The Complete Graph: Eigenvalues, Trigonometrical Unit-Equations with associated *t*-Complete-Eigen Sequences, Ratios, Sums and Diagrams

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Abstract

The complete graph is often used to verify certain graph theoretical definitions and applications. Regarding the adjacency matrix, associated with the complete graph, as a circulant matrix, we find its eigenvalues, and use this result to generate a trigonometrical unit-equations involving the sum of terms of the form $\cos[\pi a/(2t+1)];t=1,2,\ldots$, where a is odd. This gives rise to*t*-complete-eigen sequences anddiagrams, similar to the famous Farey sequence and diagram. We showthat the ratio, involving sum of the terms of the*t*-complete eigen sequence, converges to ½ , and use this ratio to find the *t*-complete eigen area. To find the eigenvalues, associated with the characteristic polynomial of complete graph, using induction, we create a general determinant equation involving the minor of the matrix associated with this characteristic polynomial.

Key words: complete graph, trigonometrical equations, eigenvalues, sequences

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1. Introduction

We use the graph-theoretical notation of Harris et. al.

Often, when a new graph-theoretical definition is introduced, the definition is tested on the complete graph. For example, a complete graph on n vertices has a minimum vertex covering consisting of any set of n-1 vertices. The number of spanning trees is well known, so is its chromatic number, radius and diameter etc. The eigenvalues of the adjacency matrix associated with the complete graph is also easy to compute (see Brouwer and Haemers, for example). They are n-1 and -1.

Considering the adjacency matrix of the complete graph as a circulant matrix, we find its eigenvalues in terms of sine and cosine. Using the cosine part and n odd, and the fact that -1 is aneigenvalue, we generatetrigonometrical-unit equations:

$$2\sum_{r=1}^{t}\cos\left(\frac{\pi(2t-2r+1)}{2t+1}\right)=1;\ \ t=1,2,3.\ldots$$

These equation resulted in t-complete-eigen sequences and, using unit mirror pairs and diagrams, similar to that of the famous Farey sequence and diagram. We show that the ratio, involving the sum of the terms of the t-complete eigen sequence, converges to ½ and evaluate area using this ratio.

There are many known methods available to find the eigenvalues associated with the complete graph (see Jessop). Some methods are short, others are long but mathematically interesting. Although the induction method can be regarded as laborious, it illustrates the variety of certain combinatorial aspects associated with the determinants, involved with the characteristic polynomial associated with the matrix of the complete graph, which we demonstrate in the theorem in section 3 below.

2. Eigenvalues of the complete graph from a circulant matrix-eigen sequences

Considering the matrix of the complete graph as a circulant matrix, we find its eigenvalues to create trigonometrical unit-equations. The results of the following Lemmas can be found in Jessop.

Lemma 2.1

Let $A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$

be a (nxn)circulant matrix.

Then the eigenvectors of the circulant matrix A are given by:

$$\underline{v_j} = \left(1, \rho_j, \rho_j^{\,2}, \ldots, \rho_j^{\,n-1}\right)^T, \quad j = 0,1,\ldots, n-1$$

where $\rho_j = \exp\left(\frac{2\pi ij}{n}\right)$ are the nth roots of unity and $i = \sqrt{-1}$.

The corresponding eigenvalues are then given by

$$\lambda_j = a_0 + a_1\rho_j + a_2\rho_j^{\,2} + \ldots + a_{n-1}\rho_j^{\,n-1},\; j = 0,\ldots, n-1.$$

Lemma 2.2

Let $A(K_n)$ be the adjacency matrix of the complete graph K_n on n vertices.

Then $A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{nxn}$,

and its eigenvalues are, for all j, where $0 \le j \le n-1$,

$$\begin{aligned}\lambda_j &= e^{\frac{2\pi ij}{n}} + e^{\frac{4\pi ij}{n}} + \ldots + e^{\frac{2(n-1)\pi ij}{n}} \\ &= \cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{4\pi j}{n}\right) + i\sin\left(\frac{4\pi j}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi j}{n}\right) + i\sin\left(\frac{2(n-1)\pi j}{n}\right) \\ &= \sum_{k=1}^{n-1}\cos\left(\frac{2\pi jk}{n}\right) + i\sum_{k=1}^{n-1}\sin\left(\frac{2\pi jk}{n}\right) \qquad (2.1)\end{aligned}$$

Using the above Lemma and the fact that the eigenvalues of the adjacency matrix associated with the complete graph are $(n-1)$(once) and -1(multiplicity$(n-1)$) we have the following theorem:

Theorem 2.1

$$2\left(\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right)=2\sum_{r=1}^{t}\cos\left(\frac{\pi(2t-2r+1)}{2t-1}\right)=1;\ t=1,2,..$$

Proof

For $j=0$ the above lemma yields the eigenvalue $n-1$.Thus for $j\neq 0$ the eigenvalues are -1.

So, for $j\neq 0$,

$$\lambda_j \quad =\sum_{k=1}^{n-1}\cos\left(\frac{2\pi jk}{n}\right)+i\sum_{k=1}^{n-1}\sin\left(\frac{2\pi jk}{n}\right)=-1$$

Now, for $j\neq 0$, we consider $n=2t+1$.

$$\begin{aligned}\sum_{k=1}^{n-1}\sin\left(\frac{2\pi jk}{n}\right) &=\sum_{k=1}^{2t}\sin\left(\frac{2\pi jk}{2t+1}\right)\\ &=\sin\left(\frac{2\pi j}{2t+1}\right)+\sin\left(\frac{4\pi j}{2t+1}\right)+...+\sin\left(\frac{4t\pi j}{2t+1}\right)\\ &=\left[\sin\left(\frac{2\pi j}{2t+1}\right)+\ldots+\sin\left(\frac{2t\pi j}{2t+1}\right)\right]+\left[\sin\left(\frac{2(t+1)\pi j}{2t+1}\right)+\ldots+\sin\left(\frac{4t\pi j}{2t+1}\right)\right]\\ &=[A]+[B]\end{aligned}$$

where A has the first t terms and B the next t terms. Adding the first term of A and the last term of B yield:

$$\begin{aligned}&\sin\left(\frac{2\pi}{2t+1}\right)+\sin\left(\frac{4\pi t}{2t+1}\right)\\ &=\sin\left(\frac{(2t+1)\pi}{2t+1}-\frac{(2t-1)\pi}{2t+1}\right)+\sin\left(\frac{(2t+1)\pi}{2t+1}+\frac{(2t-1)\pi}{2t+1}\right)\\ &=\sin\left(\frac{(2t-1)\pi}{2t+1}\right)-\sin\left(\frac{(2t-1)\pi}{2t+1}\right)=0\end{aligned}$$

Generally, adding the $r\,th$ term of A and the $t-(r-1)th$ term of B, where $r=1,2,\ldots,t$, yields

$$\sin\left(\frac{2\pi r}{2t+1}\right)+\sin\left(\frac{2\pi(2t-(r-1)}{2t+1}\right)$$

$$= \sin\left(\frac{\pi(2t+1)}{2t+1} - \frac{(2t-2r+1)\pi}{2t+1}\right) + \sin\left(\frac{\pi(2t+1)}{2t+1} + \frac{\pi(2t-2r+1)}{2t+1}\right)$$
$$= \sin\left(\frac{(2t-2r+1)\pi}{2t+1}\right) - \sin\left(\frac{(2t-2r+1)\pi}{2t+1}\right) = 0; \quad r = 1,2,...,t.$$

Therefore,for $j \neq 0$, $\sum_{k=1}^{n-1} \sin\left(\frac{2\pi jk}{n}\right) = 0$

and then

$$\lambda_j = \sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{2t}\right) + i\sum_{k=1}^{2t-1} \sin\left(\frac{2\pi jk}{2t}\right) = \sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{2t}\right) = -1.$$

Now,

$$\sum_{k=1}^{2t-1} \cos\left(\frac{2\pi jk}{2t}\right)$$
$$= \left[\cos\left(\frac{2\pi}{2t+1}\right) + \cos\left(\frac{4\pi}{2t+1}\right) + \ldots + \cos\left(\frac{2\pi t}{2t+1}\right)\right] + \left[\cos\left(\frac{2\pi(t+1)}{2t+1}\right) + \cos\left(\frac{2\pi(t+2)}{2t+1}\right) + \ldots + \cos\left(\frac{4\pi t}{2t+1}\right)\right]$$
$$= A + B$$

Ahas the first t terms and B the next t terms. Adding the first term of A and the last term of B yield:

$$\cos\left(\frac{2\pi}{2t+1}\right) + \cos\left(\frac{4\pi t}{2t+1}\right)$$
$$= \cos\left(\frac{(2t+1)\pi}{2t+1} - \frac{(2t-1)\pi}{2t+1}\right) + \cos\left(\frac{(2t+1)\pi}{2t+1} + \frac{(2t-1)\pi}{2t+1}\right)$$
$$= 2\cos\left(\frac{(2t+1)\pi}{2t+1}\right)\cos\left(\frac{(2t-1)\pi}{2t+1}\right)$$
$$= -2\cos\left(\frac{(2t-1)\pi}{2t+1}\right)$$

The t-*th* term of A and the first term of B yield:

$$\cos\left(\frac{2t\pi}{2t+1}\right) + \cos\left(\frac{2\pi(t+1)}{2t+1}\right)$$
$$= \cos\left(\frac{(2t+1)\pi}{2t+1} - \frac{\pi}{2t+1}\right) + \cos\left(\frac{(2t+1)\pi}{2t+1} + \frac{\pi}{2t+1}\right)$$
$$= -2\cos\left(\frac{\pi}{2t+1}\right).$$

Adding the second term of A and the second to last term of B:

$$\cos\left(\frac{2\pi 2}{2t+1}\right)+\cos\left(\frac{2\pi\left(2t-(2-1)\right)}{2t+1}\right)$$

$$=\cos\left(\frac{2\pi 2}{2t+1}\right)+\cos\left(\frac{2\pi(2t-1)}{2t+1}\right)$$

$$=\cos\left(\frac{(2t+1)\pi}{2t+1}-\frac{(2t-3)\pi}{2t+1}\right)+\cos\left(\frac{\pi(2t+1)}{2t+1}+\frac{\pi(2t-3)}{2t+1}\right)$$

$$=-2\cos\left(\frac{\pi(2t-3)}{2t+1}\right)$$

Generally, adding the *r-th* term of A and the $t-(r-1)$-*th* term of B; $r=1,2,\ldots,t$.

$$\cos\left(\frac{2\pi r}{2t+1}\right)+\cos\left(\frac{2\pi(2t-(r-1)}{2t+1}\right)$$

$$=\cos\left(\frac{\pi(2t+1)}{2t+1}-\frac{(2t-2r+1)\pi}{2t+1}\right)+\cos\left(\frac{\pi(2t+1)}{2t+1}+\frac{\pi(2t-2r+1)}{2t+1}\right)$$

$$=2\cos\pi\cos\left(\frac{(2t-2r+1)\pi}{2t+1}\right)$$

$$=-2\cos\left(\frac{(2t-2r+1)\pi}{2t+1}\right);\ \ r=1,2,\ldots,t.$$

Thus $2\sum_{r=1}^{t}\left[\cos\frac{(2t-2r+1)\pi}{2t+1}\right]=1;\ \ t=1,2,\ldots$ which yields

$$2\left[\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\cos\left(\frac{5\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right]=1;\ \ t=1,2,\ldots$$

which yields

$$\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\cos\left(\frac{5\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)=\frac{1}{2};\ \ t=1,2,\ldots$$

□

We therefore generate the following trigonometrical unit-equations having *t* terms involving $\cos\frac{a}{2t+1}\pi$, where $\frac{a}{2t+1}$ involves all "odd" rational numbers in the interval (0,1), i.e. *a* is also odd.There will be exactly *t* such odd rational numbers forming a *t*-sequence:

$t=1;\ 2\left[\cos\left(\frac{\pi}{3}\right)\right]=1;\ \frac{1}{3}$ is the only odd rational number between 0 and 1.

$t=2;\ 2\left[\cos\left(\frac{\pi}{5}\right)+\cos\left(\frac{3\pi}{5}\right)\right];\ \frac{1}{5},\frac{3}{5}$ are the 2 odd rational numbers between 0 and 1.

$t=3;\ 2\left[\cos\left(\frac{\pi}{7}\right)+\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{5\pi}{7}\right)\right];\ \frac{1}{7},\frac{3}{7}.\frac{5}{7}$ are the three odd rational numbers between 0 and 1.

$t=4;\ 2\left[\cos\left(\frac{\pi}{9}\right)+\cos\left(\frac{3\pi}{9}\right)+\cos\left(\frac{5\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)\right];\ \frac{1}{9},\frac{3}{9}.\frac{5}{9},\frac{7}{9}$ are the 4 terms of the sequence.

For each *t*, we therefore associate the *t*-sequence,of odd rational terms, each term belonging to the interval (0,1) and having the form $\frac{a}{2t+1}$, an odd, containing *t* terms:

$$\frac{1}{2t+1},\frac{3}{2t+1}.\frac{5}{2t+1},\frac{7}{2t+1},\ldots,\frac{2t-1}{2t+1};\ t=1,2,\ldots.$$

This sequence has similarities to the Farey sequence. The Farey sequence of order *n* is the sequence FY_n of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to *n*, arranged in order of increasing size. (seeHardy andWright). Farey sequences are named after the BritishgeologistJohn Farey, Sr., whose letter about these sequences was published in the *Philosophical Magazine* in 1816.

The sequence we derived from using the eigenvalues of the complete graph is called the

t-complete-eigen sequence.

Corollary 2.1

The sum of the terms of the *t*-complete eigen sequence:

$\frac{1}{2t+1},\frac{3}{2t+1}.\frac{5}{2t+1},\frac{7}{2t+1},\ldots,\frac{2t-1}{2t+1};\ t=1,2,\ldots$ is given by:

$$\sum_{r=1}^{t}\frac{2t-2r+1}{2t+1}=\frac{t^2}{2t+1};t=1,2,3\ldots$$

Proof

Writing each *t*-sequence down twice, with the second reversed, we get:

$$\frac{1}{2t+1},\frac{3}{2t+1},\frac{5}{2t+1},\frac{7}{2t+1},\ldots,\frac{2t-1}{2t+1}$$

$$\frac{2t-1}{2t+1},\frac{2t-3}{2t+1}.\frac{2t-6}{2t+1},\frac{2t-7}{2t+1},\ldots,\frac{1}{2t+1}$$

Adding corresponding terms we get double the sum of the terms of the sequence: $2\left(\sum_{r=1}^{t}\frac{2t-2r+1}{2t+1}\right)=t.\frac{2t}{2t+1}=\frac{2t^2}{2t+1}$.

Therefore,

$$\sum_{r=1}^{t}\frac{2t-2r+1}{2t+1}=\frac{1}{2}\left(\frac{2t^2}{2t+1}\right)=\frac{t^2}{2t+1};\ \ t=1,2,3\ldots$$

which gives the result. □

If we form the *ratio* of the t-complete-eigen sequenceby dividing each term of the original *t*-complete sequence by *t*, we obtain the sequence

$$\frac{1}{t(2t+1)},\frac{3}{t(2t+1)},\frac{5}{t(2t+1)},\frac{7}{t(2t+1)},\ldots,\frac{2t-1}{t(2t+1)};\ \ t=1,2,\ldots$$ and

$$\sum_{r=1}^{t}\frac{2t-2r+1}{t(2t+1)}=\frac{1}{t}\left(\frac{t^2}{(2t+1)}\right)=\frac{t}{(2t+1)};\ \ t=1,2,3\ldots,$$

which converges to the constant value of $\frac{1}{2}$ as *t* increases.

So, $\frac{t^2}{t(2t+1)}$ is the *t-complete-eigenratio* of $\frac{t^2}{(2t+1)}$ to *t*, which converges to the constant value of $\frac{1}{2}$.

This gives the following corollary:

Corollary 2.2

$$\lim_{t\to\infty}\sum_{r=1}^{t}\frac{2t-2r+1}{t(2t+1)}=\lim_{t\to\infty}\left[\frac{t^2}{t(2t+1)}\right]=\frac{1}{2}=\sum_{r=1}^{t}\cos\left(\frac{\pi(2t-2r+1)}{(2t+1)}\right);\ \ t=1,2,3\ldots$$

□

For the sequence $S = \frac{1}{2t+1}, \frac{3}{2t+1}, \frac{5}{2t+1}, \frac{7}{2t+1}, \ldots, \frac{2t-1}{2t+1}$ associate the *mirror image unit-pair partner* belonging to the *unit-mirror t-complete eigen sequence:*

$S' = \frac{2t}{2t+1}, \frac{2t-2}{2t+1}, \frac{2t-4}{2t+1}, \frac{2t-6}{2t+1}, \ldots, \frac{2}{2t+1}$ of the form $\frac{c}{2t+1}$ where *c* is even.

The sum of corresponding pairs of terms from S and S' yields

$$\frac{2t-2r+1}{2t+1} + \frac{2r}{2t+1} = \frac{2t-2r+1+2r}{2t+1} = \frac{2t+1}{2t+1} = 1;\ r = 1,2,3\ldots,t\,.$$

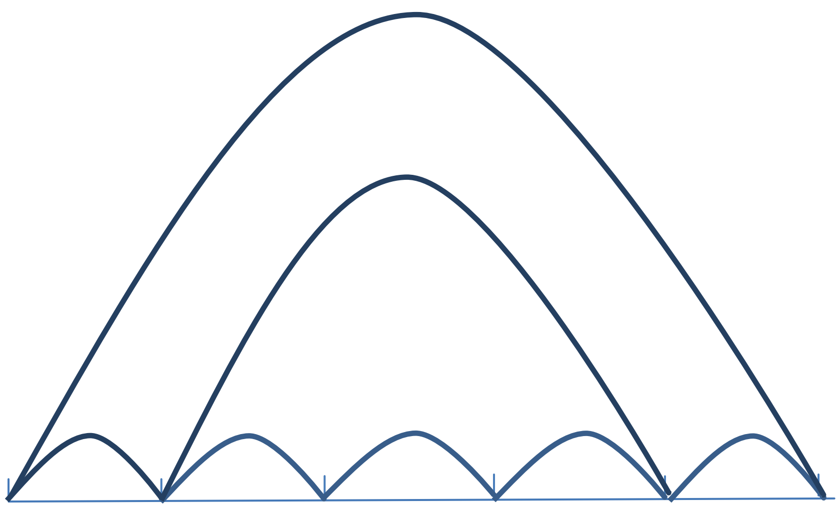
Thus $\left(\frac{2t-2r+1}{2t+1}; \frac{2r}{2t+1}\right)$; $r = 1,2,3..., t$ are *unit-mirror pairs*.

The union of S and S' yields the *total t-complete eigen sequence*:

$$S \cup S' = \frac{1}{2t+1}, \frac{2}{2t+1}, \frac{3}{2t+1}. \frac{4}{2t+1}, \frac{5}{2t+1}, \frac{6}{2t+1}, \frac{7}{2t+1}, \ldots, \frac{2t-1}{2t+1}, \frac{2t}{2t+1}$$

and $\sum_{k=1}^{2t} \frac{k}{2t+1} = t.$

Joining neighbors and unit mirror pairs we create the diagram for $t = 3$ (similar to the Farey sequence diagram):



1/7 2/7 3/7 4/7 5/7 6/7

Figure 2.1: Diagram for the total *t*-complete eigen sequence for $t = 3$

The average degree of the vertices of the complete graph on $n = 2t+1$ vertices is $n-1 = (2t+1)-1 = 2t$.

Attaching the average degree of the complete graph on $n = 2t + 1$ vertices, to the integral of the*t-complete-eigenratio* with respect to *n*, we form the *t-complete eigen area*(see Winter and Adewusi and Winter and Jessop):

$$Ar(K_{2t+1}^{\cos})$$

$$= (n-1)\int \frac{t^2}{t(2t+1)}dn$$

$$= (n-1)\int \frac{t}{(2t+1)}dn$$

$$= (n-1)\int \frac{\frac{n-1}{2}}{\left(\frac{2(n-1)}{2}+1\right)}dn$$

$$= (n-1)\int \frac{n-1}{2n}dn$$

$$= \frac{n-1}{2}(n - \ln n + c).$$

The first *t*-complete eigensequence arises when $n = 3$and $t = 1$. The sequence is

$\frac{1}{3};\frac{2}{3}$ and $Ar(K_3^{\cos}) = \frac{(3-1)}{2}(3 - \ln 3 + c) = 0$. So $c = \ln 3 - 3$, so that the *t*-complete eigen area is:

$$Ar(K_{2t+1}^{\cos}) = \frac{n-1}{2}(n - \ln n + \ln 3 - 3)$$

3. Induction and the eigenvalues of the complete graph

There are many different methods available to find the eigenvalues associated with the complete graph (see Jessop). Some methods are short, others are long but elegant. Although the induction method is long, it illustrates the intriguing aspects associated with the determinants, involved with characteristic polynomial associated with the matrix of the complete graph, which we demonstrate in the theorem below.

The following theorem is used in the proof of finding the eigenvalues of the complete graph. It involves the determinant of a minor of matrix $\lambda I - A_n$, where A_n is the adjacency matrix of the complete graph on *n* vertices.

Theorem 3.1

If $H_n = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{nxn}$, where H_n is a nxn matrix, with $n \geq 2$,

$$\text{then} \det H_n = \det \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{nxn}$$

$$= (-1)(\lambda + 1)^{n-1}.$$

Proof (by induction)

For $n=2$, $H_2 = \begin{bmatrix} -1 & -1 \\ -1 & \lambda \end{bmatrix}$

$$\begin{aligned}\det(H_2) &= -\lambda - 1 \\ &= -(\lambda+1) \\ &= -(\lambda+1)^1\end{aligned}$$

$$\det(H_3) \quad = \begin{bmatrix} -1 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}$$

$$= -\det\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \det\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} - \det\begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix}$$

$$= -1\det\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \det\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} + \det\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix}$$

$$= -\det(\lambda I - A(K_2)) + 2\det(H_2)$$

$$\begin{aligned} &= -1(\lambda^2 - 1) + 2(-\lambda - 1) \\ &= -(\lambda+1)(\lambda-1) - 2(\lambda+1) \\ &= -(\lambda+1)(\lambda - 1 + 2) \\ &= -(\lambda+1)^2 \end{aligned}$$

$$\det(H_4) \quad = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{bmatrix}$$

$$= -\det\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} + 3\det\begin{vmatrix} -1 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$$

$$\begin{aligned} &= -\det(\lambda I - A(K_3)) + 3\det(H_3) \\ &= -(\lambda\det(\lambda I - A(K_2)) + 2\det(H_2)) + 3\det(H_3) \end{aligned}$$

$$
\begin{aligned}
&= -\left(\lambda\left(\lambda^2-1\right)-2(\lambda+1)\right)-3(\lambda+1)^2 \\
&= -\left(\lambda(\lambda+1)(\lambda-1)-2(\lambda+1)\right)-3(\lambda+1)^2 \\
&= -(\lambda+1)\left(\lambda^2-\lambda-2\right)-3(\lambda+1)^2 \\
&= -(\lambda+1)(\lambda+1)(\lambda-2)-3(\lambda+1)^2 \\
&= -(\lambda+1)^2\left((\lambda-2)+3\right) \\
&= -(\lambda+1)^3
\end{aligned}
$$

$$
\begin{aligned}
\det(H_5) &= -\det(\lambda I - A(K_4)) + 4\det(H_4) \\
&= -\left(\lambda\det(\lambda I - A(K_3) + 3\det(H_3))\right) + 4\det(H_4) \\
&= -\left(\lambda(\lambda\det(\lambda I - A(K_3)) + 2\det(H_2)) + 3\det(H_3)\right) + 4\det(H_4) \\
&= -\left(\lambda^2\left(\lambda^2-1\right)-\lambda 2(\lambda+1)-3(\lambda+1)^2\right)-4(\lambda+1)^3 \\
&= -\left(\lambda^2(\lambda+1)(\lambda-1)-\lambda 2(\lambda+1)-3(\lambda+1)^2\right)-4(\lambda+1)^3 \\
&= -\left(\lambda(\lambda+1)\left(\lambda^2-\lambda-2\right)-3(\lambda+1)^2\right)-4(\lambda+1)^3 \\
&= -\left(\lambda(\lambda+1)(\lambda+1)(\lambda-2)-3(\lambda+1)^2\right)-4(\lambda+1)^3 \\
&= -(\lambda+1)^2\left[\lambda^2-2\lambda-3\right]-4(\lambda+1)^3 \\
&= -(\lambda+1)^2\left[(\lambda+1)(\lambda-3)\right]-4(\lambda+1)^3 \\
&= -(\lambda+1)^3(\lambda-3)-4(\lambda+1)^3 \\
&= -(\lambda+1)^3(\lambda-3+4) \\
&= -(\lambda+1)^3(\lambda+1) \\
&= -(\lambda+1)^4
\end{aligned}
$$

Assume the hypothesis it true for all $k \le n$, i.e., $\det(H_k) = -(\lambda+1)^{k-1}$ for all $k \le n$.

Then, for $n = k+1$,

$$
\det H_{k+1} = \det\begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}
$$

Then, expanding along the first row,

$$\det H_{k+1}$$

$$=(-1)\det\begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} + (-1)(-1)[(k+1)-1]\det\begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

The first term is obtained from the expansion of the first column (in the first row) and the second terms isobtained from the $((k+1)-1)$ identical terms obtained from the expansion of the 2nd to $((k+1)-1)$th columns.

Now,

$$\det(\lambda I - A(K_k)) = \det\begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} \text{ and } H_k = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}.$$

Then,

$$\det(H_{k+1})$$

$$=(-1)\det(\lambda I - A(K_k)) + k\det(H_k)$$

$$=(-1)\{\lambda\det(\lambda I - A(K_{k-1})) + (k-1)\det(H_{k-1})\} + k\det(H_k)$$

$$=(-1)\{\lambda(\lambda\det(\lambda I - A(K_{k-2})) + (k-2)\det(H_{k-2})) + (k-1)\det(H_{k-1})\} + k\det(H_k)$$

$$=(-1)\{\lambda^2\det(\lambda I - A_{k-2}) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\} + k\det(H_k)$$

$$=(-1)\{\lambda^2(\lambda\det(\lambda I - A_{k-3}) + (k-3)\det(H_{k-3})) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\}$$
$$+k\det(H_k)$$

$$=(-1)\{\lambda^3\det(\lambda I - A_{k-3}) + \lambda^2(k-3)\det(H_{k-3}) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\}$$
$$+k\det(H_k)$$

Now, the leading λ must have power $(k-2)$ so that we get $\det\left(\lambda I - A_{k-(k-2)}\right)$ and $\det\left(H_{k-(k-2)}\right)$ which are both known. So, continuing,

$$\begin{aligned}
&\det\left(H_{k+1}\right) \\
&= (-1)\Big[\ \lambda^{k-2}\det\left(\lambda I - A_{k-(k-2)}\right) + \lambda^{k-3}2\det\left(H_2\right) + \lambda^{k-4}3\det\left(H_3\right) + \lambda^{k-5}4\det\left(H_4\right) + \\
&\quad \ldots + \lambda^2(k-3)\det\left(H_{k-3}\right) + \lambda(k-2)\det\left(H_{k-2}\right) + (k-1)\det\left(H_{k-1}\right)\ \Big] + k\det\left(H_k\right)
\end{aligned}$$

Substituting $\det(\lambda I - A(K_2)) = \left(\lambda^2 - 1\right) = (\lambda+1)(\lambda-1)$ and $\det\left(H_k\right) = -(\lambda+1)^{k-1}$ for all $k \le n$, we get

$$\begin{aligned}
&\det\left(H_{k+1}\right) \\
&= (-1)\Big[\ \lambda^{k-2}(\lambda+1)(\lambda-1) - \lambda^{k-3}2(\lambda+1) - \lambda^{k-4}3(\lambda+1)^2 - \lambda^{k-5}4(\lambda+1)^3 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-4} - \lambda(k-2)(\lambda+1)^{k-3} - (k-1)(\lambda+1)^{k-2}\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

Factorising $(\lambda+1)$ out of the k terms in the square brackets, we get

$$\begin{aligned}
&\det\left(H_{k+1}\right) \\
&= (-1)(\lambda+1)\Big[\ \Big(\ \lambda^{k-2}(\lambda-1) - \lambda^{k-3}2 - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ \Big)\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

Working with the first two terms in square brackets, we get

$$\begin{aligned}
&\det\left(H_{k+1}\right) \\
&= (-1)(\lambda+1)\Big[\ \Big(\ \lambda^{k-3}\left(\lambda^2 - \lambda\right) - \lambda^{k-3}2 - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ \Big)\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&= (-1)(\lambda+1)\Big[\ \Big(\ \lambda^{k-3}\left(\lambda^2 - \lambda - 2\right) - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ \Big)\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&= (-1)(\lambda+1)\Big[\ \Big(\ \lambda^{k-3}(\lambda+1)(\lambda-2) - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ \Big)\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

Taking out the next factor of $(\lambda+1)$ from inside the square brackets, we get

$$\begin{aligned}
&\det\left(H_{k+1}\right) \\
&= (-1)(\lambda+1)^2\Big[\ \Big(\ \lambda^{k-3}(\lambda-2) - \lambda^{k-4}3 - \lambda^{k-5}4(\lambda+1) + \ldots
\end{aligned}$$

$$-\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ )\ ]-k(\lambda+1)^{k-1}$$

Working with the first two terms in square brackets, we get:

$$\begin{aligned}
&\det(H_{k+1})\\
&=(-1)(\lambda+1)^2[\ (\ \lambda^{k-4}(\lambda^2-2\lambda)-\lambda^{k-4}3-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ )\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^2[\ (\ \lambda^{k-4}(\lambda^2-2\lambda-3)-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ )\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^2[\ (\ \lambda^{k-4}(\lambda+1)(\lambda-3)-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ )\ ]-k(\lambda+1)^{k-1}
\end{aligned}$$

Note that the first term in the square brackets comprises of $(\lambda+1)\lambda^{k-t}(\lambda-(t-1))$.

We do the step (1) above a total of $(k-3)$ times, taking out the factor $(\lambda+1)^{k-3}$ to get

$$\begin{aligned}
&\det(H_{k+1})\\
&=(-1)(\lambda+1)^{k-3}[\ \lambda(\lambda+1)(\lambda-(k-2))-(k-1)(\lambda+1)\ ]-k(\lambda+1)^{k-1}
\end{aligned}$$

Note that the power of λ in the first term in the square brackets is $(k-2)-(k-3)=1$ and the power of $(\lambda+1)$ in the second term is also $(k-2)-(k-3)=1$. Simplifying, we get

$$\begin{aligned}
&\det(H_{k+1})\\
&=(-1)(\lambda+1)^{k-3}[\ (\lambda+1)(\lambda^2-\lambda(k-2)-(k-1))\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-2}[\ (\lambda^2-\lambda(k-2)-(k-1))\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-2}[\ (\lambda+1)(\lambda-(k-1))\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-1}[\ (\lambda-(k-1))\ ]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-1}[\ (\lambda-(k-1)+k)\ ]\\
&=(-1)(\lambda+1)^{k-1}[\ (\lambda+1)\ ]\\
&=(-1)(\lambda+1)^{k}
\end{aligned}$$

This concludes the proof, by induction, that $\det H_n=(-1)(\lambda+1)^{n-1}$, for all $n\geq 2$. □

Corollary 3.1

Let $A(K_n)$ be the adjacency matrix of the complete graph K_n on *n* vertices.

Then $A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{nxn}$

has eigenvalue $(n-1)$ with multiplicity 1, and eigenvalue -1 with multiplicity $(n-1)$.
Hence $\det(\lambda I - A(K_n)) = (\lambda+1)^{n-1}\{\lambda-(n-1)\}$.

Proof of Corollary 3.1 (by induction)

For $n=2$, $A(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{aligned} \det(\lambda I - A(K_2)) &= \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \\ &= \lambda^2 - 1 \\ &= (\lambda+1)(\lambda-1) \\ &= (\lambda+1)(\lambda-1). \end{aligned}$$

Note that the eigenvalues of A(K_2) are λ = -1 (1 time) and λ=1 (once).

Assume the hypothesis it true for $k \le n$, i.e.,

$$\det(\lambda I - A(K_k)) = \det\begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= (\lambda+1)^{k-1}\{\lambda-(k-1)\} \text{ for } k \le n$$

i.e., $\lambda = -1$ $(k-1)$ times, and $\lambda = (k-1)$ once.

Then, for $n = k+1$,

$\det[\lambda I - A(K_{k+1})]$

$$= \lambda \det \begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} + k \det \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= \lambda \det(A(K_k)) + k \det(H_k)$$

Now applying the inductive hypothesis for $\det(A(K_k))$, and Theorem 3.1 for $\det(H_k)$, we get

$$\begin{aligned} \det[\lambda I - A(K_{k+1})] &= \lambda(\lambda+1)^{k-1}\{\lambda-(k-1)\} + k(-1)(\lambda+1)^{k-1} \\ &= (\lambda+1)^{k-1}\{\lambda^2 - \lambda(k-1) - k\} \\ &= (\lambda+1)^{k-1}(\lambda+1)(\lambda-k) \\ &= (\lambda+1)^k(\lambda-k) \end{aligned}$$

i.e., $\lambda = -1$ k times and $\lambda = k$ once.

So we have proved that the eigenvalues of the adjacency matrix of the complete graph $A(K_n)$ are $\lambda = -1$ and $\lambda = n-1$, and that the characteristic polynomial is $P_{A(K_n)}(\lambda) = (\lambda+1)^{n-1}(\lambda-(n-1))$. The two factors $(\lambda+1)$ and $(\lambda-(n-1))$ give rise to the quadratic $\lambda^2-(n-2)\lambda-(n-1)$ which has the associated conjugate pairs $\lambda = \dfrac{(n-2)}{2} \pm \sqrt{\dfrac{(2-n)^2+4(n-1)}{4}}$.

4. Conclusion

Regarding the adjacency matrix, associated with the complete graph, as a circulant matrix, we formedthe unit-equations:

$$2\left[\cos\left(\frac{\pi}{2t+1}\right)+\cos\left(\frac{3\pi}{2t+1}\right)+\ldots+\cos\left(\frac{(2t-1)\pi}{2t+1}\right)\right]$$
$$=2\sum_{r=1}^{t}\cos\left(\frac{\pi(2t-2r+1)}{2t-1}\right)$$
$$=1;\ \ t=1,2,\ldots$$

For each *t*, we therefore generated the *t*-sequence, of odd rational terms, each in the interval (0,1) and having the form $\frac{a}{2t+1}$:

$$\frac{1}{2t+1},\frac{3}{2t+1},\frac{5}{2t+1},\frac{7}{2t+1},\ldots,\frac{2t-1}{2t+1};\ \ t=1,2,\ldots.$$

This sequence is referred to as the *t-complete-eigen sequence* and we showed that the sum of its terms is

$\sum_{r=1}^{t}\frac{2t-2r+1}{2t+1}=\frac{t^2}{2t+1};\ \ t=1,2,3\ldots$ and that the ratio of this sum to *t* converges to $\frac{1}{2}$.

We use the associated total *t*-complete eigen sequence to construct the diagram involving unit mirror pairs and found the *t*-complete eigen area, by using integration combined with the average degree of the complete graph on *n* vertices, to be:

$$Ar(K_{2t+1}^{\cos})=\frac{n-1}{2}(n-\ln n+\ln 3-3).$$

In order to find the eigenvalues of the adjacency matrix, associated with the complete graph, by induction, we generated an equation involving the determinant of the minor of the matrix associated with the characteristic polynomial of this adjacency matrix.

5. References

Brouwer, A. E. Haemers, 2011. W. H. *Spectra of Graphs*.Springer, New York.

Harris, J. M., Hirst, J. L. and Mossinghoff, M. 2008. *Combinatorics and Graph theory*.Springer, New York.

Hardy, G.H., Wright, E.M. 1979. *An Introduction to the Theory of Numbers* (Fifth Edition).Oxford University Press.

Jessop, C. L. *Matrices of Graphs and Designs with Emphasis on their Eigen-Pair Balanced Characteristic*. 2014, M. Sc. Dissertation, University of Kwa-Zulu Natal.

Winter, P. A. and Adewusi, F.J. 2014.Tree-cover ratio of graphs with asymptotic convergence identical to the secretary problem.*Advances in Mathematics: Scientific Journal*. 3(2): 47-61.

Winter, P. A. and Jessop, C.L. 2014.Integral eigen-pair balanced classes of graphs with their ratio, asymptote, area and involution complementary aspects.*International Journal of Graph Theory*; Article ID 148690, 16 pages.