \text{ So, if } b = c, \widetilde{(x_1, y_1)thosen}(x_2, y_2) \text{ and} \\
 &\widetilde{(x_3, y_3)thosen}(x_4, y_4) \implies (x_1, y_1) \odot (x_3, y_3)thosen(x_2, y_2) \odot (x_4, y_4).
 \end{aligned}$$

$\therefore \widetilde{thosen}$ is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

Theorem 9. Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n . Define \sim on $\mathbb{Z}_n \times \mathbb{Z}_n$ such that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $P(x_1, y_1) = P(x_2, y_2)$. Then

(a) \sim is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

(b) if $b = c$, then \sim is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

Proof. Reflexivity Clearly, $(x_1, y_1) \sim (x_1, y_1)$ since $P(x_1, y_1) = P(x_1, y_1)$.

Symmetry Suppose $(x_1, y_1) \sim (x_2, y_2)$. This implies $P(x_1, y_1) = P(x_2, y_2)$ which implies $P(x_2, y_2) = P(x_1, y_1)$ which implies $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity Suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. Then we have $P(x_1, y_1) = P(x_2, y_2)$ and $P(x_2, y_2) = P(x_3, y_3)$. These imply that $bx_1 + cy_1 - bx_2 - cy_2 = 0$ and $bx_2 + cy_2 - bx_3 - cy_3 = 0$. Also, $(x_1, y_1) \sim (x_3, y_3)$ gives $bx_1 + cy_1 - bx_3 - cy_3 = 0$. Hence, transitivity holds.

(a) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

(i) Assume that $(x_3, y_3) \odot (x_1, y_1) \sim (x_3, y_3) \odot (x_2, y_2)$. This implies

$$\begin{aligned}
 \left(P(x_3, x_1), P(y_3, y_1)\right) &\sim \left(P(x_3, x_2), P(y_3, y_2)\right) \iff \\
 bcx_1 + c^2y_1 &= bcx_2 + c^2y_2. \tag{7}
 \end{aligned}$$

$$(x_1, y_1) \sim (x_2, y_2) \iff bcx_1 + c^2y_1 = bcx_2 + c^2y_2. \tag{8}$$

Equation 7 and Equation 8 are the same. $\therefore (x_3, y_3) \odot (x_1, y_1) \sim (x_3, y_3) \odot (x_2, y_2) \implies (x_1, y_1) \sim (x_2, y_2)$.

(ii) Assume that $(x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_3, y_3)$ which implies

$$\left(P(x_1, x_3), P(y_1, y_3)\right) \sim \left(P(x_2, x_3), P(y_2, y_3)\right) \iff$$

$$b^2x_1 + bcy_1 = b^2x_2 + bcy_2. \quad (9)$$

$$(x_1, y_1) \sim (x_2, y_2) \iff b^2x_1 + bcy_1 = b^2x_2 + bcy_2. \quad (10)$$

Equation 9 and Equation 10 are the same. $\therefore (x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_3, y_3) \implies (x_1, y_1) \sim (x_2, y_2)$.

(iii) Suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_3, y_3) \sim (x_4, y_4)$. This implies $P(x_1, y_1) = P(x_2, y_2)$ and $P(x_3, y_3) = P(x_4, y_4)$ which imply

$$b^2x_1 + bcx_3 + bcy_1 + c^2y_3 - b^2x_2 - bcx_4 - bcy_2 - c^2y_4 = 0 \quad (11)$$

$$(x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_4, y_4) \iff$$

$$b^2x_1 + bcx_3 + bcy_1 + c^2y_3 - b^2x_2 - bcx_4 - bcy_2 - c^2y_4 = 0 \quad (12)$$

Equation 11 and Equation 12 are the same. $\therefore (x_1, y_1) \sim (x_2, y_2)$ and $(x_3, y_3) \sim (x_4, y_4) \implies (x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_4, y_4)$.

We therefore conclude that \sim is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

(b) We have already shown that \sim is an equivalence relation. It remains to show that if $b = c$, then \sim satisfies the three conditions of a normal congruence relative to $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in (\mathbb{Z}_n \times \mathbb{Z}_n)$.

(i) $(x_1, y_1) \odot (x_2, y_2) \sim (x_1, y_1) \odot (x_3, y_3) \implies$

$$\left(P(x_1, y_2), P(x_2, y_1) \right) \sim \left(P(x_1, y_3), P(x_3, y_1) \right) \iff$$

$$P\left(P(x_1, y_2), P(x_2, y_1) \right) = P\left(P(x_1, y_3), P(x_3, y_1) \right) \iff P(a+bx_1+cy_2, a+bx_2+cy_1) = P(a+bx_1+cy_3, a+bx_3+cy_1) \iff a+b(a+bx_1+cy_2)+c(a+bx_2+cy_1) = a+b(a+bx_1+cy_3)+c(a+bx_3+cy_1) \iff$$

$$bcx_2 + bcy_2 = bcx_3 + bcy_3 \quad (13)$$

$$(x_2, y_2) \sim (x_3, y_3) \iff P(x_2, y_2) = P(x_3, y_3) \iff a + bx_2 + cy_2 = a + bx_3 + cy_3 \iff bx_2 + cy_2 = bx_3 + cy_3. \text{ Multiplying both sides by } b \text{ gives}$$

$$b^2x_2 + bcy_2 = b^2x_3 + bcy_3 \quad (14)$$

So, if $b = c$, Equation 13 and Equation 14 are the same. $\therefore (x_1, y_1) \odot (x_2, y_2) \sim (x_1, y_1) \odot (x_3, y_3) \implies (x_2, y_2) \sim (x_3, y_3)$.

$$\begin{aligned}
 \text{(ii)} \quad (x_2, y_2) \odot (x_1, y_1) \sim (x_3, y_3) \odot (x_1, y_1) &\implies \\
 \left(P(x_2, y_1), P(x_1, y_2) \right) \sim \left(P(x_3, y_1), P(x_1, y_3) \right) &\iff \\
 P\left(P(x_2, y_1), P(x_1, y_2) \right) = P\left(P(x_3, y_1), P(x_1, y_3) \right) &\iff P(a+bx_2+cy_1, a+ \\
 bx_1+cy_2) = P(a+bx_3+cy_1, a+bx_1+cy_3) &\iff a+b(a+bx_2+cy_1)+ \\
 c(a+bx_1+cy_2) = a+b(a+bx_3+cy_1)+c(a+bx_1+cy_3) &\iff \\
 b^2x_2+c^2y_2 = b^2x_3+c^2y_3. & \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 (x_2, y_2) \sim (x_3, y_3) &\implies P(x_2, y_2) = P(x_3, y_3) \iff a+bx_2+cy_2 = a+ \\
 bx_3+cy_3 &\iff bx_2+cy_2 = bx_3+cy_3. \text{ Multiplying both sides by } b \text{ gives}
 \end{aligned}$$

$$b^2x_2+bcy_2 = b^2x_3+bcy_3. \tag{16}$$

So, if $b = c$, Equation 15 and Equation 16 are the same. $\therefore (x_2, y_2) \odot (x_1, y_1) \sim (x_3, y_3) \odot (x_1, y_1) \implies (x_2, y_2) \sim (x_3, y_3)$.

$$\begin{aligned}
 \text{(iii)} \quad (x_1, y_1) \sim (x_2, y_2) \text{ and } (x_3, y_3) \sim (x_4, y_4) &\implies P(x_1, y_1) = P(x_2, y_2) \text{ and} \\
 P(x_3, y_3) = P(x_4, y_4) &\iff a+bx_1+cy_1 = a+bx_2+cy_2 \text{ and } a+bx_3+cy_3 = \\
 a+bx_4+cy_4 &\iff bx_1+cy_1 - bx_2 - cy_2 = 0 \text{ and } bx_3+cy_3 - bx_4 - cy_4 = \\
 0 &\implies bx_1+cy_1 - bx_2 - cy_2 - bx_3 - cy_3 + bx_4 + cy_4 = 0. \text{ Multiplying both} \\
 \text{sides by } b &\text{ gives}
 \end{aligned}$$

$$b^2x_1+bcy_1 - b^2x_2 - bcy_2 - b^2x_3 - bcy_3 + b^2x_4 + bcy_4 = 0 \tag{17}$$

$$\begin{aligned}
 (x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_4, y_4) &\implies \\
 [P(x_1, y_3), P(x_3, y_1)] \sim [P(x_2, y_4), P(x_4, y_2)] &\iff P[P(x_1, y_3), P(x_3, y_1)] = \\
 P[P(x_2, y_4), P(x_4, y_2)] &\iff P(a+bx_1+cy_3, a+bx_3+cy_1) = P(a+ \\
 bx_2+cy_4, a+bx_4+cy_2) &\iff a+b(a+bx_1+cy_3)+c(a+bx_3+cy_1) = \\
 a+b(a+bx_2+cy_4)+c(a+bx_4+cy_2) &\iff
 \end{aligned}$$

$$b^2x_1+bcy_3+bcx_3+c^2y_1 - b^2x_2 - bcy_4 - bcx_4 - c^2y_2 = 0 \tag{18}$$

So, if $b = c$, Equation 17 and Equation 18 are the same. Thus, $(x_1, y_1) \sim (x_2, y_2)$ and $(x_3, y_3) \sim (x_4, y_4) \implies (x_1, y_1) \odot (x_3, y_3) \sim (x_2, y_2) \odot (x_4, y_4)$.

$\therefore \sim$ is a normal congruence over $(\mathbb{Z}_n \times \mathbb{Z}_n, \odot)$.

2.2. Quotient Quasigroups

Theorem 10. Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n .

- (a) If $\mathbb{Z}_n^2/\widetilde{\text{thosen}} = \{[K_z]\}_{z \in \mathbb{Z}_n^2}$ and for all $K_{z_1}, K_{z_2} \in \mathbb{Z}_n^2/\widetilde{\text{thosen}}$, $*$ is defined on $\mathbb{Z}_n^2/\widetilde{\text{thosen}}$ as $K_{z_1} * K_{z_2} = K_{z_1 \odot z_2}$, then $(\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$ is a quasigroup.
- (b) $\widetilde{\text{thosen}}$ induces an homomorphism from (\mathbb{Z}_n^2, \odot) to $(\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$.

Proof. (a) **Left Cancellation Law** Let $K_{z_1} * K_{z_2} = K_{z_1} * K_{z_3}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_1, y_1)} * K_{(x_2, y_2)} &= K_{(x_1, y_1)} * K_{(x_3, y_3)} \implies \\ K_{(x_1, y_1) \odot (x_2, y_2)} &= K_{(x_1, y_1) \odot (x_3, y_3)} \implies \\ (x_1, y_1) \odot (x_2, y_2) \widetilde{\text{thosen}} &= (x_1, y_1) \odot (x_3, y_3) \widetilde{\text{thosen}} \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

Right Cancellation Law Let $K_{z_2} * K_{z_1} = K_{z_3} * K_{z_1}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_2, y_2)} * K_{(x_1, y_1)} &= K_{(x_3, y_3)} * K_{(x_1, y_1)} \implies \\ K_{(x_2, y_2) \odot (x_1, y_1)} &= K_{(x_3, y_3) \odot (x_1, y_1)} \implies \\ (x_2, y_2) \odot (x_1, y_1) \widetilde{\text{thosen}} &= (x_3, y_3) \odot (x_1, y_1) \widetilde{\text{thosen}} \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

We conclude that $(\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$ is a quasigroup.

- (b) Define $\alpha : (\mathbb{Z}_n^2, \odot) \longrightarrow (\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$ by $\alpha[(x, y)] = K_{(x, y)}$. Consider

$$\alpha[(x_1, y_1) \odot (x_2, y_2)] = K_{(x_1, y_1) \odot (x_2, y_2)} = K_{(x_1, y_1)} * K_{(x_2, y_2)} = \alpha[(x_1, y_1)] * \alpha[(x_2, y_2)].$$

Thus, α is an homomorphism.

Theorem 11. Let $P(x, y) = a + bx + by$ represent a quasigroup over \mathbb{Z}_n .

- (a) If $\mathbb{Z}_n^2/\widetilde{\text{thosen}} = \{[K_z]\}_{z \in \mathbb{Z}_n^2}$ and for all $K_{z_1}, K_{z_2} \in \mathbb{Z}_n^2/\widetilde{\text{thosen}}$, $*$ is defined on $\mathbb{Z}_n^2/\widetilde{\text{thosen}}$ as $K_{z_1} * K_{z_2} = K_{z_1 \odot z_2}$, then $(\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$ is a quasigroup.
- (b) $\widetilde{\text{thosen}}$ induces an homomorphism from (\mathbb{Z}_n^2, \odot) to $(\mathbb{Z}_n^2/\widetilde{\text{thosen}}, *)$.

Proof. **(a) Left Cancellation Law** Let $K_{z_1} * K_{z_2} = K_{z_1} * K_{z_3}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_1, y_1)} * K_{(x_2, y_2)} &= K_{(x_1, y_1)} * K_{(x_3, y_3)} \implies \\ K_{(x_1, y_1) \odot (x_2, y_2)} &= K_{(x_1, y_1) \odot (x_3, y_3)} \implies \\ (x_1, y_1) \odot (x_2, y_2) \widetilde{thosen} &(x_1, y_1) \odot (x_3, y_3) \implies (x_2, y_2) \widetilde{thosen} (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

Right Cancellation Law Let $K_{z_2} * K_{z_1} = K_{z_3} * K_{z_1}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_2, y_2)} * K_{(x_1, y_1)} &= K_{(x_3, y_3)} * K_{(x_1, y_1)} \implies \\ K_{(x_2, y_2) \odot (x_1, y_1)} &= K_{(x_3, y_3) \odot (x_1, y_1)} \implies \\ (x_2, y_2) \odot (x_1, y_1) \widetilde{thosen} &(x_3, y_3) \odot (x_1, y_1) \implies (x_2, y_2) \widetilde{thosen} (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

We conclude that $(\mathbb{Z}_n^2 / \widetilde{thosen}, *)$ is a quasigroup.

(b) Define $\alpha : (\mathbb{Z}_n^2, \odot) \rightarrow (\mathbb{Z}_n^2 / \widetilde{thosen}, *)$ by $\alpha[(x, y)] = K_{(x, y)}$. Consider

$$\alpha[(x_1, y_1) \odot (x_2, y_2)] = K_{(x_1, y_1) \odot (x_2, y_2)} = K_{(x_1, y_1)} * K_{(x_2, y_2)} = \alpha[(x_1, y_1)] * \alpha[(x_2, y_2)].$$

Thus, α is an homomorphism.

Theorem 12. Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n .

(a) If $\mathbb{Z}_n^2 / \sim = \{[K_z]\}_{z \in \mathbb{Z}_n^2}$ and for all $K_{z_1}, K_{z_2} \in \mathbb{Z}_n^2 / \sim$, $*$ is defined on \mathbb{Z}_n^2 / \sim as $K_{z_1} * K_{z_2} = K_{z_1 \odot z_2}$, then $(\mathbb{Z}_n^2 / \sim, *)$ is a quasigroup.

(b) \sim induces an homomorphism from (\mathbb{Z}_n^2, \odot) to $(\mathbb{Z}_n^2 / \sim, *)$.

Proof. **(a) Left Cancellation Law** Let $K_{z_1} * K_{z_2} = K_{z_1} * K_{z_3}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_1, y_1)} * K_{(x_2, y_2)} &= K_{(x_1, y_1)} * K_{(x_3, y_3)} \implies \\ K_{(x_1, y_1) \odot (x_2, y_2)} &= K_{(x_1, y_1) \odot (x_3, y_3)} \implies \\ (x_1, y_1) \odot (x_2, y_2) \sim &(x_1, y_1) \odot (x_3, y_3) \implies (x_2, y_2) \sim (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

Right Cancellation Law Let $K_{z_2} * K_{z_1} = K_{z_3} * K_{z_1}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_2, y_2)} * K_{(x_1, y_1)} &= K_{(x_3, y_3)} * K_{(x_1, y_1)} \implies \\ K_{(x_2, y_2) \odot (x_1, y_1)} &= K_{(x_3, y_3) \odot (x_1, y_1)} \implies \\ (x_2, y_2) \odot (x_1, y_1) &\sim (x_3, y_3) \odot (x_1, y_1) \implies (x_2, y_2) \sim (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

We conclude that $(\mathbb{Z}_n^2 / \sim, *)$ is a quasigroup.

(b) Define $\alpha : (\mathbb{Z}_n^2, \odot) \rightarrow (\mathbb{Z}_n^2 / \sim, *)$ by $\alpha[(x, y)] = K_{(x, y)}$. Consider

$$\alpha[(x_1, y_1) \odot (x_2, y_2)] = K_{(x_1, y_1) \odot (x_2, y_2)} = K_{(x_1, y_1)} * K_{(x_2, y_2)} = \alpha[(x_1, y_1)] * \alpha[(x_2, y_2)].$$

Thus, α is an homomorphism.

Theorem 13. Let $P(x, y) = a + bx + by$ represent a quasigroup over \mathbb{Z}_n .

(a) If $\mathbb{Z}_n^2 / \sim = \{[K_z]\}_{z \in \mathbb{Z}_n^2}$ and for all $K_{z_1}, K_{z_2} \in \mathbb{Z}_n^2 / \sim$, $*$ is defined on \mathbb{Z}_n^2 / \sim as $K_{z_1} * K_{z_2} = K_{z_1 \odot z_2}$, then $(\mathbb{Z}_n^2 / \sim, *)$ is a quasigroup.

(b) \sim induces an homomorphism from (\mathbb{Z}_n^2, \odot) to $(\mathbb{Z}_n^2 / \sim, *)$.

Proof. (a) **Left Cancellation Law** Let $K_{z_1} * K_{z_2} = K_{z_1} * K_{z_3}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_1, y_1)} * K_{(x_2, y_2)} &= K_{(x_1, y_1)} * K_{(x_3, y_3)} \implies \\ K_{(x_1, y_1) \odot (x_2, y_2)} &= K_{(x_1, y_1) \odot (x_3, y_3)} \implies \\ (x_1, y_1) \odot (x_2, y_2) &\sim (x_1, y_1) \odot (x_3, y_3) \implies (x_2, y_2) \sim (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

Right Cancellation Law Let $K_{z_2} * K_{z_1} = K_{z_3} * K_{z_1}$, where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$. This implies

$$\begin{aligned} K_{(x_2, y_2)} * K_{(x_1, y_1)} &= K_{(x_3, y_3)} * K_{(x_1, y_1)} \implies \\ K_{(x_2, y_2) \odot (x_1, y_1)} &= K_{(x_3, y_3) \odot (x_1, y_1)} \implies \\ (x_2, y_2) \odot (x_1, y_1) &\sim (x_3, y_3) \odot (x_1, y_1) \implies (x_2, y_2) \sim (x_3, y_3) \implies \\ K_{(x_2, y_2)} &= K_{(x_3, y_3)} \implies K_{z_2} = K_{z_3}. \end{aligned}$$

We conclude that $(\mathbb{Z}_n^2 / \sim, *)$ is a quasigroup.

(b) Define $\alpha : (\mathbb{Z}_n^2, \odot) \rightarrow (\mathbb{Z}_n^2 / \sim, *)$ by $\alpha[(x, y)] = K_{(x,y)}$. Consider

$$\alpha[(x_1, y_1) \odot (x_2, y_2)] = K_{(x_1, y_1) \odot (x_2, y_2)} = K_{(x_1, y_1)} * K_{(x_2, y_2)} = \alpha[(x_1, y_1)] * \alpha[(x_2, y_2)].$$

Thus, α is an homomorphism.

Theorem 14. Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n .

(a) $K_{(x,y)} \in (\mathbb{Z}_n^2 / \sim, *)$ is a subquasigroup of $(\mathbb{Z}_n^2, \odot) \iff b(x - y) + c(P(x, x) - P(y, y)) = 0$.

(b) $K_{(x,x)} = \{(y, y)\}_{y \in \mathbb{Z}_n}$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) .

(c) $(b + cb + c^2) = 0$ if and only if $K_{(x,y)} \in (\mathbb{Z}_n^2 / \sim, *)$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) .

Proof. (a) By Theorem 6, it suffices to show that $(x, y) \sim (x, y)^2$ i.e. $(x, y) \sim (x, y) \odot (x, y)$. This implies $(x, y) \sim (P(x, x), P(y, y)) \iff P(x, P(x, x)) = P(y, P(y, y)) \iff a + bx + cP(x, x) = a + by + cP(y, y) \iff b(x - y) + c[P(x, x) - P(y, y)] = 0$ as required. Moreover,

(b) $K_{(x,x)} = \{(y, z) | (x, x) \sim (y, z)\} = \{(y, z) | P(x, y) = P(x, z)\} = \{(y, z) | a + bx + cy = a + bx + cz\} = \{(y, z) | cy = cz\} = \{(y, z) | y = z\} = \{(y, y)\}_{y \in \mathbb{Z}_n}$.

(c) $b(x - y) + c[P(x, x) - P(y, y)] = 0 \iff b(x - y) + c[a + bx + cx - (a + by + cy)] = 0 \iff (x - y)[b + bc + c^2] = 0$.

Theorem 15. Let $P(x, y) = a + bx + by$ represent a quasigroup over \mathbb{Z}_n .

(a) $K_{(x,y)} \in (\mathbb{Z}_n^2 / \sim, *)$ is a subquasigroup of $(\mathbb{Z}_n^2, \odot) \iff x = y$.

(b) $K_{(x,x)} = \{(y, y)\}_{y \in \mathbb{Z}_n}$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) .

Proof. (a) By Theorem 6, it suffices to show that $(x, y) \sim (x, y)^2$ i.e. $(x, y) \sim (x, y) \odot (x, y)$. This implies $(x, y) \sim (P(x, y), P(x, y)) \iff P(x, P(x, y)) = P(y, P(x, y)) \iff a + bx + cP(x, y) = a + by + cP(x, y) \iff x = y$ as required.

(b) $K_{(x,x)} = \{(y, z) | (x, x) \sim (y, z)\} = \{(y, z) | P(x, y) = P(x, z)\} = \{(y, z) | a + bx + cy = a + bx + cz\} = \{(y, z) | cy = cz\} = \{(y, z) | y = z\} = \{(y, y)\}_{y \in \mathbb{Z}_n}$.

Theorem 16. Let $P(x, y) = a + bx + cy$ represent a quasigroup over \mathbb{Z}_n .

- (a) $K_{(x,y)} \in (\mathbb{Z}_n^2/\widetilde{thosen}, *)$ is a subquasigroup of $(\mathbb{Z}_n^2, \odot) \iff b[x - P(x, x)] = c[y - P(y, y)]$.
- (b) If $P(x, x) = x + c$ and $P(y, y) = b + y$, then $K_{(x,y)} \in \mathbb{Z}_n^2/\widetilde{thosen}$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) .
- (c) If $P(x, x) = x$ and $P(y, y) = y$, then $K_{(x,y)} \in (\mathbb{Z}_n^2/\sim, *)$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) . Conversely, if $K_{(x,y)} \in (\mathbb{Z}_n^2/\widetilde{thosen}, \setminus *)$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) , then, $P(x, x) = x \iff P(y, y) = b$.

Proof. (a) By Theorem 6, it suffices to show that $(x, y)\widetilde{thosen}(x, y)^2$ i.e.

$$(x, y)\widetilde{thosen}(x, y) \odot (x, y). \text{ This implies } (x, y)\widetilde{thosen}(P(x, x), P(y, y)) \iff P(x, P(y, y)) = P(P(x, x), y) \iff a + bx + cP(y, y) = a + bP(x, x) + cy \iff b[x - P(x, x)] = c[y - P(y, y)] \text{ as required. Moreover,}$$

(b) If $P(x, x) = x + c$ and $P(y, y) = b + y$, the last equation is satisfied.

(c) If $P(x, x) = x$ and $P(y, y) = y$, the last equation is also satisfied.

Theorem 17. Let $P(x, y) = a + bx + by$ represent a quasigroup over \mathbb{Z}_n .

- (a) $K_{(x,y)} \in (\mathbb{Z}_n^2/\widetilde{thosen}, *)$ is a subquasigroup of $(\mathbb{Z}_n^2, \odot) \iff x = y$.
- (b) $K_{(x,x)} = \{(y, y)\}_{y \in \mathbb{Z}_n}$ is a subquasigroup of (\mathbb{Z}_n^2, \odot) .

Proof. (a) By Theorem 6, it suffices to show that $(x, y)\widetilde{thosen}(x, y)^2$ i.e.

$$(x, y)\widetilde{thosen}(x, y) \odot (x, y). \text{ This implies } (x, y)\widetilde{thosen}(P(x, y), P(x, y)) \iff P(x, P(x, y)) = P(P(x, y), y) \iff a + bx + cP(x, y) = a + bP(x, y) + cy \iff b[x - P(x, y)] - c[y - P(x, y)] = 0 \iff x = y \text{ as required.}$$

- (b) $K_{(x,x)} = \{(y, z)|(x, x)\widetilde{thosen}(y, z)\} = \{(y, z)|P(x, z) = P(y, x)\} = \{(y, z)|a + bx + cz = a + by + cx\} = \{(y, z)|b(x - y) = b(x - z)\} = \{(y, z)|y = z\} = \{(y, y)\}_{y \in \mathbb{Z}_n}$.

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