Some Properties of the Harmonic Quadrilateral

Professor Ion Patrascu, National College "Fraţii Buzeşti", Craiova, Romania
Professor Florentin Smarandache, University of New Mexico, USA

Abstract: In this article, we review some properties of the harmonic quadrilateral related to triangle simedians and to Apollonius circles.

Definition 1. A convex circumscribable quadrilateral $ABCD$ having the property $AB \cdot CD = BC \cdot AD$ is called **harmonic quadrilateral**.

Definition 2. A **triangle simedian** is the isogonal cevian of a triangle median.

Proposition 1. In the triangle ABC, the cevian AA_1, $A_1 \in (BC)$ is a simedian if and only if $\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2$. For **Proof** of this property, see *infra*.

Proposition 2. In an harmonic quadrilateral, the diagonals are simedians of the triangles determined by two consecutive sides of a quadrilateral with its diagonal.

Proof. Let $ABCD$ be an harmonic quadrilateral and $\{K\} = AC \cap BD$ (see *Fig. 1*). We prove that BK is simedian in the triangle ABC.

From the similarity of the triangles ABK and DCK, we find that:

$$\frac{AB}{DC} = \frac{AK}{DK} = \frac{BK}{CK} \ (1).$$

From the similarity of the triangles BCK şi ADK, we conclude that:

$$\frac{BC}{AD} = \frac{CK}{DK} = \frac{BK}{AK} \ (2).$$

From the relations (1) and (2), by division, it follows that:

$$\frac{AB}{BC} \cdot \frac{AD}{DC} = \frac{AK}{CK} \ (3).$$

But $ABCD$ is an harmonic quadrilateral; consequently,

$$\frac{AB}{BC} = \frac{AD}{DC};$$

substituting this relation in (3), it follows that:

$$\left(\frac{AB}{BC}\right)^2 = \frac{AK}{CK};$$

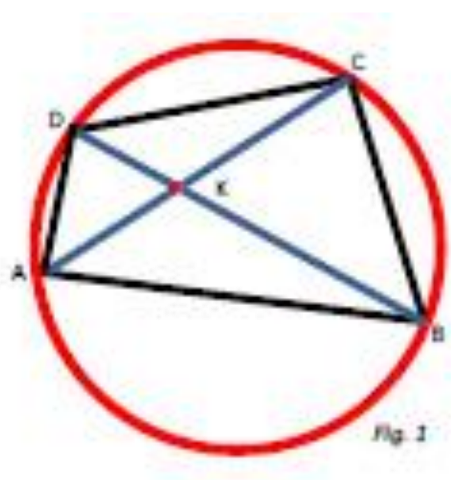

Fig. 1

as shown by Proposition 1, BK is a simedian in the triangle ABC. Similarly, it can be shown that AK is a simedian in the triangle ABD, that CK is a simedian in the triangle BCD, and that DK is a simedian in the triangle ADC.

Remark 1. The converse of the Proposition 2 is proved similarly, i.e.:

Proposition 3. If in a convex circumscribable quadrilateral a diagonal is a simedian in the triangle formed by the other diagonal with two consecutive sides of the quadrilateral, then the quadrilateral is an harmonic quadrilateral.

Remark 2. From Propositions 2 and 3 above, it results a simple way to build an harmonic quadrilateral. In a circle, let a triangle ABC be considered; we construct the simedian of *A*, be it AK, and we denote by *D* the intersection of the simedian AK with the circle. The quadrilateral $ABCD$ is an harmonic quadrilateral.

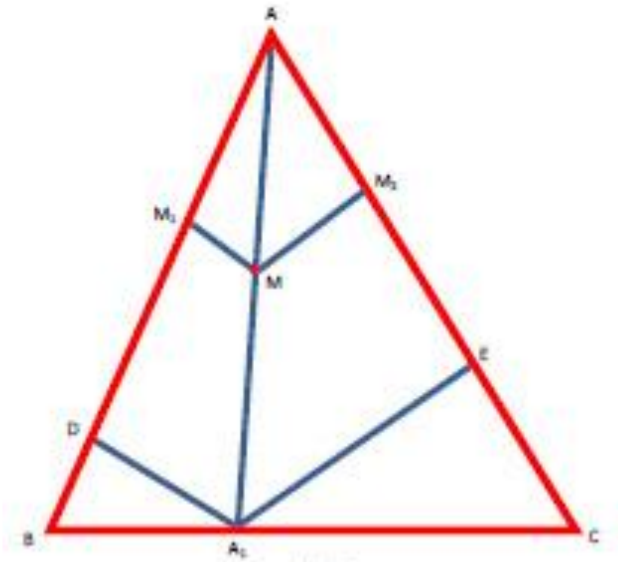

Fig. 2

Proposition 4. In a triangle ABC, the points of the simedian of *A* are situated at proportional lengths to the sides AB and AC.

Proof. We have the simedian AA_1 in the triangle ABC (see *Fig. 2*). We denote by D and E the projections of A_1 on AB, and AC respectively.
We get:

$$\frac{BA_1}{CA_1} = \frac{Area_\Delta(ABA_1)}{Area_\Delta(ACA_1)} = \frac{AB \cdot A_1D}{AC \cdot A_1E}.$$

Moreover, from Proposition 1, we know that

$$\frac{BA_1}{A_1C} = \left(\frac{AB}{AC}\right)^2.$$

Substituting in the previous relation, we obtain that:

$$\frac{A_1D}{A_1E} = \frac{AB}{AC}.$$

On the other hand, $DA_1 = AA_1$. From BAA_1 and $A_1E = AA_1 \cdot sin\widehat{CAA_1}$, hence:

$$\frac{A_1D}{A_1E} = \frac{sin\widehat{BAA_1}}{sin\widehat{CAA_1}} = \frac{AB}{AC} \quad (4).$$

If M is a point on the simedian and MM_1 and MM_2 are its projections on AB, and AC respectively, we have:

$$MM_1 = AM \cdot sin\widehat{BAA_1}, \qquad MM_2 = AM \cdot sin\widehat{CAA_1},$$

hence:

$$\frac{MM_1}{MM_2} = \frac{\sin\widehat{BAA_1}}{\sin\widehat{CAA_1}}.$$

Taking (4) into account, we obtain that:

$$\frac{MM_1}{MM_2} = \frac{AB}{AC}.$$

Remark 3. The converse of the property in the statement above is valid, meaning that, if M is a point inside a triangle, its distances to two sides are proportional to the lengths of these sides. The point belongs to the simedian of the triangle having the vertex joint to the two sides.

Proposition 5. In an harmonic quadrilateral, the point of intersection of the diagonals is located towards the sides of the quadrilateral to proportional distances to the length of these sides.
The **Proof** of this Proposition relies on Propositions 2 and 4.

Proposition 6 (R. Tucker). The point of intersection of the diagonals of an harmonic quadrilateral minimizes the sum of squares of distances from a point inside the quadrilateral to the quadrilateral sides.

Proof. Let $ABCD$ be an harmonic quadrilateral and M any point within. We denote by x, y, z, u the distances of M to the AB, BC, CD, DA sides of lenghts $a, b, c,$ and d (see *Fig. 3*).
Let S be the $ABCD$ quadrilateral area.
We have:

$$ax + by + cz + du = 2S.$$

This is true for x, y, z, u and a, b, c, d real numbers.
Following Cauchy-Buniakowski-Schwarz Inequality, we get:

$$(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + u$$

and it is obvious that:

$$x^2 + y^2 + z^2 + u^2 \geq \frac{4S^2}{a^2 + b^2 + c^2 + c}$$

We note that the minimum sum of squared distances is:

$$\frac{4S^2}{a^2 + b^2 + c^2 + d^2} = const.$$

In Cauchy-Buniakowski-Schwarz Inequality, the equality occurs if:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{u}{d}.$$

Since $\{K\} = AC \cap BD$ is the only point with this property, it ensues that $M = K$, so K has the property of the minimum in the statement.

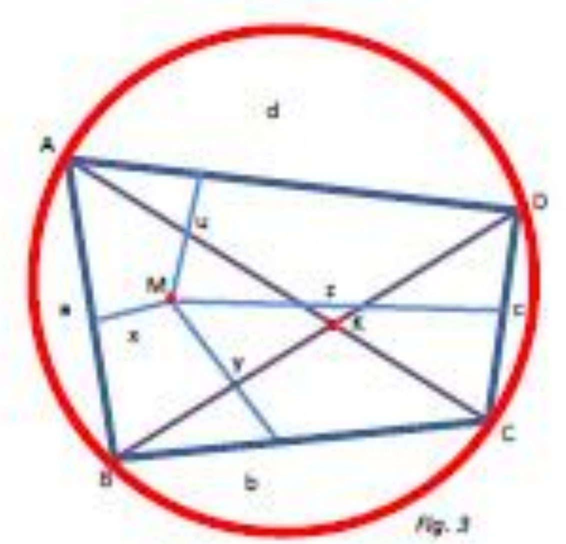

Fig. 3

Definition 3. We call external simedian of ABC triangle a cevian $AA_1{}'$ corresponding to the vertex A, where $A_1{}'$ is the harmonic conjugate of the point A_1 – simedian's foot from A relative to points B and C.

Remark 4. In *Fig. 4*, the cevian AA_1 is an internal simedian, and $AA_1{}'$ is an external simedian.
We have:

$$\frac{A_1B}{A_1C} = \frac{A_1{}'B}{A_1{}'C}.$$

In view of Proposition 1, we get that:

$$\frac{A_1{}'B}{A_1{}'C} = \left(\frac{AB}{AC}\right)^2.$$

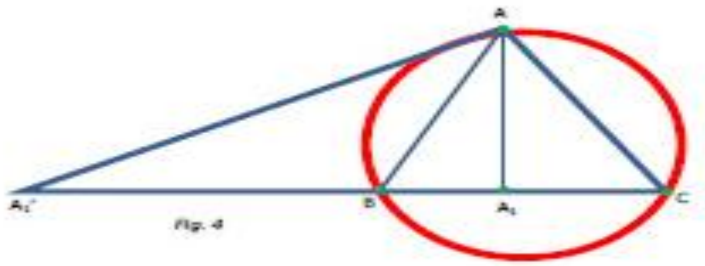
Fig. 4

Proposition 7. The tangents taken to the extremes of a diagonal of a circle circumscribed to the harmonic quadrilateral intersect on the other diagonal.

Proof. Let P be the intersection of a tangent taken in D to the circle circumscribed to the harmonic quadrilateral $ABCD$ with AC (see *Fig. 5*). Since triangles PDC and PAD are alike, we conclude that:

$$\frac{PD}{PA} = \frac{PC}{PD} = \frac{DC}{AD} \quad (5).$$

From relations (5), we find that:

$$\frac{PA}{PC} = \left(\frac{AD}{DC}\right)^2 \quad (6).$$

This relationship indicates that P is the harmonic conjugate of *K* with respect to *A* and *C*, so DP is an external simedian from *D* of the triangle ADC.
Similarly, if we denote by P' the intersection of the tangent taken in *B* to the circle circumscribed with AC, we get:

$$\frac{P\prime A}{P\prime C} = \left(\frac{BA}{BC}\right)^2 \quad (7).$$

From (6) and (7), as well as from the properties of the harmonic quadrilateral, we know that:

$$\frac{AB}{BC} = \frac{AD}{DC},$$

which means that:

$$\frac{PA}{PC} = \frac{P'A}{P'C},$$

hence $P = P'$.
Similarly, it is shown that the tangents taken to *A* and *C* intersect at point *Q* located on the diagonal BD.

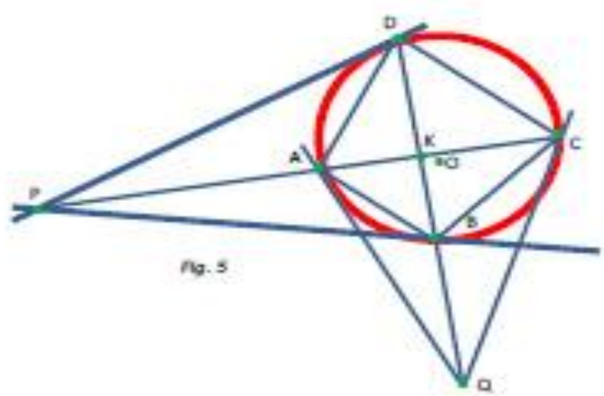
Fig. 5

Remark 5. a. The points *P* and *Q* are the diagonal poles of BD and AC in relation to the circle circumscribed to the quadrilateral.

b. From the previous Proposition, it follows that in a triangle the internal simedian of an angle is consecutive to the external simedians of the other two angles.

Proposition 8. Let $ABCD$ be an harmonic quadrilateral inscribed in the circle of center *O* and let *P* and *Q* be the intersections of the tangents taken in *B* and *D*, respectively in *A* and *C* to the circle circumscribed to the quadrilateral. If $\{K\} = AC \cap BD$, then the orthocenter of triangle PKQ is *O*.

Proof. From the properties of tangents taken from a point to a circle, we conclude that $PO \perp BD$ and $QO \perp AC$. These relations show that in the triangle PKQ, PO and QO are heights, so O is the orthocenter of this triangle.

Definition 4. The Apollonius circle related to the vertex *A* of the triangle ABC is the circle built on the segment $[DE]$ in diameter, where *D* and *E* are the feet of the internal, respectively ,external bisectors taken from *A* to the triangle ABC.

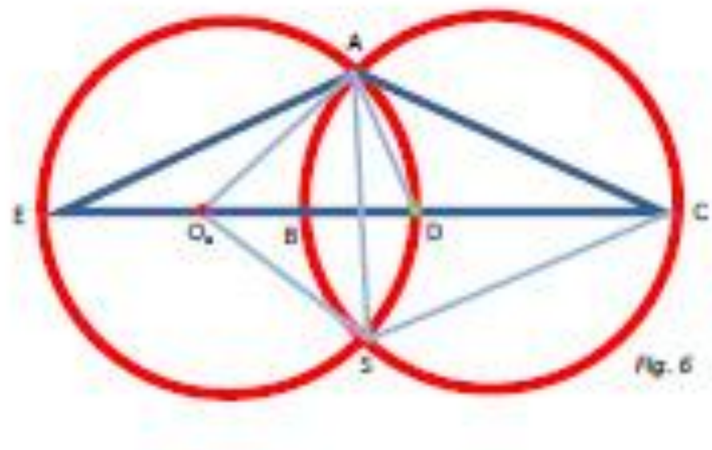

Fig. 6

Remark 6. If the triangle ABC is isosceles with $AB = AC$, the Apollonius circle corresponding to vertex *A* is not defined.

Proposition 9. The Apollonius circle relative to the vertex *A* of the triangle ABC has as center the feet of the external simedian taken from *A*.

Proof. Let O_a be the intersection of the external simedian of the triangle ABC with BC (see *Fig. 6*).

Assuming that $m(\hat{B}) > m(\hat{C})$, we find that $m(\widehat{EAB}) = \frac{1}{2}[m(\hat{B}) + m(\hat{C})]$.

O_a being a tangent, we find that $m(\widehat{O_aAB}) = m(\hat{C})$.

Withal, $m(EAO_a) = \frac{1}{2}[m(\hat{B}) - m(\hat{C})]$ and $m(AEO_a) = \frac{1}{2}[m(\hat{B}) - m(\hat{C})]$.

It results that:

$OaE = OaA$;

onward, EAD being a right angled triangle, we obtain:

$OaA = OaD,$

hence O_a is the center of Apollonius circle corresponding to the vertex A.

Proposition 10. Apollonius circle relative to the vertex *A* of triangle ABC cuts the circle circumscribed to the triangle following the internal simedian taken from *A*.

Proof. Let *S* be the second point of intersection of Apollonius circles relative to vertex *A* and the circle circumscribing the triangle ABC.

Because O_aA is tangent to the circle circumscribed in A, it results, for reasons of symmetry, that O_aS will be tangent in *S* to the circumscribed circle.
For triangle ACS, O_aA and O_aS are external simedians; it results that CO_a is internal simedian in the triangle ACS, furthermore, it results that the quadrilateral $ABSC$ is an harmonic quadrilateral.
Consequently, AS is the internal simedian of the triangle ABC and the property is proven.
Remark 7. From this, in view of *Fig. 5*, it results that the circle of center *Q* passing through *A* and *C* is an Apollonius circle relative to the vertex *A* for the triangle ABD.
This circle (of center *Q* and radius *QC*) is also an Apollonius circle relative to the vertex *C* of the triangle BCD.
Similarly, the Apollonius circles corresponding to vertexes *B* and *D* and to the triangles ABC, and ADC respectively, coincide; we can formulate the following:
Proposition 11. In an harmonic quadrilateral, the Apollonius circles - associated with the vertexes of a diagonal and to the triangles determined by those vertexes to the other diagonal - coincide.
Radical axis of the Apollonius circles is the right determined by the center of the circle circumscribed to the harmonic quadrilateral and by the intersection of its diagonals.
Proof. Referring to *Fig. 5*, we observe that the power of *O* towards the Apollonius circles relative to vertexes *B* and *C* of triangles ABC and BCU is:
$OB^2 = OC^2$.
So *O* belongs to the radical axis of the circles.
We also have $KA \cdot KC = KB \cdot KD$, relatives indicating that the point *K* has equal powers towards the highlighted Apollonius circles.

References.

[1] Roger A. Johnson – **Advanced Euclidean Geometry**, Dover Publications, Inc. Mineola, New York, USA, 2007.
[2] F. Smarandache, I. Patrascu – **The Geometry of Homological Triangles**, The Education Publisher, Inc. Columbus, Ohio, USA, 2012.
[2] F. Smarandache, I. Patrascu – **Variance on Topics of plane Geometry**, The Education Publisher, Inc. Columbus, Ohio, USA, 2013.