One Modulo $N$ Gracefullness
Of Arbitrary Supersubdivisions of Graphs

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Abstract: A function $f$ is called a graceful labelling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph $G$ is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) $\phi$ is 1 − 1 (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. In this paper we prove that the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo $N$ graceful for all positive integers $N$.

Key Words: Modulo graceful graph, Smarandache modulo graceful graph, supersubdivisions of graphs, paths, disconnected paths, cycles and stars.

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§1. Introduction

S.W.Golomb introduced graceful labelling ([1]). The odd gracefulness was introduced by R.B.Gnanajothi in [2]. C.Sekar introduced one modulo three graceful labelling ([8]) recently. V.Ramachandran and C.Sekar ([6]) introduced the concept of one modulo $N$ graceful where $N$ is any positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. We prove that the the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo $N$ graceful for all positive integers $N$.

§2. Main Results

Definition 2.1 A graph $G$ is said to be one Smarandache modulo $N$ graceful on subgraph $H < G$ with $q$ edges (where $N$ is a positive integer) if there is a function $\phi$ from the vertex set

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of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ in such a way that (i) $\phi$ is $1 - 1$ (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $H$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$, and $E(G) \setminus E(h)$ to $\{1, 2, \ldots, |E(G)| - q\}$, where $\phi^*(uv) = |\phi(u) - \phi(v)|$. Particularly, if $H = G$ such a graph is said to be one modulo $N$ graceful graph.

**Definition 2.2**([9]) In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of $m$ vertices the $m$-vertices part of $K_{2,m}$. Let $G$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a supersubdivision of $G$ if $H$ is obtained by replacing every edge $e_i$ of $G$ by the complete bipartite graph $K_{2,m}$ for some positive integer $m$ in such a way that the ends of $e_i$ are merged with the two vertices part of $K_{2,m}$ after removing the edge $e_i$ from $G$. $H$ is denoted by $SS(G)$.

**Definition 2.3**([9]) A supersubdivision $H$ of a graph $G$ is said to be an arbitrary supersubdivision of the graph $G$ if every edge of $G$ is replaced by an arbitrary $K_{2,m}$ $(m$ may vary for each edge arbitrarily). $H$ is denoted by $ASS(G)$.

**Definition 2.4** A graph $G$ is said to be connected if any two vertices of $G$ are joined by a path. Otherwise it is called disconnected graph.

**Definition 2.5** A star $S_n$ with $n$ spokes is given by $(V, E)$ where $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(S_n) = \{v_0v_i/i = 1, 2, \ldots, n\}$. $v_0$ is called the centre of the star.

**Definition 2.6** A cycle $C_n$ with $n$ points is a graph given by $(V, E)$ where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

**Theorem 2.7** Arbitrary supersubdivisions of paths are one modulo $N$ graceful for every positive integer $N$.

**Proof** Let $P_n$ be a path with successive vertices $u_1, u_2, u_3, \ldots, u_n$ and let $e_i (1 \leq i \leq n - 1)$ denote the edge $u_iu_{i+1}$ of $P_n$. Let $H$ be an arbitrary supersubdivision of the path $P_n$ where each edge $e_i$ of $P_n$ is replaced by a complete bipartite graph $K_{2,m_i}$, where $m_i$ is any positive integer such as those shown in Fig.1 for $P_6$. We observe that $H$ has $M = 2(m_1 + m_2 + \cdots + m_{n-1})$ edges.

Define $\phi(u_i) = N(i - 1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \cdots, m_i$, let

$$
\phi(u_{i,i+1}) = \begin{cases} 
N(M - 2k + 1) + 1 & \text{if } i = 1, \\
N(M - 2k + i) - 2N(m_1 + m_2 + \cdots + m_{i-1}) + 1 & \text{if } i = 2, 3, \cdots n - 1.
\end{cases}
$$

It is clear from the above labelling that the $m_i+2$ vertices of $K_{2,m_i}$ have distinct labels and the $2m_i$ edges of $K_{2,m_i}$ also have distinct labels for $1 \leq i \leq n - 1$. Therefore, the vertices of each $K_{2,m_i}, 1 \leq i \leq n - 1$ in the arbitrary supersubdivision $H$ of $P_n$ have distinct labels and also the edges of each $K_{2,m_i}, 1 \leq i \leq n - 1$ in the arbitrary supersubdivision graph $H$ of $P_n$ have distinct labels. Also the function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ is in such a way that (i) $\phi$ is $1 - 1$, and (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$, where $\phi^*(uv) = |\phi(u) - \phi(v)|$. 


Hence $H$ is one modulo $N$ graceful.

Clearly, $\phi$ defines a one modulo $N$ graceful labelling of arbitrary supersubdivision of the path $P_n$. \qed

Example 2.8 An odd graceful labelling of $ASS(P_5)$ is shown in Fig.2.

Example 2.9 A graceful labelling of $ASS(P_6)$ is shown in Fig.3.
Example 2.10 A one modulo 7 graceful labelling of $ASS(P_6)$ is shown in Fig.4.

Theorem 2.11 Arbitrary supersubdivision of disconnecte paths $P_n \cup P_r$ are one modulo $N$ graceful provided the arbitrary supersubdivision is obtained by replacing each edge of $G$ by $K_{2,m}$ with $m \geq 2$.

Proof Let $P_n$ be a path with successive vertices $v_1, v_2, \ldots, v_n$ and let $e_i$ ($1 \leq i \leq n - 1$) denote the edge $v_iv_{i+1}$ of $P_n$. Let $P_r$ be a path with successive vertices $v_{n+1}, v_{n+2}, \ldots, v_{n+r}$ and let $e_i(n + 1 \leq i \leq n + r - 1)$ denote the edge $v_iv_{i+1}$.

Let $H$ be an arbitrary supersubdivision of the disconnected graph $P_n \cup P_r$ where each edge $e_i$ of $P_n \cup P_r$ is replaced by a complete bipartite graph $K_{2,m_i}$ with $m_i \geq 2$ for $1 \leq i \leq n - 1$ and $n + 1 \leq i \leq n + r - 1$. We observe that $H$ has $M = 2(m_1 + m_2 + \cdots + m_{n-1} + m_{n+1} + \cdots + m_{n+r-1})$ edges.
Define $\phi(v_i) = N(i - 1), i = 1, 2, 3, \ldots, n$, $\phi(v_i) = N(i), i = n + 1, n + 2, n + 3, \ldots, n + r$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} 
N(M - 2k + 1) + 1 & \text{if } i = 1, \\
N(M - 2 + i) + 1 - 2N(m_1 + m_2 + \cdots + m_{i-1} + k - 1) & \text{if } i = 2, 3, \cdots n - 1, \\
N(M - 1 + i) + 1 - 2N(m_1 + m_2 + \cdots + m_{n-1} + k - 1) & \text{if } i = n + 1, \\
N(M - 1 + i) + 1 - 2N[(m_1 + m_2 + \cdots + m_{n-1}) + (m_{n+1} + \cdots + m_{i-1}) + k - 1] & \text{if } i = n + 2, n + 3, \ldots n + r - 1.
\end{cases}$$

It is clear from the above labelling that the $m_i + 2$ vertices of $K_{2,m_i}$ have distinct labels and the $2m_i$ edges of $K_{2,m_i}$ also have distinct labels for $1 \leq i \leq n - 1$ and $n + 1 \leq i \leq n + r - 1$. Therefore the vertices of each $K_{2,m_i}$, $1 \leq i \leq n - 1$ and $n + 1 \leq i \leq n + r - 1$ in the arbitrary supersubdivision $H$ of $P_n \cup P_r$ have distinct labels and also the edges of each $K_{2,m_i}$, $1 \leq i \leq n - 1$ and $n + 1 \leq i \leq n + r - 1$ in the arbitrary supersubdivision graph $H$ of $P_n \cup P_r$ have distinct labels. Also the function $\phi$ from the vertex set of $G$ to $\{0, 1, N, (N + 1), 2N, (2N + 1), \ldots, N(q - 1), N(q - 1) + 1\}$ is in such a way that (i) $\phi$ is $1 - 1$, and (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, N + 1, 2N + 1, \ldots, N(q - 1) + 1\}$, where $\phi^*(uv) = |\phi(u) - \phi(v)|$. Hence $H$ is one modulo $N$ graceful.

Clearly, $\phi$ defines a one modulo $N$ graceful labelling of arbitrary supersubdivisions of disconnected paths $P_n \cup P_r$.

**Example 2.12** An odd graceful labelling of $ASS(P_6 \cup P_3)$ is shown in Fig.6.
Example 2.13 A graceful labelling of $ASS(P_3 \cup P_4)$ is shown in Fig.7.

Example 2.14 A one modulo 4 graceful labelling of $ASS(P_4 \cup P_3)$ is shown in Fig.8.

Theorem 2.15 For any any $n \geq 3$, there exists an arbitrary supersubdivision of $C_n$ which is
one modulo $N$ graceful for every positive integer $N$.

Proof Let $C_n$ be a cycle with consecutive vertices $v_1, v_2, \cdots, v_n$. Let $G$ be a super-subdivision of a cycle $C_n$ where each edge $e_i$ of $C_n$ is replaced by a complete bipartite graph $K_{2, m_i}$ where $m_i$ is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that $G$ has $M = 2(m_1 + m_2 + \cdots + m_n)$ edges. Here the edge $v_{n-1}v_1$ is replaced by $K_{2, n-1}$ for the construction of arbitrary supersubdivision of $C_n$.

![Fig.9 Cycle $C_n$](image)

![Fig.10 An arbitrary Supersubdivision of $C_5$](image)

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \ldots, m_i$, let

$$
\phi(v_{i,i+1}^{(k)}) = \begin{cases} 
N(M - 2k + 1) + 1 & \text{if } i = 1, \\
N(M - 2k + i) + 1 - 2N(m_1 + m_2 + \cdots + m_{i-1}) & \text{if } i = 2, 3, \cdots n - 1.
\end{cases}
$$
and $\phi(v_{n,1}^{(k)}) = N(n - k + m_n - 1) + 1$.

It is clear from the above labelling that the function $\phi$ from the vertex set of $G$ to \{0, 1, $N$, $(N + 1)$, $2N$, $(2N + 1)$, $\cdots$, $N(q - 1)$, $N(q - 1) + 1$\} is in such a way that (i) $\phi$ is $1 - 1$ (ii) $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to \{1, $N + 1$, $2N + 1$, $\cdots$, $N(q - 1) + 1$\} where $\phi^*(uv)=|\phi(u) - \phi(v)|$. Hence, $H$ is one modulo $N$ graceful. Clearly, $\phi$ defines a modulo $N$ graceful labelling of arbitrary supersubdivision of cycle $C_n$.

**Example 2.16** An odd graceful labelling of $ASS(C_5)$ is shown in Fig.11.

![Fig.11](image1)

**Example 2.17** A graceful labelling of $ASS(C_5)$ is shown in Fig.12.

![Fig.12](image2)

**Example 2.18** A one modulo 3 graceful labelling of $ASS(C_4)$ is shown in Fig.13.
Theorem 2.19  Arbitrary supersubdivision of any star is one modulo $N$ graceful for every positive integer $N$.

Proof  The proof is divided into 2 cases.

Case 1  $N = 1$

It has been proved in [4] that arbitrary supersubdivision of any star is graceful.

Case 2  $N > 1$.

Let $S_n$ be a star with vertices $v_0, v_1, v_2, \ldots, v_n$ and let $e_i$ denote the edge $v_0v_i$ of $S_n$ for $1 \leq$
Let \( H \) be an arbitrary supersubdivision of \( S_n \). That is for \( 1 \leq i \leq n \) each edge \( e_i \) of \( S_n \) is replaced by a complete bipartite graph \( K_{2,m_i} \) with \( m_i \) is any positive integer for \( 1 \leq i \leq n-1 \) and \( m_n = (n-1) \). It is clear that \( H \) has \( M = 2(m_1 + m_2 + \cdots + m_n) \) edges. The vertex set and edge set of \( H \) are given by \( V(H) = \{v_0, v_1, v_2 \cdots, v_n, v_0^{(1)}, v_0^{(2)}, \cdots, v_0^{(m_1)}, v_0^{(m_2)}, \cdots, v_0^{(m_n)}\} \).

Define \( \phi : V(H) \to \{0, 1, 2 \cdots 2\sum_{i=1}^n m_i\} \) as follows:

\[
\phi(v_0) = 0. \text{ For } k = 1, 2, 3, \ldots, m_i, \text{ let }
\phi(v_0^{(k)}) = \begin{cases} 
N(M-k) + 1 & \text{if } i = 1, \\
N(M-k) + 1 - N(m_1 + m_2 + \cdots + m_{i-1}) & \text{if } i = 2, 3, \cdots n.
\end{cases}
\]

\[
\phi(v_i) = \begin{cases} 
N(M-m_i) & \text{if } i = 1, \\
NM - N(2m_1 + 2m_2 + \cdots + 2m_{i-1} + m_i) & \text{if } i = 2, 3, \cdots n.
\end{cases}
\]

It is clear from the above labelling that the function \( \phi \) from the vertex set of \( G \) to \( \{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\} \) is in such a way that (i) \( \phi \) is 1-1 (ii) \( \phi \) induces a bijection \( \phi^* \) from the edge set of \( G \) to \( \{1, N+1, 2N+1, \ldots, N(q-1)+1\} \) where \( \phi^*(uv) = |\phi(u) - \phi(v)| \). Hence \( H \) is one modulo \( N \) graceful.

Clearly, \( \phi \) defines a one modulo \( N \) graceful labelling of arbitrary supersubdivision of star \( S_n \). \( \Box \)

**Example 2.20** A one modulo 5 graceful labelling of \( \text{ASS}(S_4) \) is shown in Fig.14.
Example 2.21 An odd graceful labelling of \( AS(S_6) \) is shown in Fig.15.

![Fig.15](image)

References


