On Rotating Frames and the Relativistic Contraction of the Radius (The Rotating Disc)

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Abstract

The relativistic problem of the rotating disc or rotating frame is studied. The solution given implies the contraction of the radius and the change of the value of π depending on the type of observer. Two forms of rotation are considered. One is with constant angular velocity, independent of the radius, implying a horizon, the other is with exponentially decreasing angular velocity with respect to the radius and does not imply a horizon. In all cases the paths of signals emanating from the origin of the rotating frame advance helically in the positive and negative z direction, where they are concentrated, due to the contraction of the radius, and in some cases appear as jets.

1. Introduction

The rotating disc has been the subject of a multitude of papers since Ehrenfest [1] published what is today known as the "Ehrenfest Paradox". He noted that since the perimeter of a rotating disc would relativistically contract, while the radius remained the same, a deformation of the disc should take place. Many authors have since then contributed to the understanding of the problem by studying the geometry of the rotating disc. A historical review can be found in Rizzi and Ruggiero [2] and in Grøn [3]. The approach in the present study is closer to the idea that there is a contraction of the radius of the rotating disc. Similar ideas have been explored by Ashworth, Davies and Jennison [4], [5] and by Grünbaum and Janis [6], [7]. In particular, Ashworth, Davies and Jennison show that the radius of the rotating disc contracts according to an observer that is rotating with the disc at a distance $r > 0$ from the center. Grünbaum and Janis argue that an observer that is not rotating with the disc will see the radius of the disc contract by the same relativistic factor as the perimeter. The latter approach is closer to ours although we do not find the same results because we allow the value of π to change for the non-rotating observer, when he makes measurements regarding the rotating frame.

The paper is organized as follows: In section 2 we state our basic assumptions that will help us derive the transformations in the following sections. In sections 3 we present our notation and known relativistic results regarding the rotating frame. In section 4 we formulate and solve the problem of the contraction of the radius of the disc and the transformation of the value of π for the non rotating observer. In section 5 we present a summary of the results up to that point. In section 6 we consider the space or disc deformation as seen by the non-rotating observer and distinguish between two kinds of non- rotating observers: one within the horizon of radius *c/w* and one outside. In section 7 we discuss the results. In section 8 we generalize to rotating frames in 3 dimensions and plot the signals emanating from the origin of the rotating frame in the radial direction, as seen by the non-rotating observers. The signals gradually bend sideways until they reach a 90^o degree angle with respect to the radius, while they advance in the positive and negative z axis direction. In section 9 we examine rotation with slippage so that the angular velocity is assumed to decrease exponentially as the radius increases and as z increases. We examine both the close by non-rotating observer and the far away non-rotating observer. The signals are in this case not limited to a horizon but gradually bend sideways until they reach a maximum deflection from the radial direction and then turn asymptotically back to the radial as the radius increases, while they advance in the positive and negative z direction. In section 10 we present our conclusions.

2. Assumptions on the rotation of two concentric discs

In the following we will make three assumptions. Assumptions 2 and 3 are standard in relativity theory. Only Assumption 1 is new.

1. Discussion to justify Assumption 1

Suppose an observer O_1 is at any place on disc (disc 1) and a second observer O_2 that sits at any place of another disc (disc 2) parallel and coaxial to disc 1. O_1 sticks a pencil through his disc parallel to the axis of rotation with its point touching the other disc (disc 2). O_2 does the same thing with the tip of his pencil touching disc 1. As there is relative rotation of one disc with respect to the other, each observer will watch the perimeter of a circle being drawn on his own and the other disc. Each will see that the duration of time, according to his own clock, for a complete circle to be drawn on his own disc will be the same with the duration it takes for him to draw a complete circle on the other disc. This observation holds for half a circle or any fraction of a circle. In short, we say that the two observers will agree on epicenter angles measured as fractions of a circle (not radians). If we denote Θ the magnitude of an angle as fraction of a circle and θ the magnitude of the same angle in radians then ¯ = 2®Θ. However, we will not use ® for the moment because we are not sure the observers agree on ® .
Imagine now that the same two observers stand at the center O of their disc (one on top of the other) and let the discs rotate with respect to each other. If the first observer announces that according to his measurements the second disc is rotating with frequency ¬ , since the situation is exactly symmetrical we will expect the second observer to make the same announcement regarding the first disc. But frequency is defined as revolutions per unit time or Θ per unit time. Once they agree on Θ and ¬ , they have to agree on time rate. In fact, they may even use the same clock, since they are collocated. We may summarize in the following,

<u>Assumption 1</u>
(a)Two observers sitting at two parallel concentric discs rotating with respect to each other around an axis vertical at their center, will agree that the magnitudes of epicenter angles traveled by the other disc, measured as fractions of a circle, are equal.
(b) If they also stand at the common center of their discs, they will further agree that time rates are equal.

Remark: The situation with the two observers at the center of their discs (one on top of the other) is symmetrical and there is no point in arguing who is rotating and who is stationary. However, if they had a way to measure the centrifugal acceleration off their center, they would probably find that it is different. In that respect we may rightfully call the frame with zero centrifugal force the “preferred frame”. In what follows we will assume that one observer, usually to be denoted as O' (or O_2), sits at the center of a preferred frame K' (or K_2), which coincides with the laboratory frame unless otherwise specified.

2. Discussion to justify Assumption 2
Many experiments have verified the constancy of the speed of light, which is the basic assumption for special relativity theory. The speed of light is not the same for observers under acceleration or, according to the Principle of Equivalence, gravity.

Assumption 2
Light speed does not depend on the speed of the emitting source and its speed is constant for all observers that are not under the influence of acceleration.

3. Discussion to justify Assumption 3
An observer on a frame will agree with another observer on the same frame on the measurements of lengths. This is expected if measurements are made not by signals but by actually placing the measuring stick on the length to be measured and counting how many times it fits to it. We state then,

Assumption 3
Observers on the same frame will agree on measurements of length.

3. Time Rate and the Contraction of the Perimeter on a Rotating Disc

As we talk interchangeably about rotating discs and frames, we need to clarify that when we talk of a disc, we imagine it placed at the x-y pane of the respective frame with center at the origin.
Let two frames K_1 and K_2 with cylindrical coordinates, common origin O and common axis of rotation Z. Let observer O_1 sit at the origin of K_1 and O_2 at the origin of K_2. Let K_2 be a non rotating frame - laboratory frame- and let O_2 observe frame K_1 rotate. Let a third observer O_3 on K_1 sit at a distance from the center. When there is no danger of confusion we will rename the three observers using $O = O_1$, $O' = O_2$, $\tilde{O} = O_3$ and the frames $K_1 = K$, $K_2 = K'$.
To help clarify ideas we introduce the two subscript notation, which we will abandon shortly afterwards for economy:
A quantity Q_{ij} is defined as the quantity measured by observer i given it is stationary in the frame of observer j
For example, ΔL_{21} is the length measured by observer 2 for a line segment on the perimeter of the disc that is stationary in the frame of observer 1. Also, Δt_{21} is the time interval that observer 2 sees that a clock stationary and with observer 1 shows for the duration of an event.

Note that by Assumption 3, ΔL_{ii} is a constant for all i and will be denoted as ΔL_{stat} since all observers agree on the same segment when stationary in their frame. The same is not true for Δt_{ii}. In particular, we expect $\Delta t_{11} = \Delta t_{22} = \Delta t_{stat} \neq \Delta t_{33}$, and $\Delta t_{12} = \Delta t_{21} = \Delta t_{stat}$ because observers 1 and 2 have the same clock , while observer 3 is under the influence of centrifugal acceleration, which we suspect that will affect Δt_{33} in some unknown as yet way. Therefore, the clock of observer 1 and 2 represents stationary clock time intervals and are denoted as Δt_{stat}. Also, $\Delta t_{13} = \Delta t_{33}$ because observers 1 and 3 are on the same frame and time intervals measured by observer 1 looking at the clock of observer 3, agree with what observer 3 sees looking at his own clock. Distances in the radial direction are measured as r_{ij}, while the angular velocity of a disc is measured by w_{ij}. Specifically, $r_{21} = r_{23}$ is the radial distance as seen be the non rotating observer 2 regardless of the second subscript since both observers 1 and 3 are stationary on the rotating disc. Similarly, $w_{21} = w_{23}$ because the angular velocity of the rotating disc as seen by the non rotating observer 2, using his own clock does not depend on the second subscript since both observers 1 and 3 are stationary on the rotating disc.

Most of the authors (see for example Møller [8] pp.222-250 on rotating disc) start from the transformation between a rotating frame K_1 and a non rotating frame K_2 and the transformation relation of cylindrical coordinates $r_1 = r_2$, $z_1 = z_2$, $\theta_1 = \theta_2 - wt_1$. In this case The metric for the rotating frame is

$$ds^2 = dr_1^2 + r_1^2 d\theta_1^2 + dz_1^2 + 2wr_1^2 d\theta_1 dt_1 - (c^2 - w^2 r_1^2) dt_1^2 \tag{1}$$

For a non-moving point in space, the space differentials are null and equating the metrics of the rotating and non-rotating frames we find $c^2 dt_2^2 = (c^2 - w^2 r_1^2) dt_1^2$ from which we find that the clock of the rotating system runs slower,

$$dt_1 = \frac{dt_2}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}} \tag{2}$$

The line element $d\sigma^2$ is given by $d\sigma^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta$ where x^α takes the values $\{r_1, \theta_1, z\}$ and $\gamma_{\alpha\beta} = 0$ except for $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}}$, $\gamma_{zz} = 1$. Hence, we find,

$$d\sigma^2 = dr_1^2 + \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}} d\theta_1^2 + dz^2 \tag{3}$$

For a line segment dL_i along the perimeter this formula implies that

$$dL_2 = \frac{dL_1}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}} \tag{4}$$

Where ($dL_1 = r_1 d\theta_1$ and $dL_2 = r_2 d\theta_2$).
Whereas the result (2) is within our expectations from the special theory of relativity, the result (4) is contrary to the expected result. Since the normal Lorentz contraction would give a contraction of the perimeter instead of a lengthening as we have found above. The

result (4) is counterintuitive for another reason also: If we increase the radius, while decreasing the angular velocity so that that tangential velocity is constant ($wr_1 =$ ³) then the perimeter tends to a straight line and the transformation should approach the Lorentz length contraction. Instead, according to (4) since $wr_1 =$ ³ is constant the lengthening factor remains constant and there is no way of approaching the Lorentz contraction.
As the method with which the result is obtained is correct, the only suspect is the form of transformation assumed. We are motivated, therefore, to search for another transformation that will also satisfy the contraction of the perimeter according to the Lorentz length contraction of special relativity.

Our quest is, therefore, to find a transformation that satisfies in our notation the following,

(a) The rate of the clock of O_3 will appear slower to observer O_2:

$$\Delta t_{23} = \frac{1}{\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}}}\Delta t_{stat} \tag{5}$$

(Recall that $w_{21} = w_{23}$ and $r_{21} = r_{23}$ as we mentioned above)
(b) Line segments along the perimeter are contracted,

$$\Delta L_{21} = \Delta L_{stat}\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}} \tag{6}$$

Observe that (5) implies that the rate of the clock of O_3 will also appear slower to O_1, who has the same clock as O_2. Namely,

$$\Delta t_{13} = \Delta t_{33} = \frac{1}{\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}}}\Delta t_{stat} \tag{7}$$

But since there is no relative motion between O_1 and O_3, O_1 will think that is due to the centrifugal acceleration that O_3 feels. Requirements (a), (b) along with the assumptions 1, 2, 3 imply a transformation (contraction) on the radial distance as we will see below.

4. The Contraction of the Radius of the Rotating Disc

Refer again to observers O' (or O_2) on K' (or K_2), O (or O_1) and $\tilde{O}$ (or O_3) on K (or K_1), with K rotating with respect to K' with frequency v' according to O' (and ¬ according to O). Suppose observer O has a rod that extends radially from the center O to some point A (see Figure 1). The rod is hollow mirrored inside and infinitesimally thin so that light traveling through it follows a straight line. The rod is just an artifact to help imagine things, a statement saying that O sends a light signal radially outward is enough. Observers O and O' will agree on time rates ($dt = dt'$) (Assumption 1) and the frequencies of rotation they observe will be equal (¬ =¬ ′) (Assumption 1). They will not agree on angular velocity measured in radians per unit time, because they will in general disagree on ® and, therefore, we may say that for observer O the angular velocity is, $w = 2$®¬ ($= w_{11} = 2$®$_{11}v$), while for O', $w' = 2$®′¬ ($= w_{21} = 2$®$_{21}v$).

Observer O (who is not under acceleration because he sits at the origin although he is rotating with the disc) sends a light signal from O towards A through the rod. According to him the signal travels with velocity $\upsilon = c$ (Assumption 2) the distance OA $= r$ (see Figure 1). Until the signal reaches the end of the rod, the rod will have moved to position OB. Observer O' will see the signal travel a curved path (OCB') with constant tangential velocity $\upsilon' = c$ (Assumption 2). At the perimeter the direction of the velocity of the signal, according to O', will make an angle φ with respect to the radius OB' which will have length $OB' = r' (= r_{21})$.

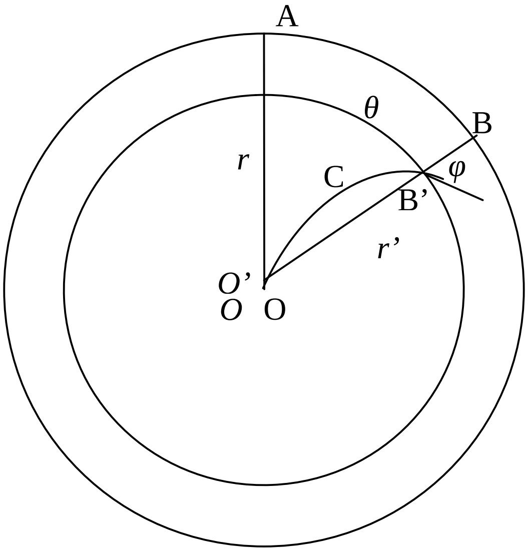


Figure 1 The path of the light signal originating at O is OCB' according to observer O' (the lab observer)

Let

r : the radius as observer O measures it (stationary length) (OA=OB)

r' : the radius according to observer O' (OB')

υ_r': the radial component of the velocity υ' according to observer O'

$\upsilon_\perp'$: the component of υ' perpendicular to the radius according to observer O'

φ : the angle between υ_r' and υ' (to be called *angle of deflection* from the radial direction)

We may write the following relationships for light signals letting $\upsilon' = c$ for :
For a light signal $\upsilon' = c$ and then

$$\upsilon_r' = c\cos\varphi \tag{8}$$

$$\upsilon_\perp' = c\sin\varphi \tag{9}$$

$$\upsilon_\perp' = w'r' \tag{10}$$

where w' is the angular velocity in radians as observed by O' and therefore,

$$w' = 2\pi\nu' \tag{11}$$

From (9) and(10)

$$\sin\varphi = \frac{w'r'}{c} \tag{12}$$

And substituting in (8)

$$\upsilon_r' = c\sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{13}$$

In order to satisfy the condition for the contraction of the perimeter (see (6)) we further require that

$$2\pi' r' = 2\pi r\sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{14}$$

Solve for π' and substitute in (11) to obtain

$$\frac{w'}{2\pi\nu} = \frac{r}{r'}\sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{15}$$

We already defined $w = 2\pi\nu$. Substitute in (15) and solve for w' to obtain

$$w'^2 = \frac{w^2 r^2 c^2}{c^2 r'^2 + r^2 w^2 r'^2} \tag{16}$$

Now substitute w'^2 in (13) noting that $\upsilon_r' \triangleq \frac{dr'}{dt'} = \frac{dr'}{dt}$ and we find

$$\frac{dr'}{dt} = c\sqrt{1-\frac{w^2 r^2}{c^2 + w^2 r^2}} \tag{17}$$

Using $r = ct$, (17) becomes

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1+w^2 t^2}} \tag{18}$$

and integrating with respect to t we find

$$r' = \frac{c}{w}\ln(wt + \sqrt{1+w^2 t^2}) = \frac{ct}{wt}\ln(wt + \sqrt{1+w^2 t^2}) = \frac{r}{wt}\operatorname{arc}\sinh(wt) \tag{19}$$

where, the constant of integration is zero because we require that $r' = 0$ for $t = 0$.
Equivalently, since $r = ct$, (19) can take the form

$$\sinh\frac{wr'}{c} = \frac{wr}{c} \tag{20}$$

Using (20), (16) becomes,

$$w'^2 = \frac{w^4 r^2}{(c^2 + r^2 w^2)\ln^2(\frac{wr}{c} + \sqrt{1+\frac{w^2 r^2}{c^2}})} = \frac{w^4 t^2}{(1+w^2 t^2)\ln^2(wt + \sqrt{1+w^2 t^2})} = \frac{w^4 t^2}{(1+w^2 t^2)\operatorname{arcsinh}^2 wt}$$

(21)

A similar relation (21) holds also between π and π' because $w' = 2\pi'\nu$ and $w = 2\pi\nu$ (see (14) and (15))
Using (12) and (16) we find

$$\cos\theta = \frac{c}{\sqrt{c^2 + w^2 r^2}} = \sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{22}$$

and we require that $w'r' \le c$.
We have shown that O' (the laboratory observer) will see a contraction of the radius of the rotating disc given by (19) or (20). Observers O' and O will not agree on the angular velocity, w and on the value of π . In fact, observer O' will perceive w' and π' as varying with the distance r . This situation arises from the fact that the contraction factor along the

perimeter is different from the contraction factor along the radius and the requirement that the speed of the signals is constant and agreed by all observers.

One remark about angular velocities is useful to clarify things. Observers O, O' and $\tilde{O}$ will agree on epicenter angle $\Delta\Theta$, measured as fraction of a circle (see Assumption 1). But according to the definition of angular velocity we may write $w = \frac{2®\Delta\Theta}{\Delta t}$, $w' = \frac{2®'\Delta\Theta}{\Delta t'}$, $\tilde{w} = \frac{2\tilde{®}\Delta\Theta}{\Delta\tilde{t}}$, while for angles $¯ = 2®\Theta$, $¯' = 2®'\Theta$, $\tilde{¯} = 2\tilde{®}\Theta$, where the tildas refer to observer $\tilde{O}$ (O_3), the primes to observer O' (O_2), and the plane letters refer to observer O (O_1). The correct notation using the subscript notation of section 3 is

$w = \frac{2®\Delta\Theta}{\Delta t} = w_{11} = \frac{2®_{11}\Delta\Theta}{\Delta t_{11}} = \frac{2®_{11}\Delta\Theta}{\Delta t_{stat}}$, $w' = \frac{2®'\Delta\Theta}{\Delta t'} = w_{21} = \frac{2®_{21}\Delta\Theta}{\Delta t_{21}} = \frac{2®_{21}\Delta\Theta}{\Delta t_{stat}}$,

$\tilde{w} = \frac{2\tilde{®}\Delta\Theta}{\Delta\tilde{t}} = w_{33} = \frac{2®_{33}\Delta\Theta}{\Delta t_{33}} = \frac{2®_{33}\Delta\Theta}{\Delta t_{stat}}\cos ¨$. But $®_{11} = ®_{22} = ®_{33} = ®$ (or $® = \tilde{®}$), because observers agree both on radial and on perimeter lengths when stationary on their frame (Assumption 3). We conclude, therefore, that

$$\frac{w'}{w} = \frac{w_{21}}{w_{11}} = \frac{®_{21}}{®_{11}} = \frac{®'}{®} \tag{23}$$

$$\frac{w}{\tilde{w}} = \frac{w_{11}}{w_{33}} = \frac{\Delta t_{33}}{\Delta t_{11}} = \frac{1}{\cos ¨} \tag{24}$$

and

$$\frac{w'}{\tilde{w}} = \frac{w_{21}}{w_{33}} = \frac{®_{21}\Delta t_{33}}{®_{33}\Delta t_{21}} = \frac{®_{21}}{®_{33}\cos ¨} = \frac{®'}{®\cos ¨} \tag{25}$$

5. Summary of Results

The angle of deflection $¨$ and in particular $\cos ¨$ takes many equivalent forms that are presented here for ease of calculations

$$\sqrt{1-\frac{w'^2 r'^2}{c^2}} = \sqrt{1-\frac{w^2 r^2}{c^2+w^2 r^2}} = \frac{c}{\sqrt{c^2+w^2 r^2}} = \frac{1}{\sqrt{1+\frac{w^2 r^2}{c^2}}} = \frac{w'r'}{wr} = \cos ¨ \tag{26}$$

$$\frac{w'r'}{c} = \sqrt{\frac{w^2 r^2}{c^2+w^2 r^2}} = \sin ¨ \tag{27}$$

$$\tan ¨ = \frac{wr}{c} = wt = ¯ = \sinh\frac{wr'}{c} \tag{28}$$

Where $¯$ is the angle of the circle traveled by the signal until it reaches the distance r from the center (see Figure 1).

The transformation among the observers O, O' and $\tilde{O}$ is summarized below in Table1(a), 1(b),1(c). However, our interest in this study will be focused on the relation between observer O and O' .

Table 1(a) Transformations between Observers $O\,(O_1)$ and $O'\,(O_2)$

Quantitities	**Transformations**
Time interval	$\Delta t = \Delta t' = \Delta t_{stat}$, $(\Delta t_{stat} = \Delta t_{21} = \Delta t_{12} = \Delta t_{11} = \Delta t_{22})$
Length segment on perimeter	$\Delta L' = \Delta L \cos \text{¨}$, $(\Delta L_{21} = \Delta L_{stat} \cos \text{¨})$
Radius	$r' = \frac{c}{w} \operatorname{arcsinh} \frac{wr}{c}$
Angular velocity	$w' = w^2 \dfrac{r \cos \text{¨}}{c \operatorname{arcsinh}(\frac{wr}{c})}$
Pi and angles	$\frac{w'}{w} = \frac{\text{¯}'}{\text{¯}} = \frac{\text{®}'}{\text{®}} = \frac{r}{r'} \cos \text{¨}$

Table 1(b) Transformations between Observers $O\ (O_1)$ and $\tilde{O}\ (O_3)$

Quantitities	**Transformations**
Time interval	$\Delta \tilde{t} = \dfrac{\Delta t}{\cos \text{¨}}$, $(\Delta t_{13} = \Delta t_{33} = \dfrac{\Delta t_{stat}}{\cos \text{¨}})$
Length segment on perimeter	$\Delta \tilde{L} = \Delta L$, $(\Delta L_{13} = \Delta L_{31} = \Delta L_{stat})$
Radius	$r = \tilde{r}$
Angular velocity	$\frac{\tilde{w}}{w} = \frac{\Delta t_{11}}{\Delta t_{33}} = \cos \text{¨}$
Pi and angles	$\text{®} = \tilde{\text{®}}$, $\text{¯} = \tilde{\text{¯}}$

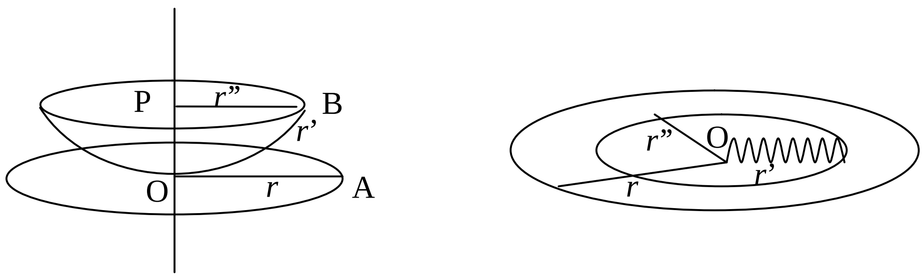
Table 1(c) Transformations between Observers $O'\ (O_2)$ and $\tilde{O}\ (O_3)$

Quantitities	**Transformations**
Time interval	$\Delta \tilde{t} = \dfrac{\Delta t'}{\cos \text{¨}}$, $(\Delta t_{23} = \Delta t_{33} = \dfrac{\Delta t_{stat}}{\cos \text{¨}})$
Length segment on perimeter	$\Delta L' = \Delta \tilde{L} \cos \text{¨}$, $(\Delta L_{23} = \Delta L_{stat} \cos \text{¨})$
Radius	$r' = \frac{c}{\tilde{w}} \cos \text{¨} \operatorname{arcsinh} \dfrac{\tilde{w}\tilde{r}}{c \cos \text{¨}}$
Angular velocity	$w' = \dfrac{\tilde{w}^2 \tilde{r}}{c \cos \text{¨} \operatorname{arcsinh}(\frac{\tilde{w}\tilde{r}}{c \cos \text{¨}})}$
Pi and angles	$\frac{\tilde{w}}{w'} = \frac{\text{®}_{33} \Delta t_{stat}}{\text{®}_{21} \Delta t_{33}} = \frac{\text{®}}{\text{®}'} \cos \text{¨} = \frac{r'}{r}$, $\frac{\tilde{\text{¯}}}{\text{¯}'} = \frac{\text{®}_{33}}{\text{®}_{21}} = \frac{\text{®}}{\text{®}'}$

6. Warp or Ripples?

From (14) and using (20) we see that $\text{®}' = \text{®}\dfrac{\frac{wr}{c}}{\operatorname{arc\,sinh}\frac{wr}{c}}\cos\text{¨}$. This implies that $\text{®}' < \text{®}$ except for $wr = 0$ for which equality holds. The decrease of ® implies a warping of the disc or the creation of ripples (see Figure 2(a) and Figure 2(b)). In the case of warping (Figure 2(a)) observer O sees the radius as the segment OA with length r, observer O' sees the curved segment OB with length r', which for him looks as straight and PB is the theoretical straight line (projection of r' on Euclidean space) with length r''.
However, we may exclude warping, because of the following argument: Consider three parallel concentric discs. Let the middle one rotate with frequency ¬ and with respect to the other two that are stationary. If indeed warping occurred the middle disc would intersect one of the other two discs, since we are allowed to bring them arbitrarily close to the middle disc. This seems unphysical. We are, therefore, inclined to exclude warping as a possibility and to consider ripples instead.
The ripples formed on the disc (Figure 2(b)) make the radius be looked in two different ways. One is the radius touching the surface of ripples (r') (the surface radius) and another is the theoretical straight line disregarding ripples (r'') (the straight radius or the projection of r' on the flat plane of rotation). The latter one satisfies the equation $2\text{®}r'' = 2\text{®}'r' = 2\text{®}r\cos\text{¨}$ and therefore,

$$r'' = r\cos\text{¨} \tag{29}$$



(a) (b)

Figure 2 (a) The rotating disc is warped. The radius on the surface of the warped disc is r'. The straight line (Euclidean) radius is r''.The stationary radius is r. (b) The rotating disc forms ripples. The length of the radius on the rippled surface is r'. The straight line (Euclidean) radius of the disc is r''. The stationary radius is r.

As we will see in the discussion below (section 7), if w is finite but $r \to \infty$ then $r' \to \infty$, $r'' \to \frac{c}{w}$, $w' \to 0$, $\cos\text{¨} \to 0$. The lab observer O' will see the surface radius (r') tend very slowly (logarithmically) to infinity and $w'r' \to c$, while the straight (on the flat Euclidean surface) radius r'' tends to $\frac{c}{w}$.
Physically, r'' is the radius that an observer O'' on the laboratory frame will observe, who is located at a distance greater than $\frac{c}{w}$ from the axis of rotation of the disc. If O'', enters into the region of distance less than $\frac{c}{w}$ from the axis of rotation, then his geometry ceases

to be Euclidean. He becomes observer of type O'. He is now on the rippled surface (which he perceives as flat) his pi is now ®$'$ and the radius of the disc is now given by r', which is not limited by any boundary. Since rotation of the disc does not affect the time rate of their clocks, because they do not participate in the rotation, we expect that the time rate for O' and O'' after O'' crosses the boundary to remain equal or that $\Delta t' = \Delta t'' = \Delta t_{stat}$, although they cannot exchange light signals. If we denote by double primes the quantities observed by O'', the above implies that

$$\Delta t'' = \Delta t' = \Delta \tilde{t} \frac{1}{\sqrt{1 - \frac{w'^2 r'^2}{c^2}}} \tag{30}$$

Regarding lengths (perimeter and radial) observer O'' sees the lengths of observer O smaller by a factor of cos ¨ because of (29). Hence,

$$\Delta L'' = \frac{r''}{r} \Delta L = \Delta L \cos \text{¨} \tag{31}$$

Further, epicenter angles are not affected,

$$\Delta\Theta = \Delta\Theta'' \tag{32}$$

Hence, there is agreement on frequency of revolution measurements, that is $v = v' = v''$ because $v'' = \frac{\Delta\Theta''}{\Delta t_{stat}}$

Also from (14) and (29) ®$'r' =$ ®r''. Since observer O'' lives in Euclidean space, his pi denoted as ®$''$ must be equal to the normal pi or ® $=$ ®$''$. It follows that $w'' = 2$®$''$¬$'' = 2$®¬ $= w$. From these observations one easily deduces using (26) that $w'r' = w''r'' = wr''$. Substituting in (26) we find,

$$\cos \text{¨} = \sqrt{1 - \frac{w''^2 r''^2}{c^2}} = \sqrt{1 - \frac{w^2 r''^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}} \tag{33}$$

This was expected since the Lorentz contraction of the perimeter for O'' is given by $\sqrt{1 - \frac{w''^2 r''^2}{c^2}}$. Finally, substituting (33) into (29) we obtain,

$$r'' = r \sqrt{1 - \frac{w^2 r''^2}{c^2}} \tag{34}$$

And solving we find

$$r'' = r \frac{c}{\sqrt{c^2 + w^2 r^2}} \tag{35}$$

Which was also expected since by (26) $\cos$ ¨ $= \frac{c}{\sqrt{c^2 + w^2 r^2}}$

7. Discussion of Results

First we note that from (26), (29) and because $w'r' = w''r'' = wr''$

$$\lim_{wr \to \infty} w'r' = \lim_{wr \to \infty} w''r'' = c \tag{36}$$

$$\lim_{w \to \infty} r'' = \lim_{w \to \infty} r' = \lim_{w \to \infty} \left(\frac{r}{\sqrt{1 + w^2 t^2}} \right) = 0 \tag{37}$$

And similarly,

$$\lim_{w\to 0} r'' = \lim_{w\to 0} r' = \lim_{w\to 0}\left(\frac{r}{\sqrt{1+w^2t^2}}\right) = r \tag{38}$$

Also

$$\lim_{r\to\infty} w' = 0 \tag{39}$$

And

$$\lim_{w\to\infty} w' = \infty \tag{40}$$

1. As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos\text{¨} \to 0$ and $\text{³}_r{}' \to 0$ and $\text{³}_{–}{}' \to c$. This says that as the tangential velocity of the rotating disc becomes big, the rays at the circumference are almost tangential and their tangential velocity approaches the speed of light, while their radial velocity tends to zero. In other words, light signals emanating from the center, O, will bend and turn around in circles expanding very slowly as they will be bend almost entirely tangentially.
2. In particular if $w \to \infty$ while r (and hence t) remains finite, $w' \to \infty$, $\cos\text{¨} \to 0$, $r' \to 0$, $r'' \to 0$, $\text{³}_r{}' \to 0$ and $\text{³}_{–}{}' \to c$. In this case, when the angular velocity (w) becomes big, while the rest radius (r) remains finite, the radius (r' and r'') for the lab observers O' and O'' shrinks to become very small (but $wr'' = w'r' \to c$), the light signals starting from the center, O, bend to turn in circles ($\cos\text{¨} \to 0$) with tangential velocity $\text{³}_{–}{}' \to c$ and the radial velocity of the light signals tends to zero ($\text{³}_r{}' \to 0$)
3. If w is finite but $r \to \infty$ then $r' \to \infty$, $r'' \to \frac{c}{w}$, $w' \to 0$, $\cos\text{¨} \to 0$. Observer O' will see the surface radius (r') tend very slowly (logarithmically) to infinity (while $w'r' \to c$), while O'' will see the radius (r'') tend to $\frac{c}{w}$. The light signals bend and go around in tighter and tighter circles with tangential velocity that tends to c ($\text{³}_{–}{}' \to c$), while the radial velocity drops to zero.
4. It is straightforward that when there is no rotation $w = 0$ we end back in frame K with $r = r' = r''$, $\Delta L' = \Delta L = \Delta\tilde{L}$, $\text{³}_r{}' = \text{³}_r$ as expected.
5. To show that the light signals will spiral out in tighter and tighter circles we may examine

$$\frac{dr'}{dr} = \frac{1}{\left(1+\frac{w^2r^2}{c^2}\right)^{\frac{1}{2}}} = \cos\text{¨} \tag{41}$$

which is increasing with diminishing rate as r increases. Similar observations hold for r'' where

$$\frac{dr''}{dr} = \frac{1}{\left(1+\frac{w^2r^2}{c^2}\right)^{\frac{3}{2}}} \tag{42}$$

6. Angle ¯ $= \measuredangle AOB$ in Figure 1 is traversed by the signal whilst it travels the length r, and is given by ¯ $= wt$ where $t = \frac{r}{c}$ or ¯ $= \frac{wr}{c} = \tan$ ¨ $= \sinh \frac{wr'}{c}$. Angle ¯ can become very big and even be the result of many revolutions.
7. If the signal originates from the perimeter towards the center, then its path will be symmetric with respect to the radius OB' in Figure 1.
8. A plot of r' for increasing time will look like the following Figure 3

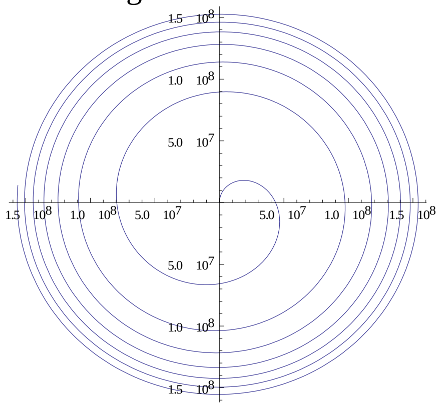


Figure 3 The path of a light signal originating from the center of a rotating frame as seen by the non rotating observer O'. Numbers on the axes are nonessential since scaling changes with w.

8. Generalization to three Dimensions

An observer O at the center O of a rotating frame K is rotating with the frame. He carries a rod (similar to the one we used in the 2 dimensional case above) pointing radially but with an angle ¶ with respect to the z axis. As we said in the 2-dimensionalcase, the rod is simply an artifact to help us imagine the situation. It suffices to say that observer O sends a signal from the origin with an angle ¶ to the z axis. The situation is depicted in Figure 4

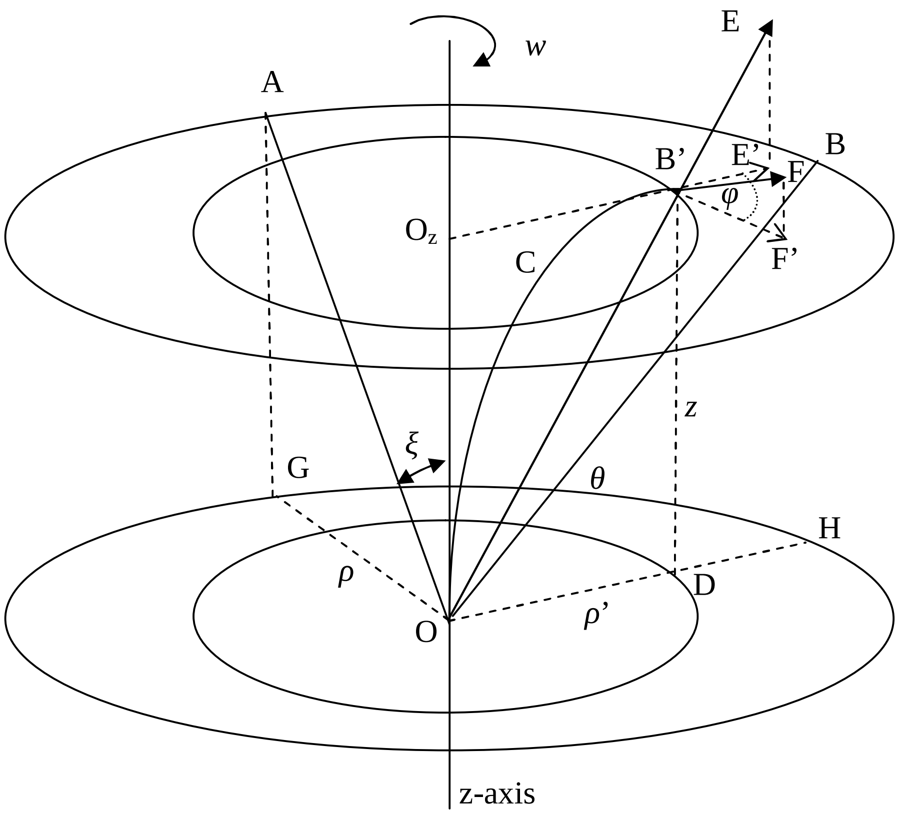


Figure 4 The signals originate from O and move along the rod OA for observer O, who rotates with the frame. The non-rotating observer O' sitting at O will see the signal travel a helical path OCB' while the rod travels from position OA to OB until the signal traverses the rod. The projection of the velocity vector B'F of the signal on the plane of rotation is B'F', as observer O' perceives it, has magnitude $c\sin¶$. The angle $\mathrm{E'\hat{B}'F'} = ¨$ is the angle of deflection from the radius $\mathrm{O_z B'}$=OD=$°'$ where $O_z B'$ is drawn from B' perpendicular to the z axis. The angle traversed by the rod is $\mathrm{G\hat{O}H}$=$¯$

8.1 Non-rotating nearby observer O'

Suppose a signal with velocity c originates from O with angle ¶ with respect to the axis of rotation as seen by observer O and is directed towards A through the rod OA (see Figure 4). The radial (in cylindrical coordinates $(°, ¯, z)$) velocity of the signals for observer O is $c\sin¶ = c\frac{°}{\sqrt{°^2 + z^2}}$ and the z component is $c\cos¶$. Suppose now that O rotates with the rod OA with frequency $¬'$ as seen by another observer O' that sits on top of O but does not rotate with O. Let also ¬ be the frequency that O thinks his frame, K, rotates with respect to K' of observer O'. Because of Assumption 1, $¬' = ¬$ and it has meaning to define the angular velocity of the frame K as $w \equiv 2®¬$. Observer O' will see the signal travel the helical path OCB' in the same time that it takes to traverse the rod for observer O, while the rod moves from position OA to OB. For him light travels along the helical path with the same velocity c and the z component equals that of observer O ($c\cos¶$). The velocity vector for observer O' is B'F and it makes an angle ¶ with the z-axis. The projection B'F' of the velocity vector B'F (tangential to the helical path for observer O') on the plane of rotation is denoted as v'_{proj} and

$$v'_{proj} = c\sin\theta \tag{43}$$

The angle between it and the radial (in cylindrical coordinates) B'E' for Observer O' is φ . This angle is called *the angle of deflection* of the velocity vector of the signal from the radial direction. Let us denote the velocity in the radial direction (in cylindrical coordinates) as observer O' sees it by v'_ρ . Then $v'_\rho = v_{proj}\cos\varphi$ and therefore,

$$v'_\rho = c\sin\theta\cos\varphi \tag{44}$$

The tangential component of the light signal for observer O' on the plane of rotation will be perpendicular to v'_ρ and will be given by

$$c\sin\theta\sin\varphi = w'\rho' \tag{45}$$

Also, by the definition of angular velocity,

$$v'_\phi = w'\rho' \tag{46}$$

As usual, the primed quantities v'_{proj}, v'_ρ, v'_ϕ, ρ', w' are as observer O' perceives them. For the same reasons (Lorentz contraction of perimeter) as in the two dimensional case we require that (14) holds. Namely,

$$2\pi'\rho' = 2\pi\rho\sqrt{1-\frac{w'^2\rho'^2}{c^2}} \tag{47}$$

Solving for π' and substituting in $w' = 2\pi'\nu$ we find

$$w'^2 = \frac{w^2\rho^2c^2}{c^2\rho'^2+\rho^2w^2\rho'^2} \tag{48}$$

where $w = 2\pi\nu$ and we note that (48) is the same as (16), as expected.
Finally,

$$v'_\rho = \frac{d\rho'}{dt} \tag{49}$$

Equations (44), (45), (46), (48), (49) are five equations in five unknowns: $\varphi, w', \rho', v'_\rho, v'_\phi$ given w, ρ, θ .
From (45) and (46)

$$\sin\varphi = \frac{w'\rho'}{c\sin\theta} \tag{50}$$

and hence,

$$\cos\varphi = \sqrt{1-\frac{w'^2\rho'^2}{c^2\sin^2\theta}} \tag{51}$$

with the condition $w'\rho' \le c\sin\theta$. And using (48)

$$\cos\varphi = \sqrt{1-\frac{w^2\rho^2}{(c^2+\rho^2w^2)\sin^2\theta}} \tag{52}$$

Since $\rho = ct\sin\theta$ and $z = ct\cos\theta$, we may write the above relation as,

$$\cos\varphi = \sqrt{\frac{1-w^2t^2\cos^2\theta}{1+w^2t^2\sin^2\theta}} = \sqrt{\frac{c^2-w^2z^2}{c^2+w^2\rho^2}} \tag{53}$$

with the condition that $1-w^2t^2\cos^2\P \ge 0$ (or $z \le \frac{c}{w}$) (or $w'^{\circ}{}' \le c\sin\P$)

(Note that $\cos\text{¨} = 0$ either when $z = \frac{c}{w}$ or when w° goes to ∞)

Substituting in (44)

$$^{3}{}'_{\circ} = c\sin\P\cos\text{¨} = c\sin\P\sqrt{\frac{1-w^2t^2\cos^2\P}{1+w^2t^2\sin^2\P}} \tag{54}$$

and since $^{3}{}'_{\circ} = \frac{d^{\circ}{}'}{dt}\frac{dt}{dt'} = \frac{d^{\circ}{}'}{dt}$,

$$^{\circ}{}' = c\sin\P\int_0^t\sqrt{\frac{1-w^2t^2\cos^2\P}{1+w^2t^2\sin^2\P}}dt \tag{55}$$

It is convenient to represent the integral in the RHS of (55) as a function of $\P$ and t. So we define,

$$I(\P,t) \equiv \int_0^t\sqrt{\frac{1-w^2t^2\cos^2\P}{1+w^2t^2\sin^2\P}}dt \tag{56}$$

Then we may rewrite (55) as

$$^{\circ}{}' = c\sin\P\, I(\P,t) \tag{57}$$

Note that for $\P = \frac{®}{2}$, (57) becomes

$^{\circ}{}'\Big]_{\P=\frac{®}{2}} = c\int_0^t\frac{dt}{\sqrt{1+w^2t^2}} = \frac{c}{w}\operatorname{arcsinh} wt = \frac{c}{w}\operatorname{arcsinh}\frac{wr}{c}$, which is what we found for the two dimensional case (recall (19)).

Below we summarize the following equations that are useful for calculations,

$$\sin\text{¨} = \frac{w'^{\circ}{}'}{c\sin\P} \overset{\text{(see (48))}}{=} \frac{w^{\circ}}{\sin\P\sqrt{c^2+{}^{\circ 2}w^2}} = \frac{wt}{\sqrt{1+w^2t^2\sin^2\P}} \tag{58}$$

$$\cos\text{¨} = \sqrt{1-\frac{w'^{2\,\circ}{}'^{2}}{c^2\sin^2\P}} = \sqrt{\frac{c^2-w^2z^2}{c^2+{}^{\circ 2}w^2}} = \sqrt{\frac{1-w^2t^2\cos^2\P}{1+w^2t^2\sin^2\P}} \tag{59}$$

$$\tan\text{¨} = \frac{wt}{\sqrt{1-w^2t^2\cos^2\P}} = \frac{w^{\circ}}{\sin\P\sqrt{c^2-w^2z^2}} = \frac{w'^{\circ}{}'}{\sqrt{c^2\sin^2\P - w'^{2\,\circ}{}'^{2}}} \tag{60}$$

$$\frac{w'^{\circ}{}'}{w^{\circ}} = \frac{2®\neg'^{\circ}{}'}{2®\neg^{\circ}} = \frac{®'^{\circ}{}'}{®^{\circ}} = \frac{c}{\sqrt{c^2+w^{2\,\circ 2}}} = \cos\text{¨}\Big]_{z=0} = \cos\text{¨}\Big]_{\P=\frac{®}{2}} \tag{61}$$

$$I(\P,t) = \int_0^t \cos\text{¨}\, dt \tag{62}$$

8.1.1 Plot of signals for observer O'

Under the condition that $1-w^2t^2\cos^2\P \ge 0$ (which is required for $\cos\text{¨}$ to be a real number) and $w \ne 0$, and $0 < \P \le \frac{®}{2}$ we want to calculate $I(\P,t)$

Make the substitution

$$k = i\cot\P \tag{63}$$

And

$$x = iwt\sin\P \tag{64}$$

Where $\S = \sqrt{-1}$ and then (56) becomes

$$I(\P,t) = \frac{1}{iw\sin\P}\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}dx \tag{65}$$

But the integral on the RHS is an incomplete Elliptic integral of the second kind denoted as $E(k,x)$. Therefore, (65) can be written as

$$I(\P,t) = \frac{1}{iw\sin\P}E(i\cot\P, iwt\sin\P) \tag{66}$$

which can be used for calculations.

If we take the definite integral of (55) for $0 \le t \le \frac{1}{w\cos\P}$ (the limit allowed by the condition $1 - w^2t^2\cos^2\P \ge 0$) we find the max value that $\circ'$ can take for each particular w and $\P$. Namely,

$$\circ'_{\mathrm{m}}(\P) = c\sin\P\int_0^{\frac{1}{w\cos\P}} \sqrt{\frac{1-w^2t^2\cos^2\P}{1+w^2t^2\sin^2\P}}dt = c\sin\P\, I(\P, \frac{1}{w\cos\P}) \tag{67}$$

This says that for any fixed $\P$, the signals in the radial direction are bounded (see Figure 5). The signals bend and rotate until they reach $z = \frac{c}{w}$ at $t_{\mathrm{m}} = \frac{1}{w\cos\P}$, the angle of deflection becomes 90^{o} degrees, the radial velocity diminishes to zero and the radial distance of the signal becomes.

$$\circ'_{\mathrm{m}}(\P) = c\sin\P\, I_{\mathrm{m}}(\P) \tag{68}$$

where, we denote $I_{\mathrm{m}}(\P) = I(\P, \frac{1}{w\cos\P})$

However, for $\P = \frac{®}{2}$ we come back to the two dimensional case and the signal is not bounded. It expands slowly all the time logarithmically and again the radial velocity tends to zero.

We said that all signals, when they reach $\circ'_{\mathrm{m}} = c\sin\P\, I_{\mathrm{m}}(\P)$, they are at $z_{\mathrm{m}} = \frac{c}{w}$. After that, the signals are not observable by O'

Further, because of (50) $w'\circ' = c\sin\P\sin¨ \le c\sin\P$ and for $\circ'_{\mathrm{m}}$, where $\sin¨ = 1$, we have

$$w'_{\mathrm{m}}\circ'_{\mathrm{m}} = c\sin\P \tag{69}$$

and hence at $\circ'_{\mathrm{m}}$ the angular velocity of the signals w'_{m} for observer O' is

$$w'_{\mathrm{m}} = \frac{c}{\circ'_{\mathrm{m}}} \tag{70}$$

In order for the normal Lorentz contraction of the perimeter to hold we must satisfy (47) and (48) which lead to

$$\frac{\text{®}'}{\text{®}} = \frac{\text{°}}{\text{°}'}\frac{c}{\sqrt{c^2 + w^2\text{°}^2}} \tag{71}$$

The effect of w is to scale down distances as it increases. The faster the body rotates, the faster the signals bend making tighter revolutions closer to the body.
The signals are not limited in the radial direction but cover the whole space allowed by $|z| \le \frac{c}{w}$, as angle ¶ varies from 0 to 2®

In Figure 5 we plot the path of the signals in the region $0 \le z \le \frac{c}{w}$ for several signals with different values of ¶ .The paths are for the region $-\frac{c}{w} \le z \le 0$ are symmetric with respect to the plane of rotation.

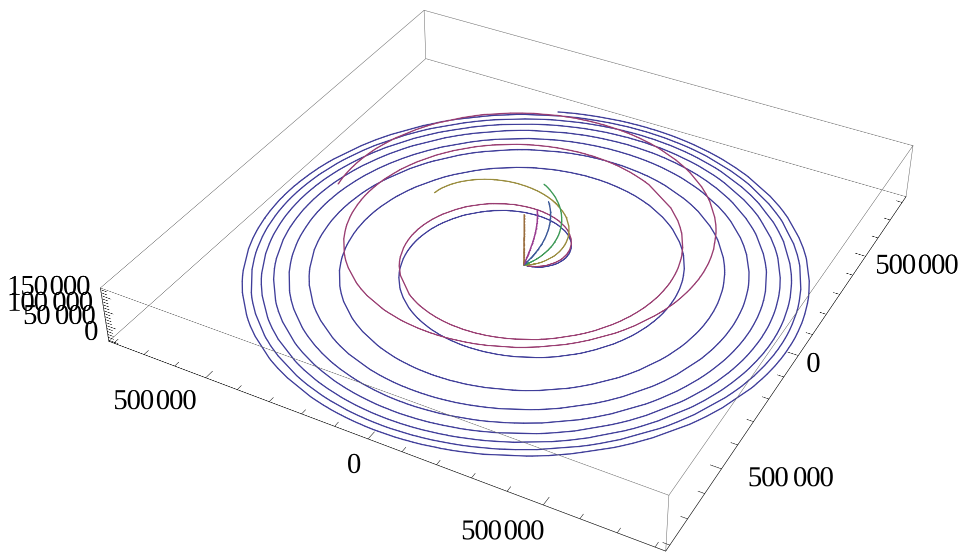


Figure 5 Plot of signals emanating from O for observer O' as they rotate with radius °′ while advancing in the z direction for the region $0 \le z \le c / w$ for different values of ¶ . The numbers on the axes are not essential since they depend on the value of w.

8.2 Introduction of precession

Up till now we talked of K as rotating frame. However, as we plan later to place a point mass at the origin and imagine the signals it sends out, it is useful to add precession to , K since rotating bodies also precess as they rotate unless they are symmetrical in their axes of inertia. It is also possible to add nutation as well (if the rotating body is subject to an external force). However, we will not consider nutation in this study as it does not add much to what we want to demonstrate. The effect of precession (and nutation) is the same as if observer O oscillates his rod, through which the light ray travels, up and down around an angle ¶ from the z axis. The addition of precession (and nutation) to the case of observer O' will make him see a wavy and curved path for the signal, instead of only curved. The precession thus also justifies our preference for “ripples” in the two dimensional case that was studied above. Referring to Figure 6, a point body at O is rotating around the axis OC. The axes OC rotates around the z axis with the angular velocity of precession Ω. The angle of inclination of OC with the z axis is $¸_0$. The projection of AC on AB which is drawn parallel to the x axis is AD. Hence, $AD = AC\cos\Omega t$. It follows that $\frac{AD}{OA} = \frac{AC}{OA}\cos\Omega t$ or $\tan ¸ = \tan ¸_0 \cos\Omega t$ where $¸_0 = \measuredangle AOC$ and $¸ = \measuredangle AOD$

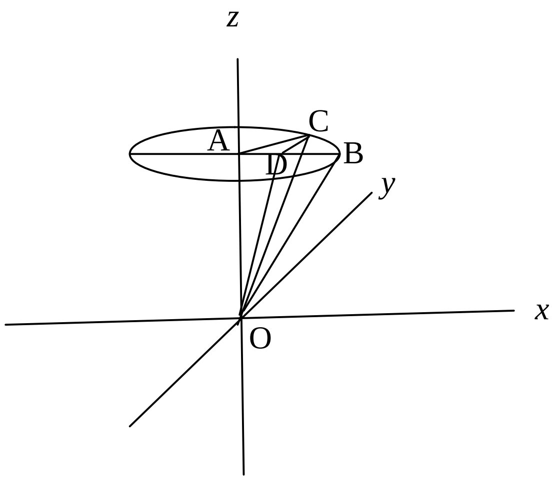


Figure 6 A point body at O rotates with angular velocity *w* around axis OC which itself rotates around the z axis with inclination $¸_0 = \measuredangle AOC$ and angular velocity Ω. If we let $¸ = \measuredangle AOD$ then $\tan ¸ = \tan ¸_0 \cos\Omega t$.

A signal that is emitted by the point body at angle $¶$ from the z axis when the body does not precess (when $¸_0 = 0$), when it precesses it will have an angle $¶ + ¸$ to the z axis. The average of angle $¶ + ¸$ over time is $¶$. In this formulation the problem for observer O' is given by

$$2®'°' = 2®° \sqrt{1 - \frac{w'^2 °'^2}{c^2}} \tag{72}$$

$$c\sin(¶ + ¸)\sin ¨ = w'°' \tag{73}$$

$$\frac{d°'}{dt} = c\sin(¶ + ¸)\cos ¨ \tag{74}$$

$$° = ct\sin(¶ + ¸) \tag{75}$$

$$w' = 2®'v \tag{76}$$

$$z = ct\cos(¶ + ¸) \tag{77}$$

$$w = 2®v \tag{78}$$

Solving the same way we did for the no precession case we find,

$$\cos \text{¨} = \sqrt{\frac{1-w^2t^2\cos^2(\text{¶}+\text{¸})}{1+w^2t^2\sin^2(\text{¶}+\text{¸})}} \tag{79}$$

And

$$\text{°}' = c\int_0^t \sin(\text{¶}+\text{¸})\cos\text{¨}\,dt \tag{80}$$

8.3 Non-rotating far away observer O''

Assuming we have precession, the far away observer O'' does not see ripples, or even if he observes them he cares about the straight line average for the radius $\text{°}''$. This average path is given by a signal that has inclination $\text{¶}''$ but no precession, as if it originates from a signal with inclination ¶ and $\text{¸} = 0$ in the world of observer O. The light signal starting from the center with radial direction that will follow a wavy and curved path (as if the space has ripples) with velocity c according to observer O', will appear to observer O'' as curved but non wavy having velocity ³_c and inclination $\text{¶}''$.

The equations describing the problem of observer O'' are:

Lorentz contraction of the perimeter,

$$2\text{®}\text{°}'' = 2\text{®}\bar{\text{°}}\sqrt{1-\frac{w^2\text{°}''^2}{c^2}} \tag{81}$$

The tangential velocity must equal $w\text{°}''$,

$$\text{³}_c\sin\text{¶}''\sin\text{¨}'' = w\text{°}'' \tag{82}$$

The rate of change of $\text{°}''$ must equal the radial velocity,

$$\frac{d\text{°}''}{dt} = \text{³}_c\sin\text{¶}''\cos\text{¨}'' \tag{83}$$

The distance traveled in the z direction is equal for observers O and O'',

$$c\cos\text{¶} = \text{³}_c\cos\text{¶}'' \tag{84}$$

Where $\text{°} = ct\sin(\text{¶}+\text{¸})$ and the average over time is $\bar{\text{°}}$ given by

$$\bar{\text{°}} = ct\sin\text{¶} \tag{85}$$

Since the average of $\text{¸} = 0$.

Solving (81) and denoting for economy $\bar{\text{°}}$ as simply ° we find

$$\text{°}'' = \text{°}\frac{c}{\sqrt{c^2+w^2\text{°}^2}} \tag{86}$$

Now solving (86) for ° we find that

$$\text{°} = \frac{\text{°}''c}{\sqrt{c^2-w^2\text{°}''^2}} \tag{87}$$

From (87) we see that $\text{°}'' < \frac{c}{w}$ and as $\text{°} \to \infty$, $\text{°}'' \to \frac{c}{w}$. This is also obvious by setting $\text{°} = ct\sin\text{¶}$ in (86) and letting $t \to \infty$ thus $\text{°}'' \to \frac{c}{w}$ regardless of the value of ¶.

From (86) using (85) and taking the derivative with respect to t we find

$$\frac{d\circ''}{dt} = \frac{c\sin\P}{(1+w^2t^2\sin^2\P)^{\frac{3}{2}}} \tag{88}$$

From (82) and (83) we have,

$$\tan\text{¨}'' = \frac{w\circ''}{\frac{d\circ''}{dt}} \tag{89}$$

And using (86) and (88) we find

$$\tan\text{¨}'' = wt(1+w^2t^2\sin^2\P) \tag{90}$$

Dividing (82) by (84) we find

$$\tan\P''\sin\text{¨}'' = \frac{w\circ''}{c\cos\P} \tag{91}$$

And using (86) and (90) we obtain

$$\tan\P'' = \frac{wt\sin\P}{\cos\P\sqrt{1+w^2t^2\sin^2\P}}\sqrt{1+\cot^2\text{¨}''} = \tan\P\,\frac{\sqrt{1+w^2t^2(1+w^2t^2\sin^2\P)^2}}{(1+w^2t^2\sin^2\P)^{\frac{3}{2}}} \tag{92}$$

Finally from (84) and using $\cos\P'' = \frac{1}{\sqrt{1+\tan^2\P''}}$ we find

$$\text{³}_c = \frac{c\cos\P}{\cos\P''} = c\cos\P\sqrt{1+\frac{\tan^2\P(1+w^2t^2(1+w^2t^2\sin^2\P)^2)}{(1+w^2t^2\sin^2\P)^3}} \tag{93}$$

This equation allows ³_c to take values greater and smaller than c.
The signals according to observer O'' are limited in the radial direction to c/w which they approach asymptotically, as we remarked in (87) and therefore lie within the cylindrical surface of radius c/w without being limited in the z direction since $z'' = z$ due to (84).

9. Rotation with Slippage

9.1 The two-Dimensional Case for observer O'

If we allow the angular velocity w to vary as a function of the radius it is like having a disc that consists of rings with small width that slide one after the other. Let for example

$$w = w_0 f(r) \tag{94}$$

where, $0 < f(r) < 1$ and non-increasing in r
In this setup there is a multitude of O type observers, who stand at the origin O, one for each particular value of the radius r, who rotate with the angular velocity of that particular ring. Observer O' is standing on the center on top of O but is not rotating with the disc. The clocks of the two types of observers will run at the same rate. Again equations (11) to (18) continue to hold. Substituting (94) into (18) we find,

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1+w_0^2t^2f(r)^2}} = c\cos\text{¨} \tag{95}$$

and

$$\cos\varphi = \frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}} \tag{96}$$

If we let $f(r) = e^{-\alpha r}$ we find

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2\alpha r}}} \tag{97}$$

or

$$r' = c\int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2\alpha r}}} + const \tag{98}$$

To see how r' behaves take the derivative of $\cos\varphi$

$$\frac{d}{dr}\cos\varphi = \frac{1}{c}\frac{d}{dt}\cos\varphi = -\frac{w_0^2 t e^{-2\alpha ct}(1 - \alpha ct)}{c(1 + w_0^2 t^2 e^{-2\alpha ct})^{\frac{3}{2}}} \tag{99}$$

At $t = 0$ the derivative is negative then as t increases the derivative increases and at $t = \frac{1}{\alpha c}$ it becomes zero and then positive and finally for big t it tends to zero. A plot of $\cos\varphi$ appears in Figure 7

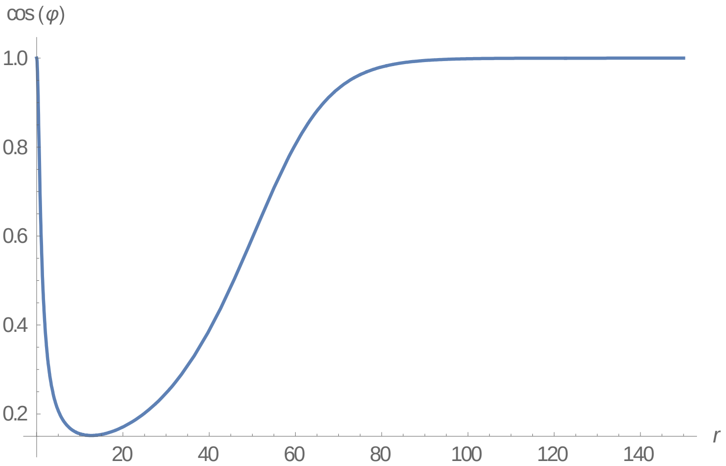


Figure 7 The cosine of deflection angle φ versus *r*. The deflection angle starts at zero (cos0°=1). Then it increases reaching almost 90° degrees (for big enough w_0) at $r = \frac{1}{\alpha}$. Then it falls again to zero (cos90°=1) asymptotically.

The deflection angle initially at 0° increases approaching 90° degrees (closer to 90° for higher w_0) and then drops again to zero. This behavior is similar to the behavior we have examined for the rotation without slippage. Namely, as the angle of deflection increases the signals start rotating in tighter circles until they reach $r = \frac{1}{\alpha}$. Then the signals rotate in less and less tight circles until they are directed asymptotically radially outward ($\varphi = 0$).

The choice of slippage according to the rule $f(r) = e^{-\alpha r}$ can be justified by the assumption that for a disc consisting of slipping rings each rings slips with respect to the previous by the same proportion in angular velocity. Consider for example a width r of n layers. The first has velocity w_0, the second $w_0 \sim^{\frac{r}{n}}$, the third $w_0(\sim^{\frac{r}{n}})^2$ the nth $w_0 \;\; ^n$ where $\;\; = \sim^{\frac{r}{n}}$. Each layer slips with respect to the previous by proportion forming a geometric series. Therefore, letting $n \to \infty$ the ring at radius r will have angular velocity $w_0 \sim^r$. For the case that $0 < \sim < 1$, where we are interested, we may substitute $\sim = e^{-ª}$ where $ª > 0$ and then $w_0 \sim^r = w_0 e^{-ªr}$ or $f(r) = e^{-ªr}$.

9.2 The three-Dimensional Case for observer O'

For the three dimensional case we assume that there is slippage both in the radial and the z direction. To achieve this we assume that

$$w = w_0 e^{-ª° - «z} = w_0 e^{-ct(ª \sin¶ + « \cos¶)} \tag{100}$$

where, $ª \geq 0$, $« \geq 0$. We disregard precession (assuming the amplitude of precession is very small) otherwise $¶$ must be replaced by $¶ + ¸$ complicating the problem since $¸$ is a function of t. Relations (44) to (49) continue to hold, with

$w = w_0 e^{-ª° - «z} = w_0 e^{-ct(ª \sin¶ + « \cos¶)}$

$$\frac{d°'}{dt} = c \sin¶ \cos¨ = c \sin¶ \sqrt{\frac{1 - e^{-2tc(ª \sin¶ + « \cos¶)} w_0^2 t^2 \cos^2 ¶}{1 + e^{-2ct(ª \sin¶ + « \cos¶)} w_0^2 t^2 \sin^2 ¶}} \tag{101}$$

where

$$\cos¨ = \sqrt{\frac{c^2 - e^{-2(ª° + «z)} w_0^2 z^2}{c^2 + e^{-2(ª° + «z)} w_0^2 °^2}} = \sqrt{\frac{1 - e^{-2tc(ª \sin¶ + « \cos¶)} w_0^2 t^2 \cos^2 ¶}{1 + e^{-2ct(ª \sin¶ + « \cos¶)} w_0^2 t^2 \sin^2 ¶}} \tag{102}$$

because $z = ct \cos¶$ and $° = ct \sin¶$. Therefore,

$$°' = c \sin¶ \, I(¶, t, ª, «) \tag{103}$$

where

$$I(¶, t, ª, «) \equiv \int_0^t \sqrt{\frac{1 - e^{-2tc(ª \sin¶ + « \cos¶)} w_0^2 t^2 \cos^2 ¶}{1 + e^{-2ct(ª \sin¶ + « \cos¶)} w_0^2 t^2 \sin^2 ¶}} dt \tag{104}$$

For $¶ = 90^\circ$ we have,

$$I(\frac{®}{2}, t, ª, «) = \int_0^t \sqrt{\frac{1}{1 + w_0^2 t^2 e^{-2ctª}}} dt \tag{105}$$

Observe that (102) is the same as (59), where $w = w_0 e^{-ª° - «z}$. The same is true for (58) (60), (61). The required condition for (102) to be real, is

$$te^{-tc(ª \sin¶ + « \cos¶)} \leq \frac{1}{w_0 \cos¶} \tag{106}$$

It is obvious that for *t* close to 0 and *t* very big the above condition is satisfied, while the left hand side has only one maximum. Therefore, for each $¶$, it is either satisfied for all *t* or there are positive t_1 and t_2 with $t_1 \leq t_2$ so that it is satisfied for $t \notin (t_1, t_2)$ and not satisfied within the interval ($t \in (t_1, t_2)$).

Given a ¶ , if it is satisfied for all *t*, then the angle of deflection ¨ does not become 90º degrees (except perhaps at a single point). Therefore, the signal is allowed to increase its radius for all *t* and asymptotically become radial.
If it is not satisfied for an interval ($t \in (t_1, t_2)$), it means that at t_1 the signal has reached $\cos ¨ = 0$ (angle of deflection 90º degrees) and cannot increase its radius anymore. So after t_1 and until it reaches t_2 the signal is not observable in the interval (t_1, t_2). After t_2, the signal is again allowed to increase its radius, increase $\cos ¨$ and become asymptotically radial.
What is the condition so that given ¶ , (106) is satisfied for all *t*? It is that the maximum of $te^{-tc(ª \sin ¶ + « \cos ¶)}$ is less than or equal to $\dfrac{1}{w_0 \cos ¶}$. And what is the maximum? Taking the derivative $\dfrac{d}{dt} te^{-tc(ª \sin ¶ + « \cos ¶)} = e^{-tc(ª \sin ¶ + « \cos ¶)}(1 - tc(ª \sin ¶ + « \cos ¶))$, and maximum occurs at $t = \dfrac{1}{c(ª \sin ¶ + « \cos ¶)}$. So substituting in (106), if

$$\frac{\cos ¶}{ª \sin ¶ + « \cos ¶} \le \frac{ce}{w_0} \qquad (107)$$

condition (106) is satisfied for all *t* and there is no t_1, t_2
In the opposite case, when

$$\frac{\cos ¶}{ª \sin ¶ + « \cos ¶} > \frac{ce}{w_0} \qquad (108)$$

In order to find t_1 and t_2 we solve (106)
$-c(ª \sin ¶ + « \cos ¶) te^{-tc(ª \sin ¶ + « \cos ¶)} \ge -\dfrac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶}$ and using the Lambert function ($W(.)$) we obtain $-c(ª \sin ¶ + « \cos ¶)t \ge W(-\dfrac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶})$ and finally

$$t \le -\frac{1}{c(ª \sin ¶ + « \cos ¶)} W(-\frac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶}) \qquad (109)$$

This, by the theory on Lambert functions, gives two solutions, when
$-\dfrac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶} > -\dfrac{1}{e}$, which by the way is the same as condition (108) as expected.
In particular, $t_1 = -\dfrac{1}{c(ª \sin ¶ + « \cos ¶)} W_0(-\dfrac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶})$ and
$t_2 = -\dfrac{1}{c(ª \sin ¶ + « \cos ¶)} W_{-1}(-\dfrac{c(ª \sin ¶ + « \cos ¶)}{w_0 \cos ¶})$ where $W_0(.)$ is the solution near the origin and $W_{-1}(.)$ is the solution further away from -1 on the negative branch of the Lambert function.

Looking now at $\cos ¨$ we take the derivative with respect to t to find its interior minimum (that corresponds to a maximum of ¨) along the path of the signal. After some straight forward manipulation we find,

$$\frac{d}{dt}\cos ¨ = \begin{cases} -\dfrac{w_0^2 t e^{-2ct(ª\sin¶+«\cos¶)}(1-ct(ª\sin¶+«\cos¶))}{\sqrt{1+e^{-2ct(ª\sin¶+«\cos¶)}w_0^2t^2\sin^2¶}} \\ \left(\dfrac{\cos^2¶}{\sqrt{1-e^{-2ct(ª\sin¶+«\cos¶)}w_0^2t^2\cos^2¶}}+\dfrac{\sin^2¶\sqrt{1-e^{-2ct(ª\sin¶+«\cos¶)}w_0^2t^2\cos^2¶}}{\sqrt{1+e^{-2ct(ª\sin¶+«\cos¶)}w_0^2t^2\sin^2¶}}\right)\end{cases}$$
(110)

The second factor, in big parentheses, is positive. Therefore, looking at the first factor we see that it starts negative for $t=0$ and then changes sign at

$$t_{¨\,max} = \frac{1}{c(ª\sin¶+«\cos¶)} = \frac{\sqrt{°^2+z^2}}{c(ª°+«z)} \tag{111}$$

This corresponds to

$$°_{¨\,max} = ct_{¨\,max}\sin¶ = \frac{\sin¶}{ª\sin¶+«\cos¶} \tag{112}$$

$$z_{¨\,max} = ct_{¨\,max}\cos¶ = \frac{\cos¶}{ª\sin¶+«\cos¶} \tag{113}$$

Equations (112) and (113) can be combined using the definition of $\cos¶$ and $\sin¶$ in a single equation giving the locus of points where $¨$ attains its maximum along the path of a signal,

$$ª°_{¨\,max} + «z_{¨\,max} = 1 \tag{114}$$

which is a straight line. The value of $\cos¨$ at the minimum is in $(°, z)$ space,

$$\cos¨_{max} = \cos¨\big|_{t=t_{max}} = \sqrt{\frac{(ª\sin¶+«\cos¶)^2c^2 - w_0^2e^{-2}\cos^2¶}{(ª\sin¶+«\cos¶)^2c^2 + w_0^2e^{-2}\sin^2¶}} \text{ for } (ª\sin¶+«\cos¶)^2c^2 - w_0^2e^{-2}\cos^2¶ \ge 0$$
(115)

which is the same as

$$\cos¨_{max} = \sqrt{\frac{c^2 - w_0^2e^{-2}z^2_{¨\,max}}{c^2 + w_0^2e^{-2}°^2_{¨\,max}}} \tag{116}$$

The condition in (115) is a condition on $¶$,

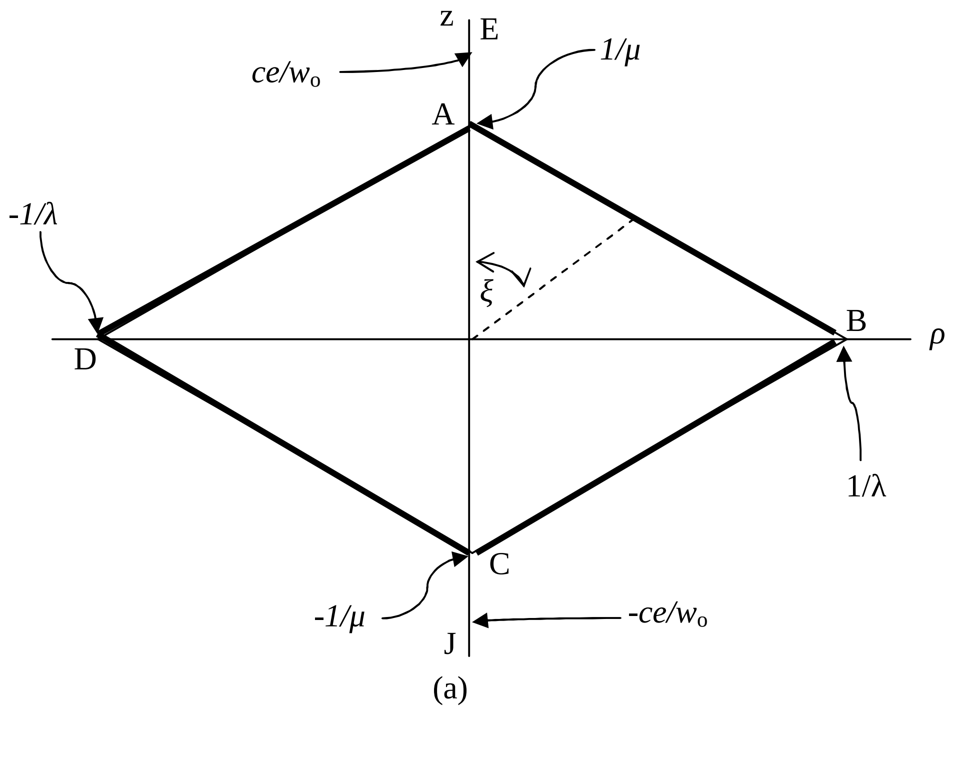
$$z_{¨\,max} = \frac{\cos¶}{ª\sin¶+«\cos¶} \le \frac{ce}{w_0} \tag{117}$$

This by the way is the same condition as (107) that is required for the non existence of the solutions t_1, t_2.

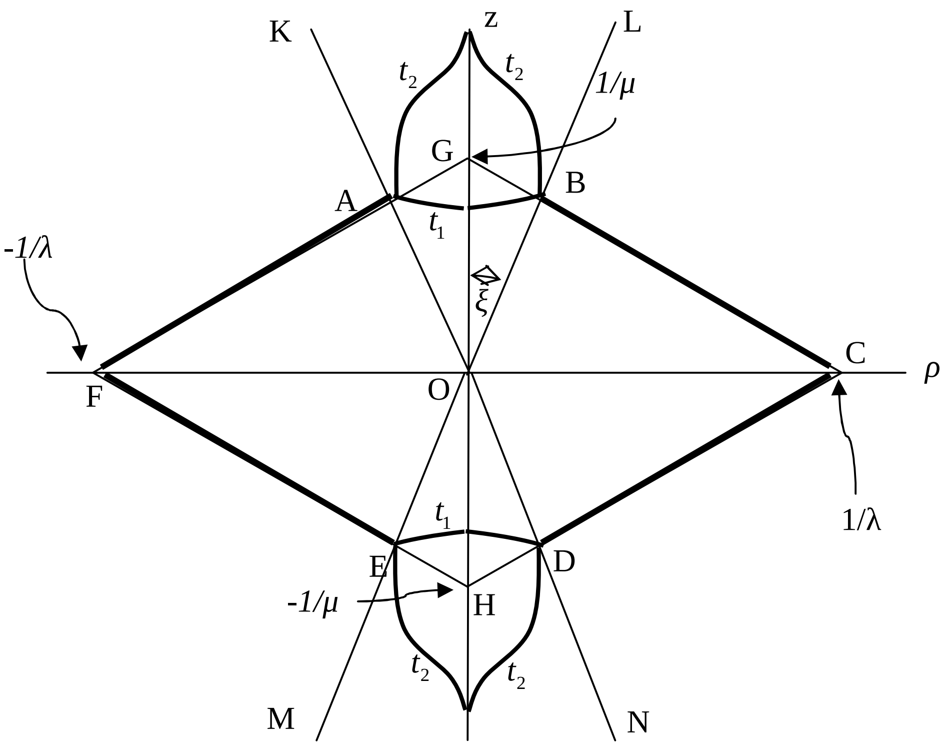
The parametric plot of $z_{¨\,max}$ versus $°_{¨\,max}$ is a straight line given by (114). The plot appears in Figure 8 where two cases are shown. In Figure 8 (a) the condition (117) is satisfied for all z and all $¶$. In order for this to be true, we require the maximum over $¶$ of the left hand side of (117) to be less or equal to the right hand side. This maximum occurs at $¶=0$ and at this point condition (117) becomes

$$\frac{1}{«} \le \frac{ce}{w_0} \tag{118}$$

In this case the minimum of $\cos$ ¨ in the direction of the path of the signal occurs on the rhombus by revolution ABCD for all ¶ .
In Figure 8 (b), (118) does not hold. This means that for some ¶ (117) is valid and for the rest it does not hold. In particular, talking about the first quadrant, because the same hold for the rest by symmetry, for ¶ $\leq G\hat{O}B$ it does not hold, but it holds for ¶ $> G\hat{O}B$. So for ¶ $\leq G\hat{O}B$, $\cos$ ¨ does not attain a minimum on the rhombus, because it becomes zero before reaching it, as it encounters the curve AB (marked as t_1), which is the solution of t_1 . This solution as well as t_2 (the curve between AB marked as t_2) exist, when condition (117), which the same as (107), holds. Between t_1 and t_2 the signal is not observable by O' , because $\cos$ ¨ becomes imaginary, unless we allow O' to observe signals travelling with speed greater than *c*. After t_2 the radius starts to increase again and the angle of deflection ¨ returns asymptotically to zero. Observe that at A and B $\frac{\cos ¶}{ª \sin ¶ + « \cos ¶} = \frac{ce}{w_0}$. In the opposite case, when ¶ $> G\hat{O}B$, the local minimum of $\cos$ ¨ is attained on the remaining of the rhombus BCD and EFA. After that the signal returns slowly to the radial direction.
Note that in Figure 8 we plot z_{max} vs °$_{max}$. But observer O' sees °$'$ instead of ° , which is contracted with respect to ° . Therefore, the shape that observer O' will see will not be a rhombus but will be deformed since it will be contracted in the ° direction.



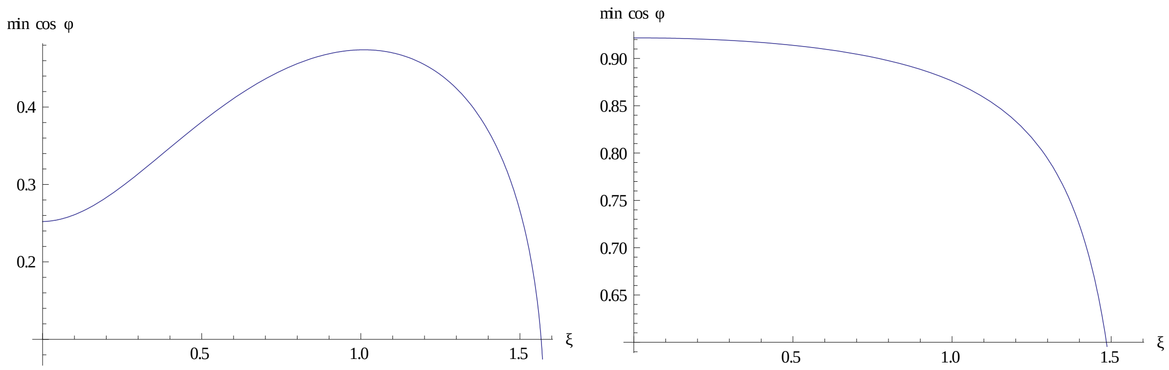
(a)



(b)

Figure 8 In (a) $1/« \leq ce/w_0$. The rhombus ABCD (bold lines), describes the locus of points, where $\cos ¨$ is minimum. In (b) $1/« > ce/w_0$. In this case for $¶ \leq G\hat{O}B$, the signals reach $\cos ¨ = 0$, when they arrive at the curve marked t_1. So the deflection angle ¨ has reached 90º degrees and cannot increase any more. Between t_1 and t_2 the signal is not observable by observer O'. After that, the deflection starts decreasing and asymptotically becomes zero, thus the signal returns to the radial direction. If on the other hand, $¶ > G\hat{O}B$, then the signal attains its maximum deflection on the line of the rest of the rhombus BCD and EFA and after that it decreases asymptotically towards zero returning to the radial direction. The diagrams show $z_{¨\ max}$ vs $°_{¨\ max}$, while observer O' sees $°'$. So we must imagine a contraction in the ° direction to reflect what observer O' sees, and then the rhombus will be deformed accordingly.

In Figure 9 we plot $\cos ¨_{max}$ as a function of ¶ when (101) does hold. In (a) for a small value of « and in (b) for a big value of « .



(a) (b)

Figure 9 Plot of $\cos ¨_{\max}$ as a function of the inclination of the signal, $¶$, as it varies from 0 to $® / 2$ when condition (118) is valid. (a) corresponds to small value of $«$ while (b) to a bigger one. In (b) the deflection of the signal in the z direction is smallest and it increases (the cosine decreases) gradually as we approach the radial direction.

9.3 Rotation with Slippage for Observer O''

Observer O'' is the far away observer. For him the problem is described by equations (81) to (87) with the only difference that the angular velocity w is now not constant but varies with the distance from the origin according to $w = w_0 e^{-ª° - «z} = w_0 e^{-ct(ª \sin¶ + « \cos¶)}$
From (86) and using $° = ct \sin ¶$ (recall from (85) that we denote $\bar{°}$ which is the average of $°$ by simply $°$)

$$°'' = \frac{°c}{\sqrt{c^2 + w^2 °^2}} = \frac{°c}{\sqrt{c^2 + w_0^2 °^2 e^{-2(ª° + «z)}}} = \frac{ct \sin ¶}{\sqrt{1 + w_0^2 t^2 \sin^2 ¶\, e^{-2ct(ª \sin¶ + « \cos¶)}}} \qquad (119)$$

Taking the time derivative and equating to the radial velocity using equation (83) we obtain,

$$c \sin ¶ \frac{(1 + c(ª \sin ¶ + « \cos ¶) t^3 w^2 \sin^2 ¶)}{(1 + w^2 t^2 \sin^2 ¶)^{\frac{3}{2}}} = ³_c \sin ¶'' \cos ¨'' \qquad (120)$$

By dividing (82) by (120) we find

$$\tan ¨'' = wt \frac{1 + w^2 t^2 \sin^2 ¶}{1 + c \quad t^3 w^2 \sin^2 ¶} \qquad (121)$$

Where $\quad = ª \sin ¶ + « \cos ¶$
Dividing (82) by (84) and using (86) and (121) we find

$$\tan ¶'' = \frac{w°''}{c \cos ¶ \sin ¨''} = \tan ¶ \frac{wt}{\sqrt{1 + w^2 t^2 \sin^2 ¶}} \sqrt{1 + \frac{(1 + c \quad t^3 w^2 \sin^2 ¶)^2}{w^2 t^2 (1 + w^2 t^2 \sin^2 ¶)^2}} \qquad (122)$$

Using (84) we obtain

$$³_c = c \frac{\cos ¶}{\cos ¶''} = c \cos ¶ \sqrt{1 + \tan^2 ¶''} = c \cos ¶ \sqrt{1 + \tan^2 ¶ \frac{w^2 t^2}{1 + w^2 t^2 \sin^2 ¶} \left(1 + \frac{(1 + c \quad t^3 w^2 \sin^2 ¶)^2}{w^2 t^2 (1 + w^2 t^2 \sin^2 ¶)^2}\right)}$$
(123)

The ratio $³_c / c$ tends to 1 when $t \to 0$ or $t \to \infty$. Also for $t \geq \frac{1}{c}$, $\frac{³_c}{c} \geq 1$. (Recall that $\frac{1}{c}$ is the time where $¨$ attains its maximum for observer O' as we found in (111)). Since $³_c / c$ starts at 1 and tends at 1 at infinity and for $t \geq \frac{1}{c}$ it is $\frac{³_c}{c} \geq 1$, it must either be flat equal to 1 or have at least one maximum for $t \geq \frac{1}{c}$.

The plot of $\tan\varphi''$ vs t appears in Figure 10(a) and shows that it has a maximum that corresponds to a max for φ''. In Figure 10(b) we plot V_c / c for small angular velocity w_0 and in Figure 10(c) V_c / c for big w_0.

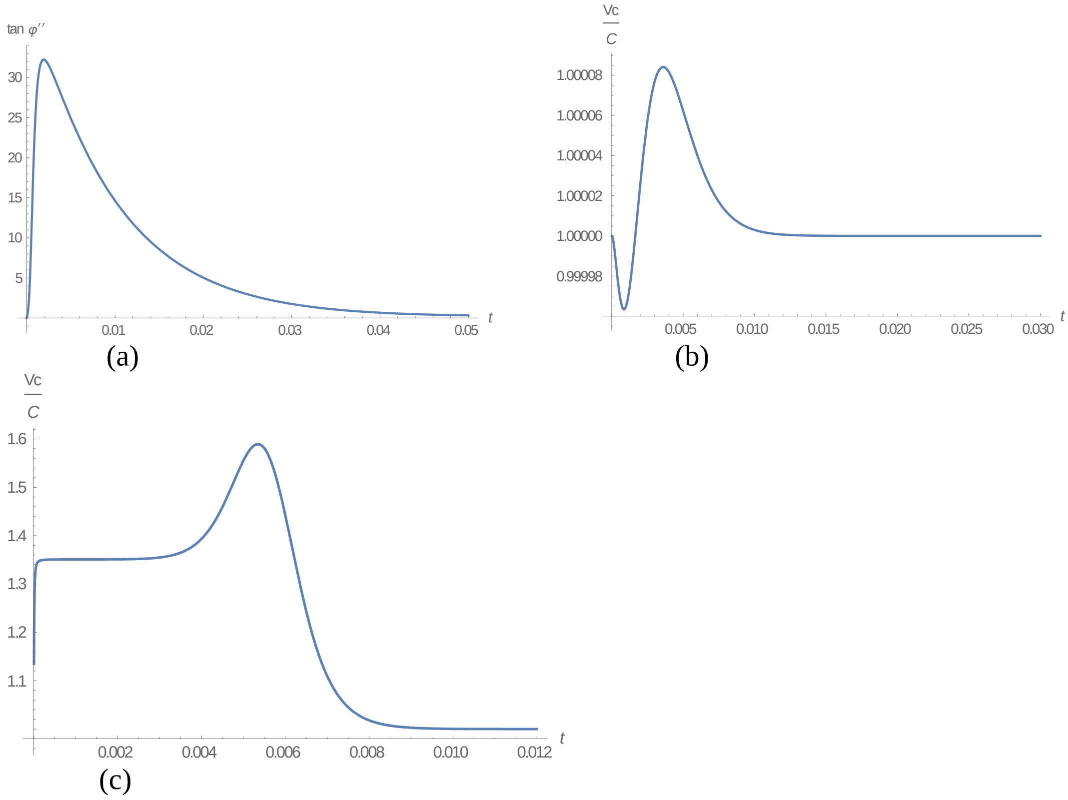


Figure 10 In (a) is the plot of $\tan\varphi''$ vs t. In (b) and (c) is the ratio V_c / c vs t, for small w_0 small (about10 rad/sec) and for w_0 big (about 3*10[5]rad/sec) respectively.

Taking $\dfrac{d\circ''}{d\circ}$ we see that it is positive:

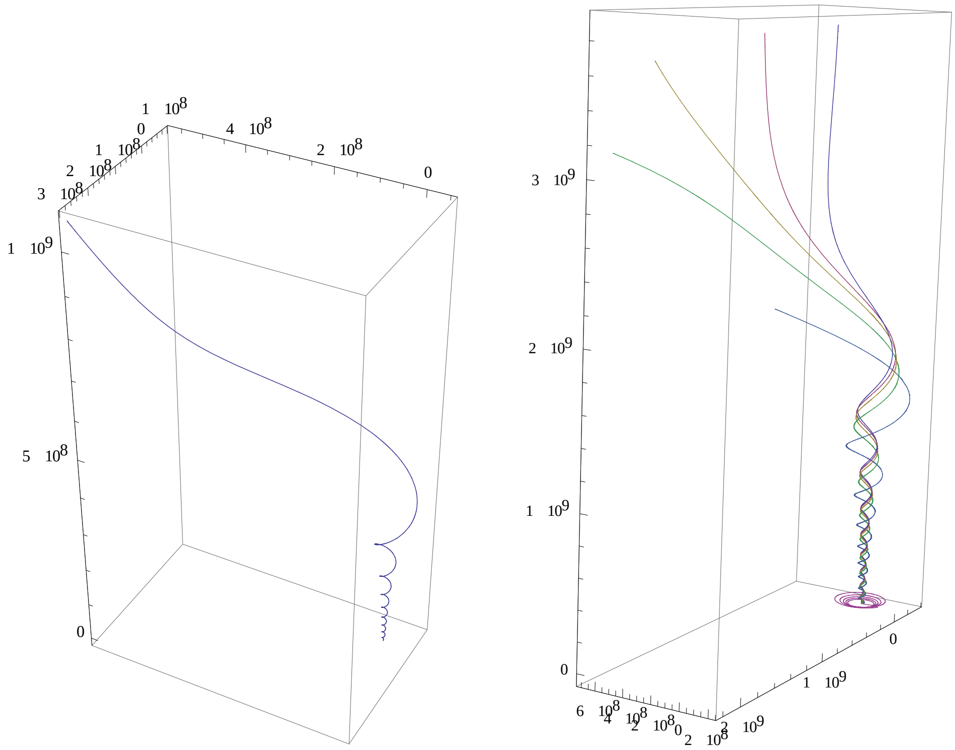
$$\frac{d\circ''}{d\circ} = c\frac{c^2 + \text{ª}\circ{}^3 w_0^2 e^{-2(\text{ª}\circ+\text{«}z)}}{(c^2 + \circ^2 w_0^2 e^{-2(\text{ª}\circ+\text{«}z)})^{\frac{3}{2}}} \tag{124}$$

and hence $\lim\limits_{\circ\to\infty} \circ'' = \infty$ unless $\text{ª} = 0$ in which case $\lim\limits_{\circ\to\infty} \circ'' = \dfrac{c}{w_0}$, which agrees with the result for the no slippage case. Further, for t big, $\circ'' \to \circ$.

The plot of the signal as a function of time, t, as it is given by (119), while z advances according to $z = ct\cos\P$ and ¯ given by,

$$\text{¯} = \int_0^t w_0 e^{-ct(\text{ª}\sin\P+\text{«}\cos\P)}dt = \frac{w_0}{c(\text{ª}\sin\P + \text{«}\cos\P)}\left(1 - e^{-ct(\text{ª}\sin\P+\text{«}\cos\P)}\right) \tag{125}$$

appears in Figure 11, where the inclination $\P$ is a parameter.



(a) (b)

Figure 11 The signal path as it advances in time upward in the z direction, while revolving around the z axis at increasing in time radial distance °″. (a) A single signal path. (b) Many signal paths for the same time interval with different ¶. The signals paths towards the positive z semi axis only are drawn. To complete the picture one must imagine the same jet of signals towards the negative z direction.

We see how most of the signal except those close to $¶ \approx 90^0$ travel tight to each other in the z direction until they break up to return asymptotically to the radial direction. The jet like formation is symmetric with respect to the plane of rotation and another jet emanates towards the negative z direction.

The maximum of tan¨″ (or max for ¨″) is hard to calculate although manipulation of the graph that appears in Figure 10(a) shows clearly that a maximum occurs very close to $t_{¨\,\max} = \dfrac{1}{c(ª\sin¶ + «\cos¶)}$, the minimum of cos¨ (or max of ¨) as we found in (111), when we studied the case of observer O'. It is logical that they must be very close since observer O'' sees an average of what O' sees. We will, therefore, proceed to a first approximation as if they are the same so that

$$t_{¨''\max} \simeq t_{¨\,\max} = \frac{1}{c(ª\sin¶ + «\cos¶)} \tag{126}$$

This, as we mentioned in (111) to (114) implies that our maximum tan¨″ is very close to the locus of points ($°_{¨\,\max}, z_{¨\,\max}$), where the minimum of cos¨ is attained, namely, the rhombus described by

$$ª°_{¨\,\max} + «z_{¨\,\max} = 1 \tag{127}$$

Where

$$°_{¨\,\max} = ct_{¨\,\max}\sin ¶ = \frac{\sin ¶}{ª\sin ¶ + «\cos ¶} \tag{128}$$

$$z_{¨\,\max} = ct_{¨\,\max}\cos ¶ = \frac{\cos ¶}{ª\sin ¶ + «\cos ¶} \tag{129}$$

Then we calculate what shape observer O'' sees as the rhombus where ¨ is maximized. For this we use (119) and calculate it at $t_{¨\,\max}$ to find

$$°''_{¨\,\max} = \frac{°_{¨\,\max}}{\sqrt{1+\dfrac{w_0^2\,°^2_{¨\,\max}}{c^2e^2}}} = \frac{\sin ¶}{\sqrt{(ª^2+\dfrac{w_0^2}{c^2e^2})\sin^2 ¶ + «^2\cos^2 ¶}} \tag{130}$$

Using (130) along with (128) and (129) we plot the line or locus of points ($°''_{¨\,\max}, z_{¨\,\max}$), (recall this is an approximation as we said above) where ¨ attains its maximum, in observer's O'' space $(°'', z)$ as a function of ¶ . This plot appears in Figure 12 and it forms a cigar like locus but it may be wider close to a rectangle with rounded corners depending on the values of ª and « .The width of the locus decreases with increasing w_0 . Observe that $\mathrm{OA{=}OB} = \dfrac{1}{\sqrt{ª^2+\dfrac{w_0^2}{c^2e^2}}}$ (substitute (128) into (130) and set $¶ = ® / 2$) and that the locus crosses the z axis at $\dfrac{1}{«}$ and at $-\dfrac{1}{«}$.

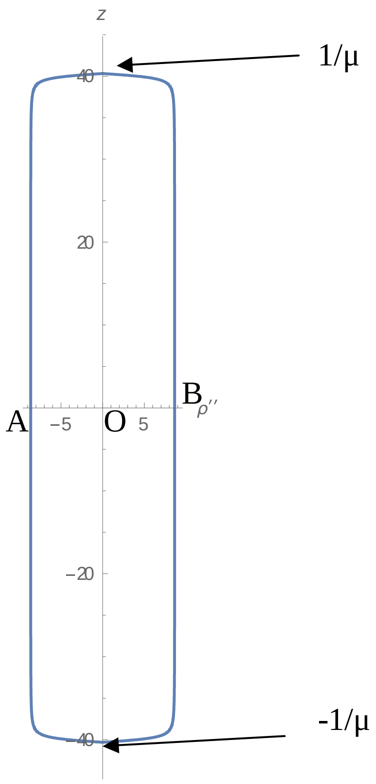


Figure 12 The locus of points $(°_{¨''\max}, z_{¨\ \max})$, (where ¨ attains its maximum) as parameter ¶ varies from 0 to $® / 2$. The other quadrants are obtained by symmetry. The width narrows dramatically as w increases. The shape and roundness also varies with parameters ª and «. For ¶ $= 90^\circ$ the distance $\text{OA=OB} = \dfrac{1}{\sqrt{ª^2 + \dfrac{w_0^2}{c^2 e^2}}}$.

The deflection angle at its approximate maximum at

$$t_{¨''\max} \simeq t_{¨\ \max} = \frac{1}{c(ª \sin ¶ + « \cos ¶)} = 1/c$$

Assumes the approximate value

$$\tan ¨''_{\max} \simeq \frac{w}{c(ª \sin ¶ + « \cos ¶)} = \frac{w_0 e^{-ct(ª \sin ¶ + « \cos ¶)}}{c(ª \sin ¶ + « \cos ¶)} \tag{131}$$

10. Conclusion

Starting from the assumption that two observers rotating with respect to the other around a common axis will agree on epicenter angles as fractions of a circle but not necessarily on the value of *π*, we find the length of the radius of a rotating disc as seen by the non rotating observer. The radius will be contracted but not with the same factor as the perimeter because we allow the value of *π* to change for the non rotating observer with regards to his measurements on the rotating disc. We argued that we have to consider two types of non rotating observers. One within the radius *c/w* and one outside. For the non rotating observer O' within *c/w*, a light signal starting radially from the origin of the rotating disc (frame) that rotates with the frame at an angle ¶ $= 90^\circ$ from the z axis, will gradually turn sideways forming tighter circles until it asymptotically reaches 90º degrees deflection from the radial as the radius tends to infinity. His space is distorted and *π* is different. If a signal is emitted from the origin at an angle ¶ $< 90^\circ$ from the z axis it again expands rotating in ever tighter to one another circles, until it reaches $|z| = \dfrac{c}{w}$. After that the signal is not observable by observer O'.

For the outside observer O'', who stands outside the cylinder with radius c/w, the light rays follow helical paths with increasing radius, which asymptotically tends to c/w.

Next we examined rotation with slippage (rotation is not uniform but decreases exponentially as the radius and z increases with slippage parameters ª and « respectively). In this case, the space is not divided by a cylinder of radius *c/w*, but still there is contraction of the radial distances. The light rays originating from the origin at an angle ¶ from the z axis change their initial direction sideways until they reach a maximum deflection from the radial direction and then asymptotically turn back to the radial. The locus of points, where the maximum deflection occurs is determined. For an observer O' within or near this locus it (the locus) looks close to a deformed rhombus by revolution

with the radial distance contracted, while for an observer O'' located far away it looks like a cigar depending on the slippage parameters.
For the slippage case and for observer O'', it is worth noting the jet like formations, in the direction of the positive and negative z axis, of the signals that emanate from the origin of the rotating frame. Similar jet effects, but less pronounced, should appear for observer O' for the slippage case.

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