International J.Math. Combin. Vol.3(2014), 55-69

Friendly Index Sets and Friendly Index Numbers of Some Graphs

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider the set $A = \{0, 1\}$. A labeling $f : V(G) \to A$, induces a partial edge labeling $f^* : E(G) \to A$, defined by $f^*(xy) = f(x)$ if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and we denote $e_{Bf^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. In this paper we define friendly index number(FIN) and full friendly index number(FFIN) of graph G as the cardinality of the distinct elements of friendly index set and full friendly index set respectively and obtaining these numbers along with their sets of some families graphs.

Key Words: Friendly index set, full friendly index set, friendly index number and full friendly index number, Smarandache friendly index number.

AMS(2010): 05C76

§1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V, E)$. Here elements of set V and E are known as vertices and edges respectively. For all other terminologies and notations we follow Harary [2].

In 1986 Cahit [1] introduced cordial graph labeling. A function f from $V(G)$ to $\{0, 1\}$, where for each edge xy, $f^*(xy) = |f(x) - f(y)|$, $v_f(i)$ is the number of vertices v with $f(v) = i$ and $e_{f^*}(i)$ is the number of edges e with $f^*(e) = i$, is called friendly if $|v_f(1) - v_f(0)| \leq 1$. A friendly labeling f is called cordial if $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$.

In [6] Lee and Ng defined the friendly index set of a graph G as $FI(G) = \{|e_{f^*}(1) - e_{f^*}(0)| :$ f^* runs over all friendly labeling f of $G\}$. The concept was extended by Harris and Kwong [7] to full friendly index set for the graph G, denoted $FFI(G)$, defined as $FFI(G) = \{e_{f^*}(1) - e_{f^*}(0)$: f^* runs over all friendly labeling f of $G\}$.

Lee, Liu and Tan [5] considered a new labeling problem of graph theory. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0, 1\}$. For each vertex labeling f of G, a partial edge labeling f^* of G is defined in the following way.

[1]Received February 18, 2014, Accepted August 26, 2014.

For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f^* respectively.

In [4] Kim, Lee, and Ng defined the balance index set of a graph G as $\mathrm{BI}(G) = \{|e_{f^*}(1) - e_{f^*}(0)| : f^*$ runs over all friendly labelings f of G $\}$.

Definition 1.1 *The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 1.2 *The crown $C_n \odot K_1$ is obtained by joining a pendant edge to each vertex of C_n.*

Definition 1.3 *A chord of cycle C_n is an edge joining two non-adjacent vertices of cycle C_n.*

Definition 1.4 *The shell S_n is the graph obtained by taking $n-3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called fan f_{n-1}. Thus $S_n = f_{n-1} = P_{n-1} + K_1$.*

Definition 1.5 *The wheel W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex, the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges and edges joining apex and vertices of cycle are spoke edges.*

Definition 1.6 *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.7 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

More details of known results of graph labelings given in Gallian [3].

In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways

$$4+0, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

In this paper we are using the idea of integer partition of numbers. Let G be any graph with p vertices. Partition of p in to (p_0, p_1), where p_0 and p_1 are the number of vertices labeled by 0 and 1 respectively.

§2. Main Results

Here we are introducing two new parameters $e_{Bf^*}(i)$ and $e_{Ff^*}(i)$, which are the number of edges labeled i under balanced labeling and cordial labeling respectively. While proving our results, FI(G) and FFI(G) are used as below:

$FI(G) = \{|e_{Ff^*}(1) - e_{Ff^*}(0)| : Ff^*$ runs over all friendly labeling f of $G\}$;
$FFI(G) = \{e_{Ff^*}(1) - e_{Ff^*}(0) : Ff^*$ runs over all friendly labeling f of $G\}$.

Theorem 2.1 *Let $G(V,E)$ be a graph with $|E(G)| = q$ and $e_{Bf^*}(i)$ is the number of edges labeled i under the balanced labeling, where $i = 0, 1$. Then*

(1) $FI(G) = \{|q - 2(e_{Bf^*}(0) + e_{Bf^*}(1))|$: *the partial edge labeling Bf^* runs over all friendly labeling f of $G\}$;*

(2) $FFI(G) = \{q - 2(e_{Bf^*}(0) + e_{Bf^*}(1))$: *the partial edge labeling Bf^* runs over all friendly labeling f of $G\}$.*

Definition 2.2 *For a graph G with a subgraph $H \leq G$, the Smarandache friendly index number $SFIN$ is the number of distinct elements runs over all labeling $f : V(G) \to A$ with friendly index set $FIN(H)$, particularly, if $H = G$, such number is called friendly index number on G and denoted by FIN.*

Definition 2.2 *The full friendly index number is the number of distinct elements in the full friendly index set and it is denoted as FFIN.*

We are using Theorem 2.1 to prove the following results.

Theorem 2.4 *In a shell graph S_n with $n \geq 4$ vertices,*

$$FI(S_n) = \begin{cases} \{1, 3, 5, \cdots, n-2\}, & \textit{if } n \textit{ is odd} \\ \{1, 3, 5, \cdots, n-1\}, & \textit{if } n \textit{ is even} \end{cases}$$

Proof In a shell graph S_n, $|V(S_n)| = n$ and $|E(S_n)| = 2n - 3$.

Case 1 n is odd.

To satisfy friendly labeling, the possible compositions of n are

$$\left(\frac{n-1}{2}, \frac{n+1}{2}\right) \text{ and } \left(\frac{n+1}{2}, \frac{n-1}{2}\right).$$

Consider the composition $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ of n. If the apex vertex labeled 0, then $e_{Bf^*}(0) = \frac{n-3}{2} + i$, where $i = 0, 1, 2, \cdots, \frac{n-5}{2}$; $e_{Bf^*}(1) = j$, where $j = i+1, i+2, i+3, \cdots, \frac{n-1}{2}$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = \left|(2n-3) - 2\left(\frac{n-3}{2} + i + j\right)\right| = |n - 2(i+j)|,$$

where $i = 0, 1, 2, \cdots, \dfrac{n-5}{2}$ and $j = i+1, i+2, i+3, \cdots, \dfrac{n-1}{2}$. If we consider the composition $\left(\dfrac{n+1}{2}, \dfrac{n-1}{2}\right)$ of n and the apex vertex labeled 0, then $e_{Bf^*}(0) = \dfrac{n-1}{2} + i$, where $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$; $e_{Bf^*}(1) = j$, where if $i = 0, 1, 2, \cdots, \dfrac{n-5}{2}$, then $j = 0, 1, 2, \cdots, \dfrac{n-3}{2}$; if $i = \dfrac{n-3}{2}$, then $j = 0, 1, 2, \cdots, \dfrac{n-5}{2}$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = \left|(2n-3) - 2\left(\frac{n-1}{2} + i + j\right)\right| = |n - 2(i+j+1)|,$$

where if $i = 0, 1, 2, \cdots, \dfrac{n-5}{2}$, then $j = 0, 1, 2, \cdots, \dfrac{n-3}{2}$; if $i = \dfrac{n-3}{2}$, then $j = 0, 1, 2, \cdots, \dfrac{n-5}{2}$.

Considering all possible values of i and j, we get $\mathrm{BI}(S_n) = \{1, 3, 5, \cdots, n-2\}$. Also if the apex vertex labeled 1, then $\mathrm{FI}(S_n)$ will be same.

Case 2 n is even.

To satisfy friendly labeling, the possible partition of n is $\left(\dfrac{n}{2}, \dfrac{n}{2}\right)$. If the apex vertex labeled 0, then, $e_{Bf^*}(0) = \dfrac{n}{2} - 1 + i$, where $i = 0, 1, 2, \cdots, \dfrac{n}{2} - 2$; $e_{f^*}(1) = j$, where if $i = 0$, then $j = i, i+1, i+2, \cdots, \dfrac{n}{2} - 1$; if $i = 1, 2, \cdots, \dfrac{n}{2} - 2$, then $j = i+1, i+2, i+3, \cdots, \dfrac{n}{2} - 1$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = \left|(2n-3) - 2\left(\frac{n}{2} - 1 + i + j\right)\right| = |n - (2i + 2j + 1)|,$$

where if $i = 0$, then $j = i, i+1, i+2, \cdots, \dfrac{n}{2} - 1$; if $i = 1, 2, \cdots, \dfrac{n}{2} - 2$, then $j = i+1, i+2, i+3, \cdots, \dfrac{n}{2} - 1$.

Considering all possible values of i and j, we get $\mathrm{FI}(S_n) = \{1, 3, 5, \cdots, n-1\}$. Also if the apex vertex labeled 1, then $\mathrm{FI}(S_n)$ will be same. □

Corollary 2.5 *The graph S_n is cordial.*

Corollary 2.6 *The friendly index set of the graph S_n forms an arithmetic progression with common difference* 2.

Corollary 2.7

$$FIN(S_n) = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd} \\ \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.8 *In a shell graph S_n with $n \geq 4$ vertices,*

$$FFI(S_n) = \begin{cases} \{-n+6, -n+8, -n+10, ..., n-2\}, & \text{if } n \text{ is odd} \\ \{-n+5, -n+7, -n+9, ..., n-1\}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.9

$$FFIN(S_n) = \begin{cases} n-3, & \text{if } n \text{ is odd} \\ n-2, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.10 *The full friendly index set of the graph S_n forms an arithmetic progression with common difference* 2*.*

Example 2.11 Friendly index set of shell graph S_5 is $\{1, 3\}$.

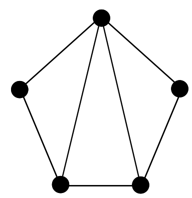

Figure 1: The shell graph S_5

Table 1: Compositions of integer 5 for friendly labeling with elements of friendly index set.

Compositions of integer 5	Corresponding elements friendly index set
(2, 3)	1, 3
(3, 2)	1, 3

Theorem 2.12 *In a crown graph $C_n \odot K_1$ with $n \geq 3$,*

$$FI(C_n \odot K_1) = \begin{cases} \{0, 4, 8, \cdots, 2n\}, & \text{if } n \text{ is even} \\ \{0, 4, 8, \cdots, 2n-2\}, & \text{if } n \text{ is odd} \end{cases}$$

Proof Consider the crown graph $C_n \odot K_1$, $|V(C_n \odot K_1)| = 2n$ and $|E(C_n \odot K_1)| = 2n$.

Case 1 n is even.

To satisfy friendly labeling, the possible partitions of number of vertices of cycle and pendent vertices of $C_n \odot K_1$ are $(n-i, i)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, \frac{n}{2}$.

If i=0, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = n$. Therefore friendly index is '0'. If $i = 1, 2, 3, \cdots, \frac{n}{2}$, then $e_{Bf^*}(0) = n-i-1-j+k$, where $j = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$; $e_{Bf^*}(1) = l + k$, where $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$ such that $j + l = i - 1$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2n - 2[(n-i-1-j+k) + (l+k)]| = |4(i-l-k)|,$$

where if $i = 1, 2, \cdots, \frac{n}{2}$, then $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$.

Considering all possible values of i, l and k, we get FI=$\{0, 4, 8, \cdots, 2n\}$.

Case 2 n is odd.

To satisfy friendly labeling, the possible partitions of number of vertices of cycle and pendent vertices of $C_n \odot K_1$ are $(n-i, i)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, \frac{n-1}{2}$.

If i=0, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = n$. Therefore friendly index is $'0'$. If $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $e_{Bf^*}(0) = n-i-1-j+k$, where $j = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$, $e_{Bf^*}(1) = l+k$, where $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$ such that $j+l = i-1$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2n - 2[(n-i-1-j+k) + (l+k)]| = |4(i-l-k)|,$$

where $i = 1, 2, \cdots, \frac{n-1}{2}$, $l = 0, 1, 2, \cdots, i-1$ and $k = 0, 1, 2, \cdots, i$.

Considering all possible values of i, l and k, we get FI$(C_n \odot K_1) = \{0, 4, 8, ..., 2n-2\}$. $\square$

Corollary 2.13 *The graph $C_n \odot K_1$ is cordial.*

Corollary 2.14 *The friendly index set of the graph $C_n \odot K_1$ forms an arithmetic progression with common difference* 4.

Corollary 2.15

$$FIN(C_n \odot K_1) = \begin{cases} \frac{n+1}{2}, & \textit{if } n \textit{ is odd} \\ \frac{n}{2}+1, & \textit{if } n \textit{ is even} \end{cases}$$

Corollary 2.16 *In a crown graph $C_n \odot K_1$ with $n \geq 3$,*

$$FFI(C_n \odot K_1) = \begin{cases} \{-2n+4, -2n+8, -2n+12, ..., 2n\}, & \textit{if } n \textit{ is even} \\ \{-2n+6, -2n+10, -2n+14, ..., 2n-2\}, & \textit{if } n \textit{ is odd} \end{cases}$$

Corollary 2.17 *The full friendly index set of the graph $C_n \odot K_1$ forms an arithmetic progression with common difference* 4.

Corollary 2.18

$$FFIN(C_n \odot K_1) = \begin{cases} n, & \textit{if } n \textit{ is even} \\ n-1, & \textit{if } n \textit{ is odd} \end{cases}$$

Example 2.19 Friendly index set of crown graph $C_5 \odot K_1$ is $\{0, 4, 8\}$.

Theorem 2.20 *In a helm graph H_n,*

$$FI(H_n) = \begin{cases} \{1, 3, 5, \cdots, 2n-1\}, & \textit{if } n \textit{ is odd} \\ \{0, 2, 4, \cdots, 2n\}, & \textit{if } n \textit{ is even} \end{cases}$$

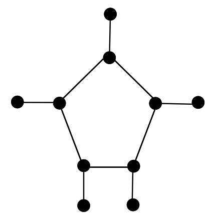

Figure 2: The crown graph $C_5 \cdot K_1$

Table 2: Compositions of integer 5 for friendly labeling with elements of friendly index set.

Partition of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	0
(4, 1) and (1, 4)	0, 4
(3, 2) and (2, 3)	0, 4, 8

Proof Consider the helm graph H_n. $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

Case 1 n is odd.

First we label the apex vertex as 0.

Subcase 1.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = n$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n.$$

Subcase 1.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i, i)$ and $(i, n-i)$, where $i = 1, 2, 3, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-i)+l$, where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$; $e_{Bf^*}(1) = k+l$, where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |2(i+j-k-2l) - n|,$$

where if $i = 1, 2, 3, \cdots, \frac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i, k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$, $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$ such that $j+k = 2i$. Therefore,

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l|,$$

where if $i = 1, 2, 3, \cdots, \dfrac{n-1}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$.

Subcase 1.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n, 0)$ respectively, then $e_{Ff^*}(0) = n$ and $e_{Ff^*}(1) = 2n$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n.$$

Subcase 1.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, n-1$ respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+l$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |3n - 2[(n-j) + (n-(i+1)) + k + 2l + 1]|,$$

where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j + k = 2(i+1)$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = |n + 2i - 4j + 4l + 4|,$$

where if $i = 0, 1, 2, \cdots, \dfrac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \dfrac{n-1}{2}, \dfrac{n+1}{2}, \dfrac{n+3}{2}, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 1.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = n+1$ and $e_{Ff^*}(1) = 2n-1$. Therefore

$$|e_{Ff^*}(1) - e_{Ff^*}(0)| = n - 2.$$

Considering all the above sub cases and all possible values of i, j and l, we get $BI(H_n) = \{1, 3, 5, \cdots, 2n-1\}$. If we label the apex vertex as 1 and considering all possible compositions of number of vertices for friendly labeling, then also the friendly index set will be same.

Case 2 n is even.

First we label the apex vertex as 0.

Subcase 2.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = n$.

Subcase 2.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i,i)$ and $(i,n-i)$, where $i=1,2,3,\cdots,n-1$, respectively. Then $e_{Bf^*}(0)=(n-j)+(n-i)+l$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-i$; $e_{Bf^*}(1)=k+l$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $k=0,1,2,\cdots,i-1$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $k=2i-n,2i-(n-1),2i-(n-2),\cdots,i-1$ and $l=0,1,2,\cdots,n-i$ such that $j+k=2i$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|2(i+j-k-2l)-n|$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$, $k=0,1,2,\cdots,i-1$ and $l=0,1,2,3,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$, $k=2i-n,2i-(n-1),2i-(n-2),\cdots,i-1$ and $l=0,1,2,\cdots,n-i$ such that $j+k=2i$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|n+2i-4j+4l|$, where if $i=1,2,3,\cdots,\frac{n}{2}$, then $j=i+1,i+2,i+3,\cdots,2i$ and $l=0,1,2,3,\cdots,i$; if $i=\frac{n}{2}+1,\frac{n}{2}+2,\frac{n}{2}+3,\cdots,n-1$, then $j=i+1,i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-i$.

Subcase 2.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n,0)$, respectively, then $e_{Ff^*}(0)=n$ and $e_{Ff^*}(1)=2n$. Therefore, $|e_{Ff^*}(0)-e_{Ff^*}(1)|=n$.

Subcase 2.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1),i+1)$ and $(i,n-i)$, where $i=0,1,2,\cdots,n-1$, respectively. Then $e_{Bf^*}(0)=(n-j)+(n-(i+1))+l$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $j=i+2,i+3,\cdots,n$ and $l=0,1,2,\cdots,n-(i+1)$; $e_{Bf^*}(1)=k+l+1$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $k=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $k=2(i+1)-n,2i-(n-1),2i-(n-2),\cdots,i$ and $l=0,1,2,\cdots,n-(i+1)$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|3n-2[(n-j)+(n-(i+1))+k+2l+1]|$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$, $k=0,1,2,\cdots,i$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1$, then $j=i+2,i+3,i+4,\cdots,n$, $k=2(i+1)-n,2(i+1)-(n-1),2(i+1)-(n-2),\cdots,i$ and $l=0,1,2,\cdots,n-(i+1)$ such that $j+k=2(i+1)$. Therefore, $|e_{Ff^*}(1)-e_{Ff^*}(0)|=|n+2i-4j+4l+4|$, where if $i=0,1,2,\cdots,\frac{n}{2}-1$, then $j=i+2,i+3,i+4,\cdots,2(i+1)$ and $l=0,1,2,\cdots,i$; if $i=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-2$, then $j=i+2,i+3,i+4,\cdots,n$ and $l=0,1,2,\cdots,n-(i+1)$.

Subcase 2.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n-1,1)$, respectively, then $e_{Ff^*}(0)=n+1$ and $e_{f^*}(1)=2n-1$. Therefore $|e_{Ff^*}(1)-e_{Ff^*}(0)|=n-2$. Considering all the above sub cases and all possible values of i, j and l, we get $BI(H_n)=\{0,2,4,\cdots,2n\}$.

If we label the apex vertex as 1 and considering all possible compositions of number of vertices for friendly labeling , then also the friendly index set will be same. □

Corollary 2.21 *The graph H_n is cordial.*

Corollary 2.22 *The friendly index set of helm graph H_n forms an arithmetic progression with* *common difference* 2.

Corollary 2.23 *In a helm graph H_n,*

$$FIN(H_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.24 *In a helm graph H_n,*

$$FFI(H_n) = \begin{cases} \{-2n+5, -2n+7, -2n+9, \cdots, 2n-1\}, & \text{if } n \text{ is odd} \\ \{-2n+6, -2n+8, -2n+10, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.25 *The full friendly index set of helm graph H_n forms an arithmetic progression with common difference* 2.

Corollary 2.26 *In a Helm graph H_n, $FFIN(H_n) = 2n - 2$.*

Table 3: Compositions of number of rim vertices of wheel and pendent vertices of H_5 for friendly labeling with elements of friendly index set.

Compositions of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	5
(4, 1) and (1, 4)	1, 3
(3, 2) and (2, 3)	1, 3, 5, 7
(2, 3) and (3, 2)	1, 3, 5, 9
(1, 4) and (4, 1)	3, 7
(0, 5) and (5, 0)	5
(4, 1) and (0, 5)	1
(3, 2) and (1, 4)	1, 3, 5
(2, 3) and (2, 3)	1, 3, 5, 7
(1, 4) and (3, 2)	1, 5
(0, 5) and (4, 1)	3

Example 2.27 Friendly index set of helm graph H_5 is $\{1, 3, 5, 7, 9\}$.

Theorem 2.28 *In a flower graph Fl_n,*

$$FI(Fl_n) = \begin{cases} \{0, 4, 8, \cdots, 2n-2\}, & \text{if } n \text{ is odd} \\ \{0, 4, 8, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

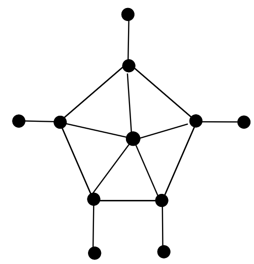

Figure 3: The helm graph H_5

Proof Consider the flower graph Fl_n. $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$.

Case 1 n is odd.

First we label the apex vertex as 0.

Subcase 1.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n,0)$ and $(0,n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 1.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i,i)$ and $(i,n-i)$, where $i = 1,2,3,\cdots,n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-i)+i+l$, where if $i = 1,2,3,\cdots,\frac{n-1}{2}$, then $j = i+1,i+2,i+3,\cdots,2i$ and $l = 0,1,2,\cdots,i$; if $i = \frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\cdots,n-1$, then $j = i+1,i+2,i+3,\cdots,n$ and $l = 0,1,2,\cdots,n-i$; $e_{Bf^*}(1) = k+l$, where if $i = 1,2,3,\cdots,\frac{n-1}{2}$, then $k = 0,1,2,\cdots,i-1$ and $l = 0,1,2,\cdots,i$; if $i = \frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\cdots,n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2),\cdots,i-1$ and $l = 0,1,2,\cdots,n-i$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j)+(n-i)+i+k+2l)|$, where if $i = 1,2,3,\cdots,\frac{n-1}{2}$, then $j = i+1,i+2,i+3,\cdots,2i$, $k = 0,1,2,\cdots,i-1$ and $l = 0,1,2,\cdots,i$; if $i = \frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\cdots,n-1$, then $j = i+1,i+2,i+3,\cdots,n$, $k = 2i-n, 2i-(n-1), 2i-(n-2),\cdots,i-1$ and $l = 0,1,2,\cdots,n-i$ such that $j+k = 2i$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 4\,|i-j+l|$, where if $i = 1,2,3,\cdots,\frac{n-1}{2}$, then $j = i+1, i+2, i+3,\cdots,2i$ and $l = 0,1,2,\cdots,i$; if $i = \frac{n+1}{2},\frac{n+3}{2},\frac{n+5}{2},\cdots,n-1$, then $j = i+1,i+2,i+3,\cdots,n$ and $l = 0,1,2,\cdots,n-i$.

Subcase 1.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0,n)$ and $(n,0)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 1.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0,1,2,\cdots,n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+i+l$, where if $i = 0,1,2,\cdots,\frac{n-3}{2}$, then $j = i+2,i+3,i+4,\cdots,2(i+1)$ and $l = 0,1,2,\cdots,i$; if $i = \frac{n-1}{2},\frac{n+1}{2},\frac{n+3}{2},\cdots,n-1$, then $j = i+2,i+3,\cdots,n$ and $l = 0,1,2,\cdots,n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i =$

$0, 1, 2, \cdots, \frac{n-3}{2}$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-1$, then $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j) + (n-(i+1)) + i + 2l + k + 1)|$, where if $i = 0, 1, 2, \cdots, \frac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j + k = 2(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4(i-j+l+1)|$, where if $i = 0, 1, 2, \cdots, \frac{n-3}{2}$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 1.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Considering all the above sub cases and all possible values of i, j and l, we get $FI(Fl_n) = \{0, 4, 8, \cdots, 2n-2\}$. If we label the apex vertex '$1'$' and consider all possible compositions of number of vertices for friendly labeling , then the balance index set will be same.

Case 2. n is even.

First we label the apex vertex as '$0'$'.

Subcase 2.1 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n, 0)$ and $(0, n)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 2.2 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-i, i)$ and $(i, n-i)$, where $i = 1, 2, 3, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j) + (n-i) + i + l$, where if $i = 1, 2, 3, \cdots, \frac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$; $e_{Bf^*}(1) = k + l$, where if $i = 1, 2, 3, \cdots, \frac{n}{2}$, then $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, ..., i$; if $i = \frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, n-1$, then $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$. Therefore, $|e_{Ff^*}(0) - e_{Ff^*}(1)| = |4n - 2((n-j) + (n-i) + i + k + 2l)|$, where if $i = 1, 2, 3, \cdots, \frac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$, $k = 0, 1, 2, \cdots, i-1$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, ..., n$, $k = 2i-n, 2i-(n-1), 2i-(n-2), \cdots, i-1$ and $l = 0, 1, 2, \cdots, n-i$ such that $j + k = 2i$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 4\,|i-j+l|$, where if i=1, 2, 3, . . .,$\frac{n}{2}$, then $j = i+1, i+2, i+3, \cdots, 2i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}+1, \frac{n}{2}+2, \frac{n}{2}+3, \cdots, n-1$, then $j = i+1, i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-i$.

Subcase 2.3 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n, 0)$, respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Subcase 2.4 If the compositions of rim vertices of wheel and pendent vertices of helm are $(n-(i+1), i+1)$ and $(i, n-i)$, where $i = 0, 1, 2, \cdots, n-1$, respectively. Then $e_{Bf^*}(0) = (n-j)+(n-(i+1))+i+l$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $j = i+2, i+3, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$; $e_{Bf^*}(1) = k+l+1$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4n - 2((n-j)+(n-(i+1))+i+2l+k+1)|$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$, $k = 0, 1, 2, \cdots, i$ and $l = 0, 1, 2, ..., i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1$, then $j = i+2, i+3, i+4, \cdots, n$, $k = 2(i+1)-n, 2(i+1)-(n-1), 2(i+1)-(n-2), \cdots, i$ and $l = 0, 1, 2, \cdots, n-(i+1)$ such that $j+k = 2(i+1)$. Therefore, $|e_{Ff^*}(1) - e_{Ff^*}(0)| = |4(i-j+l+1)|$, where if $i = 0, 1, 2, \cdots, \frac{n}{2}-1$, then $j = i+2, i+3, i+4, \cdots, 2(i+1)$ and $l = 0, 1, 2, \cdots, i$; if $i = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-2$, then $j = i+2, i+3, i+4, \cdots, n$ and $l = 0, 1, 2, \cdots, n-(i+1)$.

Subcase 2.5 If the compositions of rim vertices of wheel and pendent vertices of helm are $(0, n)$ and $(n-1, 1)$ respectively, then $e_{Ff^*}(0) = 2n$ and $e_{Ff^*}(1) = 2n$. Therefore $|e_{Ff^*}(1) - e_{Ff^*}(0)| = 0$.

Considering all the above subcases and all the possible values of i, j and l, we get $FI(Fl_n) = \{0, 4, 8, \cdots, 2n\}$. If we label the apex vertex '1' and consider all possible compositions of number of vertices for friendly labeling, then the balance index set will be same. □

Corollary 2.29 *The flower graph Fl_n is cordial.*

Corollary 2.30 *The friendly index set of the graph Fl_n forms an arithmetic progression with common difference* 4.

Corollary 2.31 *In a flower graph Fl_n,*

$$FIN(Fl_n) = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{n+2}{2}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.32 *In a flower graph Fl_n,*

$$FFI(Fl_n) = \begin{cases} \{-2n+6, -2n+10, -2n+14, \cdots, 2n-2\}, & \text{if } n \text{ is odd} \\ \{-2n+4, -2n+8, -2n+12, \cdots, 2n\}, & \text{if } n \text{ is even} \end{cases}$$

Corollary 2.33 *The full friendly index set of the flower graph Fl_n forms an arithmetic progression with common difference* 4.

Corollary 2.34 *In a flower graph Fl_n,*

$$FFIN(Fl_n) = \begin{cases} n-1, & \text{if } n \text{ is odd} \\ n, & \text{if } n \text{ is even} \end{cases}$$

Example 2.35 Friendly index set of flower graph Fl_5 is $\{0, 4, 8\}$.

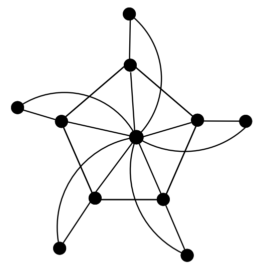

Figure 4: The flower graph Fl_5

Table 4: Compositions of number of rim vertices of wheel and number of vertices with degree two of Fl_5 for friendly labeling and corresponding elements of friendly index set.

Compositions of integers 5 and 5	Corresponding elements of friendly index set
(5, 0) and (0, 5)	0
(4, 1) and (1, 4)	0,4
(3, 2) and (2, 3)	0, 4, 8
(2, 3) and (3, 2)	0, 4, 8
(1, 4) and (4, 1)	0, 4
(0, 5) and (5, 0)	0
(4, 1) and (0, 5)	4
(3, 2) and (1, 4)	0, 4, 8
(2, 3) and (2, 3)	0, 4, 8
(1, 4) and (3, 2)	0, 4
(0, 5) and (4, 1)	0

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