Non-Congruent Triangles with Equal Perimeters and Arias

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In [1] Professor I. Ivănescu from Craiova has proposed the following

Open problem
Construct, using a ruler and a compass, two non-congruent triangles, which have equal perimeters and arias.

In preparation for the proof of this problem we recall several notions and we prove a Lemma.

Definition
An A-ex-inscribed circle to a given triangle $ABC$ is the tangent circle to the side $(BC)$ and to the extended sides $(AB), (AC)$.

The center of the A-ex-inscribed triangle is the intersection of the external bisectors of the angles $B$ and $C$, which we note it with $I_a$ and its radius with $r_a$.

Observation 1.
To a given triangle correspond three ex-inscribed circles. In figure 1 we represent the A-ex-inscribed circle to triangle $ABC$.
Lemma 1
The length of the tangent constructed from one of the triangle’s vertexes to the corresponding ex-inscribed circle is equal with the triangle’s semi-perimeter.

Proof
Let $D_a, E_a, F_a$ the points of contact of the $A$-ex-inscribed triangle with $(BC), AC, AB$. We have $AE_a = AF_a$, $BD_a = BF_a$, $CD_a = CE_a$ (the tangents constructed from a point to a circle are congruent). We note $BD_a = x$, $CD_a = y$ and we observe that $AE_a = AC + CE_a$, therefore $AE_a = b + y$, $AF_a = AB + BF_a$, it results that $AF_a = c + x$. We resolve the system:

\[
\begin{cases}
x + y = a \\
x + c = y + b
\end{cases}
\]

and we obtain

\[
x = \frac{1}{2}(a + b - c) \]
\[
y = \frac{1}{2}(a + c - b)
\]

Taking into consideration that the semi-perimeter $p = \frac{1}{2}(a + b + c)$ we have $x = p - c; y = p - b$, and we obtain that $AF_a = AE_a = p$ thus the lemma is proved.

The proof of the open problem

Fig.2
Let $ABC$ a given triangle. We construct $C(I, r)$ its inscribed circle and $C(I_a, r_a)$ its A-ex-inscribed circle, see figure 2. In conformity with the Lemma we have that $AF_a = p$ - the semi-perimeter of triangle $ABC$.

We construct the point $F' \in (AF')$ and the circle of radius $r$ tangent in $F'$ to $AB$, that is $C(I', r)$. It is easy to justify that angle $F'AI' > \angle FAI$ and therefore angle $F'AE' > \angle A$ (we noted $E$ the contact point with the circle $C(I', r)$ of the tangent constructed from $A$). We note $I'_a$ the intersection point of the lines $AI', IA_F$.

We construct the circle $C(I'_aI_aF_a)$ and then the internal common tangent to this circle and to the circle $C(I', r)$; we note $B' , C'$ the intersections of this tangent with $AB$ respectively with $AE'$. From these constructions it result that the circle $C(I', r)$ is inscribed in the triangle $AB'C'$ and the circle $C(I'_aI_aF_a)$ ex-inscribed to this triangle.

The Lemma states that the semi-perimeter of the triangle $AB'C'$ is equal with $AF_a$ therefore it is equal to $p$ - the semi-perimeter of triangle $ABC$.

On the other side the inscribed circles in the triangles $ABC$ and $AB'C'$ are congruent. Because the aria $S$ of the triangle $ABC$ is given by the formula $S = p \cdot r$, we obtain that also the aria of triangle $AB'C'$ is equal with $S$.

The constructions listed above can be executed with a ruler and a compass without difficulty, and the triangles $ABC$ and $AB'C'$ are not congruent.

Indeed, our constructions are such that the angle $B'AC'$ is greater than angle $BAC$. Also we can choose $F'$ on $(AF')$ such that $F'AI'$ is different of $\frac{1}{2} C$ and of $\frac{1}{2} B$. In this way the angle $A$ of the triangle $AB'C'$ is not congruent with any angle of the triangle $ABC$.

**Observation 2**

We practically proved much more than the proposed problem asked, because we showed that for any given triangle $ABC$ we can construct another triangle which will have the same aria and the same perimeter with the given triangle without being congruent with it.

**Observation 3**

In [2] the authors find two isosceles triangles in the conditions of the hypothesis.

**Note**

The authors thank to Professor Ştefan Brânzan from the National College “Fraţii Buzesti” – Craiova for his suggestions, which made possible the enrichment of this article.

**References**
