On Rotating Frames and the Relativistic Contraction of the Radius (The Rotating Disc)

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Abstract

The relativistic problem of the rotating disc or rotating frame is studied. The solution given implies the contraction of the radius and the change of the value of π depending on the type of observer. Two forms of rotation are considered. One is constant, independent of the radius, implying a horizon, the other is exponentially decreasing with the radius and does not imply a horizon. In all cases the paths of signals emanating from the origin of the rotating frame advance helically in the positive and negative z direction, where they are concentrated, due to the contraction of the radius, and in some cases appear as jets.

1. Introduction

The rotating disc has been the subject of a multitude of papers since Ehrenfest [1] published what is today known as the "Ehrenfest Paradox". He noted that since the perimeter of a rotating disc would relativistically contract, while the radius remained the same, a deformation of the disc should take place. Many authors have since then contributed to the understanding of the problem by studying the geometry of the rotating disc. A historical review can be found in Rizzi and Ruggiero [2] and in Grøn [3]. The approach in the present study is closer to the idea that there is a contraction of the radius of the rotating disc. Similar ideas have been explored by Ashworth, Davies and Jennison [4], [5] and by Grünbaum and Janis [6], [7]. In particular, Ashworth, Davies and Jennison show that the radius of the rotating disc contracts according to an observer that is rotating with the disc at a distance $r > 0$ from the center. Grünbaum and Janis argue that an observer that is not rotating with the disc will see the radius of the disc contract by the same relativistic factor as the perimeter. The latter approach is closer to ours although we do not find the same results because we allow the value of π to change for the non-rotating observer, when he makes measurements regarding the rotating frame.

The paper is organized as follows: In section 2 we state our basic assumptions that will help us derive the transformations in the following sections. In sections 3 we present our notation and known relativistic results regarding the rotating frame. In section 4 we formulate and solve the problem of the contraction of the radius of the disc and the transformation of the value of π for the non rotating observer. In section 5 we present a summary of the results up to that point. In section 6 we consider the space or disc deformation as seen by the non-rotating observer and distinguish between two kinds of non- rotating observers: one within the horizon of radius c/w and one outside. In section 7 we discuss the results. In section 8 we generalize to rotating frames in 3 dimensions. In section 9 we plot the signals emanating from the origin of the rotating frame in the radial direction, as seen by the non-rotating observers. The signals gradually bend sideways until they reach a 90º degree angle with respect to the radius, while they advance in the positive and negative z axis direction. In section 10 we examine rotation with slippage so that the angular velocity is assumed to decrease exponentially as the radius increases. We find that the signals gradually bend sideways until they reach a maximum deflection from the radial direction and then turn asymptotically back to the radial as the radius increases, while they advance in the positive and negative z direction. In section 11 we present our conclusions.

2. Assumptions on the rotation of two concentric discs

In the following we will make three assumptions. Assumptions 2 and 3 are standard in relativity theory. Only Assumption 1 is new.

1. Discussion to justify Assumption 1
Suppose an observer O_1 is at any place on disc (disc 1) and a second observer O_2 that sits at any place of another disc (disc 2) parallel and coaxial to disc 1. O_1 sticks a pencil through his disc parallel to the axis of rotation with its point touching the other disc (disc 2). O_2 does the same thing with the tip of his pencil touching disc 1. As there is relative rotation of one disc with respect to the other, each observer will watch the perimeter of a circle being drawn on his own and the other disc. Each will see that at the moment a complete circle is drawn on his disc, so does on the other. This observation holds for half a circle or any fraction of a circle. In short, we say that the two observers will agree on angles measured as fractions of a circle (not radians). If we denote Θ the magnitude of an angle as fraction of a circle and θ the magnitude of the same angle in radians then $\theta = 2\pi\Theta$. However, we will not use π for the moment because we are not sure the observers agree on π.
Imagine now that the same two observers stand at the center O of their disc (one on top of the other) and let the discs rotate with respect to each other. If the first observer announces that according to his measurements the second disc is rotating with frequency ν, since the situation is exactly symmetrical we will expect the second observer to make the same announcement regarding the first disc. But frequency is defined as revolutions per unit time or Θ per unit time. Once they agree on Θ and ν, they have to agree on time rate. In fact, they may even use the same clock, since they are collocated. We may summarize in the following,

<u>Assumption 1</u>
(a)Two observers sitting at two parallel concentric discs rotating with respect to each other around an axis vertical at their center, will agree that the magnitudes of epicenter angles traveled by the other disc, measured as fractions of a circle, are equal.
(b) If they also stand at the common center of their discs, they will further agree that time rates are equal.

Remark: The situation with the two observers at the center of their discs (one on top of the other) is symmetrical and there is no point in arguing who is rotating and who is stationary. However, if they had a way to measure the centrifugal acceleration off their center, they would probably find that it is different. In that respect we may rightfully call the frame with zero centrifugal force the "preferred frame". In what follows we will assume that one observer, usually to be denoted as O' (or O_2), sits at the center of a preferred frame K' (or K_2), which coincides with the laboratory frame unless otherwise specified.

2. Discussion to justify Assumption 2
Many experiments have verified the constancy of the speed of light, which is the basic assumption for special relativity theory. The speed of light is not the same for observers under acceleration or, according to the Principle of Equivalence, gravity.

Assumption 2
Light speed does not depend on the speed of the emitting source and its speed is constant for all observers that are not under the influence of acceleration.

3. Discussion to justify Assumption 3
An observer on a frame will agree with another observer on the same frame on the measurements of lengths. This is expected if measurements are made not by signals but by actually placing the measuring stick on the length to be measured and counting how many times it fits to it. We state then,

Assumption 3
Observers on the same frame will agree on measurements of length.

3. Time Rate and the Contraction of the Perimeter on a Rotating Disc

As we talk interchangeably about rotating discs and frames, we need to clarify that when we talk of a disc, we imagine it placed at the x-y pane of the respective frame with center at the origin.
Let two frames K_1 and K_2 with cylindrical coordinates, common origin O and common axis of rotation Z. Let observer O_1 sit at the origin of K_1 and O_2 at the origin of K_2. Let K_2 be a non rotating frame - laboratory frame- and let O_2 observe frame K_1 rotate. Let a third observer O_3 on K_1 sit at a distance from the center. When there is no danger of confusion we will rename the three observers using $O = O_1$, $O' = O_2$, $\tilde{O} = O_3$ and the frames $K_1 = K$, $K_2 = K'$.
A quantity Q_{ij} is defined as the quantity measured by observer i given it is stationary in the frame of observer j
For example, ΔL_{21} is the length measured by observer 2 for a line segment on the perimeter of the disc that is stationary in the frame of observer 1. Also, Δt_{21} is the time interval that observer 2 sees that a clock stationary and with observer 1 shows for the duration of an event.
Note that by Assumption 3, ΔL_{ii} is a constant for all i and will be denoted as ΔL_{stat} since all observers agree on the same segment when stationary in their frame. The same is not true for Δt_{ii}. In particular, we expect $\Delta t_{11} = \Delta t_{22} = \Delta t_{stat} \neq \Delta t_{33}$, and $\Delta t_{12} = \Delta t_{21} = \Delta t_{stat}$ because observers 1 and 2 have the same clock , while observer 3 is under the influence of centrifugal acceleration, which we suspect that will affect Δt_{33} in some unknown as yet

way. Therefore, the clock of observer 1 and 2 represents stationary clock time intervals and are denoted as Δt_{stat}. Distances in the radial direction are measured as r_{ij}, while the angular velocity of a disc is measured by w_{ij}. Specifically, $r_{21} = r_{23}$ is the radial distance as seen be the non rotating observer 2 regardless of the second subscript since both observers 1 and 3 are stationary on the rotating disc. Similarly, $w_{21} = w_{23}$ because the angular velocity of the rotating disc as seen by the non rotating observer 2, using his own clock does not depend on the second subscript since both observers 1 and 3 are stationary on the rotating disc.

Most of the authors (see for example Møller [8] pp.222-250 on rotating disc) start from the transformation between a rotating frame K_1 and a non rotating frame K_2 and the transformation relation of cylindrical coordinates $r_1 = r_2$, $z_1 = z_2$, $\theta_1 = \theta_2 - wt_1$. In this case The metric for the rotating frame is

$$ds^2 = dr_1^2 + r_1^2 d\theta_1^2 + dz_1^2 + 2wr_1^2 d\theta_1 dt_1 - (c^2 - w^2 r_1^2) dt_1^2 \tag{1}$$

For a non-moving point in space, the space differentials are null and equating the metrics of the rotating and non-rotating frames we find $c^2 dt_2^2 = (c^2 - w^2 r_1^2) dt_1^2$ from which we find that the clock of the rotating system runs slower,

$$dt_1 = \frac{dt_2}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}} \tag{2}$$

The line element $d\sigma^2$ is given by $d\sigma^2 = \gamma_{\iota\kappa} dx^\iota dx^\kappa$ where x^ι takes the values $\{r_1, \theta_1, z\}$ and $\gamma_{\iota\kappa} = 0$ except for $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}}$, $\gamma_{zz} = 1$. Hence, we find,

$$d\sigma^2 = dr_1^2 + \frac{r_1^2}{1 - \frac{w^2 r_1^2}{c^2}} d\theta_1^2 + dz^2 \tag{3}$$

For a line segment dL_i along the perimeter this formula implies that

$$dL_2 = \frac{dL_1}{\sqrt{1 - \frac{w^2 r_1^2}{c^2}}} \tag{4}$$

Where ($dL_1 = r_1 d\theta_1$ and $dL_2 = r_2 d\theta_2$).

Whereas the result (2) is within our expectations from the special theory of relativity, the result (4) is contrary to the expected result. Since the normal Lorentz contraction would give a contraction of the perimeter instead of a lengthening as we have found above. The result (4) is counterintuitive for another reason also: If we increase the radius, while decreasing the angular velocity so that that tangential velocity is constant ($wr_1 = \upsilon$) then the perimeter tends to a straight line and the transformation should approach the Lorentz length contraction. Instead, according to (4) since $wr_1 = \upsilon$ is constant the lengthening factor remains constant and there is no way of approaching the Lorentz contraction.

As the method with which the result is obtained is correct, the only suspect is the form of transformation assumed. We are motivated, therefore, to search for another transformation that will also satisfy the contraction of the perimeter according to the Lorentz length contraction of special relativity.

Our quest is, therefore, to find a transformation that satisfies in our notation the following,

(a) The rate of the clock of O_3 will appear slower to observer O_2:

$$\Delta t_{23} = \frac{1}{\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}}}\Delta t_{stat} \tag{5}$$

(Recall that $w_{21} = w_{23}$ and $r_{21} = r_{23}$ as we mentioned above)
(b) Line segments along the perimeter are contracted,

$$\Delta L_{21} = \Delta L_{stat}\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}} \tag{6}$$

Observe that (5) implies that the rate of the clock of O_3 will appear slower to O_1, who has the same clock as O_2. Namely,

$$\Delta t_{13} = \Delta t_{33} = \frac{1}{\sqrt{1-\frac{w_{21}^2 r_{21}^2}{c^2}}}\Delta t_{stat} \tag{7}$$

But since there is no relative motion between O_1 and O_3, O_1 will think that is due to the centrifugal acceleration that O_3 feels,

4. The Contraction of the Radius of the Rotating Disc

Refer again to observers O' (or O_2) on K' (or K_2), O (or O_1) and $\tilde{O}$ (or O_3) on K (or K_1) , with K rotating with respect to K' with frequency ν' according to O' (and ν according to O). Suppose observer c has a rod that extends radially from the center O to some point A (see Figure 1). The rod is hollow mirrored inside and infinitesimally thin so that light traveling through it follows a straight line. The rod is just an artifact to help imagine things, a statement saying that O sends a light signal radially outward is enough. Observers O and O' will agree on time rates ($dt = dt'$) (Assumption 1) and the frequencies of rotation they observe will be equal ($\nu = \nu'$) (Assumption 1). They will not agree on angular velocity measured in radians per unit time, because they will in general disagree on π and, therefore, we may say that for observer O the angular velocity is, $w = 2\pi\nu$ ($= w_{11} = 2\pi_{11}\nu$), while for O', $w' = 2\pi'\nu$ ($= w_{21} = 2\pi_{21}\nu$).
Observer O (who is not under acceleration because he sits at the origin although he is rotating with the disc) sends a light signal from O towards A through the rod. According to him the signal travels with velocity $\upsilon = c$ (Assumption 2) the distance OA $= r$ (see Figure 1). Until the signal reaches the end of the rod, the rod will have moved to position OB. Observer O' will see the signal travel a curved path (OCB') with constant tangential velocity $\upsilon' = c$ (Assumption 2). At the perimeter the direction of the velocity of the signal, according to O', will make an angle φ with respect to the radius OB' which will have length $OB' = r'$ ($= r_{21}$).

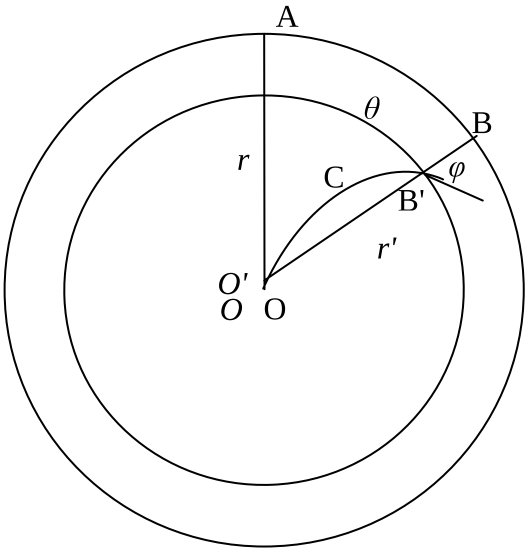


Figure 1 The path of the light signal originating at O is OCB' according to observer O' (the lab observer)

Let
r : the radius as observer O measures it (stationary length) (OA=OB)
r' : the radius according to observer O' (OB')
$\upsilon_r{}'$: the radial component of the velocity υ' according to observer O'
$\upsilon_\theta{}'$: the component of υ' perpendicular to the radius according to observer O'
φ : the angle between $\upsilon_r{}'$ and υ'

We may write the following relationships for light signals letting $\upsilon' = c$ for :
For a light signal $\upsilon' = c$ and then

$$\upsilon_r{}' = c\cos\varphi \tag{8}$$

$$\upsilon_\theta{}' = c\sin\varphi \tag{9}$$

$$\upsilon_\theta{}' = w'r' \tag{10}$$

where w' is the angular velocity in radians as observed by O' and therefore,

$$w' = 2\pi'\nu \tag{11}$$

From (9) and(10)

$$\sin\varphi = \frac{w'r'}{c} \tag{12}$$

And substituting in (8)

$$\upsilon_r{}' = c\sqrt{1-\frac{w'^2r'^2}{c^2}} \tag{13}$$

In order to further satisfy the condition for the contraction of the perimeter (see (6)) we further require that

$$2\pi'r' = 2\pi r\sqrt{1-\frac{w'^2r'^2}{c^2}} \tag{14}$$

Solve for π' and substitute in (11) to obtain

$$\frac{w'}{2\pi\nu} = \frac{r}{r'}\sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{15}$$

We already defined $w = 2\pi\nu$. Substitute and solve (15) for w' to obtain

$$w'^2 = \frac{w^2 r^2 c^2}{c^2 r'^2 + r^2 w^2 r'^2} \tag{16}$$

Now substitute w'^2 in (13) noting that $\upsilon_r{}' \triangleq \frac{dr'}{dt'} = \frac{dr'}{dt}$ and we find

$$\frac{dr'}{dt} = c\sqrt{1-\frac{w^2 r^2}{c^2 + w^2 r^2}} \tag{17}$$

Using $r = ct$ (17) becomes

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1+w^2 t^2}} \tag{18}$$

and integrating with respect to t we find

$$r' = \frac{c}{w}\ln(wt + \sqrt{1+w^2t^2}) = \frac{ct}{wt}\ln(wt + \sqrt{1+w^2t^2}) = \frac{r}{wt}\operatorname{arc\,sinh}(wt) \tag{19}$$

where the constant of integration is zero because we require that $r' = 0$ for $t = 0$.
Equivalently since $r = ct$ (19) can take the form

$$\sinh\frac{wr'}{c} = \frac{wr}{c} \tag{20}$$

Using (20), (16) becomes,

$$w'^2 = \frac{w^4 r^2}{(c^2 + r^2 w^2)\ln^2(\frac{wr}{c} + \sqrt{1+\frac{w^2 r^2}{c^2}})} = \frac{w^4 t^2}{(1+w^2t^2)\ln^2(wt + \sqrt{1+w^2t^2})} = \frac{w^4 t^2}{(1+w^2t^2)\operatorname{arcsinh}^2 wt} \tag{21}$$

A similar relation (21) holds also between π and π' because $w' = 2\pi'\nu$ and $w = 2\pi\nu$ (see (14) and (15))
Using (12) and (16) we find

$$\cos\varphi = \frac{c}{\sqrt{c^2 + w^2 r^2}} = \sqrt{1-\frac{w'^2 r'^2}{c^2}} \tag{22}$$

and we require that $w'r' \le c$.
We have shown that O' (the laboratory observer) will see a contraction of the radius of the rotating disc given by (19) or (20). Observers O' and O will not agree on the angular velocity, w and on the value of π. In fact, observer O' will perceive w' and π' as varying with the distance r. This situation arises from the fact that the contraction factor along the perimeter is different from the contraction factor along the radius and the requirement that the speed of the signals is constant and agreed by all observers.

One remark about angular velocities is useful to clarify things. Observers O, O' and $\tilde{O}$ will agree on epicenter angle $\Delta\Theta$, measured as fraction of a circle (see Assumption 1).
But according to the definition of angular velocity we may write $w = \frac{2\pi\Delta\Theta}{\Delta t}$, $w' = \frac{2\pi'\Delta\Theta}{\Delta t'}$, $\tilde{w} = \frac{2\tilde{\pi}\Delta\Theta}{\Delta\tilde{t}}$, while for angles $\theta = 2\pi\Theta$, $\theta' = 2\pi'\Theta$, $\tilde{\theta} = 2\tilde{\pi}\Theta$, where the tildas refer to

observer $\tilde{O}(O_3)$, the primes to observer $O'(O_2)$, and the plane letters refer to observer $O(O_1)$. The correct notation using the subscript notation of section 3 is
$w=\frac{2\pi\Delta\Theta}{\Delta t}=w_{11}=\frac{2\pi_{11}\Delta\Theta}{\Delta t_{11}}=\frac{2\pi_{11}\Delta\Theta}{\Delta t_{stat}}$, $w'=\frac{2\pi'\Delta\Theta}{\Delta t'}=w_{21}=\frac{2\pi_{21}\Delta\Theta}{\Delta t_{21}}=\frac{2\pi_{21}\Delta\Theta}{\Delta t_{stat}}$,
$\tilde{w}=\frac{2\tilde{\pi}\Delta\Theta}{\Delta\tilde{t}}=w_{33}=\frac{2\pi_{33}\Delta\Theta}{\Delta t_{33}}=\frac{2\pi_{33}\Delta\Theta}{\Delta t_{stat}}\cos\varphi$. But $\pi_{11}=\pi_{22}=\pi_{33}=\pi$ (or $\pi=\tilde{\pi}$), because observers agree both on radial and on perimeter lengths when stationary on their frame (Assumption 3). We conclude, therefore, that

$$\frac{w'}{w}=\frac{w_{21}}{w_{11}}=\frac{\pi_{21}}{\pi_{11}}=\frac{\pi'}{\pi} \tag{23}$$

$$\frac{w}{\tilde{w}}=\frac{w_{11}}{w_{33}}=\frac{\Delta t_{33}}{\Delta t_{11}}=\frac{1}{\cos\varphi} \tag{24}$$

and

$$\frac{w'}{\tilde{w}}=\frac{w_{21}}{w_{33}}=\frac{\pi_{21}\Delta t_{33}}{\pi_{33}\Delta t_{21}}=\frac{\pi_{21}}{\pi_{33}\cos\varphi}=\frac{\pi'}{\pi\cos\varphi} \tag{25}$$

5. Summary of Results

The angle of deflection φ and in particular $\cos\varphi$ takes many equivalent forms that are presented here for ease of calculations

$$\sqrt{1-\frac{w'^2r'^2}{c^2}}=\sqrt{1-\frac{w^2r^2}{c^2+w^2r^2}}=\frac{c}{\sqrt{c^2+w^2r^2}}=\frac{1}{\sqrt{1+\frac{w^2r^2}{c^2}}}=\frac{w'r'}{wr}=\cos\varphi \tag{26}$$

$$\frac{w'r'}{c}=\sqrt{\frac{w^2r^2}{c^2+w^2r^2}}=\sin\varphi \tag{27}$$

$$\tan\varphi=\frac{wr}{c}=wt=\theta=\sinh\frac{wr'}{c} \tag{28}$$

Where θ is the angle of the circle traveled by the signal until it reaches the distance r from the center (see Figure 1).

The transformation among the observers O, O' and $\tilde{O}$ is summarized below in Table1(a), 1(b),1(c). However, our interest in this study is focused on the relation between observer O and O' .

Table 1(a) Transformations between Observers $O(O_1)$ and $O'(O_2)$

Quantitities	**Transformations**
Time interval	$\Delta t=\Delta t'=\Delta t_{stat}$, ($\Delta t_{stat}=\Delta t_{21}=\Delta t_{12}=\Delta t_{11}=\Delta t_{22}$)
Length segment on perimeter	$\Delta L'=\Delta L\cos\varphi$, ($\Delta L_{21}=\Delta L_{stat}\cos\varphi$)
Radius	$r'=\frac{c}{w}\text{arcsinh}\frac{wr}{c}$

Angular velocity	$w' = w^2 \dfrac{r\cos\varphi}{c\,\mathrm{arcsinh}(\frac{wr}{c})}$
Pi and angles	$\dfrac{w'}{w} = \dfrac{\theta'}{\theta} = \dfrac{\pi'}{\pi} = \dfrac{r}{r'}\cos\varphi$

Table 1(b) Transformations between Observers O (O_1) and $\tilde{O}$ (O_3)

Quantitities	**Transformations**
Time interval	$\Delta\tilde{t} = \dfrac{\Delta t}{\cos\varphi}$, $(\Delta t_{13} = \Delta t_{33} = \dfrac{\Delta t_{stat}}{\cos\varphi})$
Length segment on perimeter	$\Delta\tilde{L} = \Delta L$, $(\Delta L_{13} = \Delta L_{31} = \Delta L_{stat})$
Radius	$r = \tilde{r}$
Angular velocity	$\dfrac{\tilde{w}}{w} = \dfrac{\Delta t_{11}}{\Delta t_{33}} = \cos\varphi$
Pi and angles	$\pi = \tilde{\pi}$, $\theta = \tilde{\theta}$

Table 1(c) Transformations between Observers O' (O_2) and $\tilde{O}$ (O_3)

Quantitities	**Transformations**
Time interval	$\Delta\tilde{t} = \dfrac{\Delta t'}{\cos\varphi}$, $(\Delta t_{23} = \Delta t_{33} = \dfrac{\Delta t_{stat}}{\cos\varphi})$
Length segment on perimeter	$\Delta L' = \Delta\tilde{L}\cos\varphi$, $(\Delta L_{23} = \Delta L_{stat}\cos\varphi)$
Radius	$r' = \dfrac{c}{\tilde{w}}\cos\varphi\,\mathrm{arcsinh}\dfrac{\tilde{w}\tilde{r}}{c\cos\varphi}$
Angular velocity	$w' = \dfrac{\tilde{w}^2\tilde{r}}{c\cos\varphi\,\mathrm{arcsinh}(\frac{\tilde{w}\tilde{r}}{c\cos\varphi})}$
Pi and angles	$\dfrac{\tilde{w}}{w'} = \dfrac{\pi_{33}\Delta t_{stat}}{\pi_{21}\Delta t_{33}} = \dfrac{\pi}{\pi'}\cos\varphi = \dfrac{r'}{r}$, $\dfrac{\tilde{\theta}}{\theta'} = \dfrac{\pi_{33}}{\pi_{21}} = \dfrac{\pi}{\pi'}$

6. Warp or Ripples?

From (14) and using (20) we see that $\pi' = \pi \dfrac{\frac{wr}{c}}{\mathrm{arc\,sinh}\frac{wr}{c}}\cos\varphi$. This implies that $\pi' < \pi$ except for $wr = 0$ for which equality holds. The decrease of π implies a warping of the disc or the creation of ripples (see Figure 2(a) and Figure 2(b)). In the case of warping (Figure 2(a)) observer O sees the radius as the segment OA with length r, observer O' sees the curved segment OB with length r', which for him looks as straight and PB is the theoretical straight line (projection of r' on Euclidean space) with length r''.

However, we may exclude warping, because of the following argument: Consider three parallel concentric discs. Let the middle one rotate with frequency ν and with respect to the other two that are stationary. If indeed warping occurred the middle disc would intersect one of the other two discs, since we are allowed to bring them arbitrarily close to the middle disc. This seems unphysical. We are, therefore, inclined to exclude warping as a possibility and to consider ripples instead. Anyway, whether warp or ripples the results in this paper are not affected.

The ripples formed on the disc (Figure 2(b)) make the radius be looked in two different ways. One is the radius touching the surface of ripples (r') (the surface radius) and another is the theoretical straight line disregarding ripples (r'') (the straight radius or the projection of r' on the flat plane of rotation). The latter one satisfies the equation $2\pi r'' = 2\pi r\cos\varphi$ and therefore,

$$r'' = r\cos\varphi \tag{29}$$

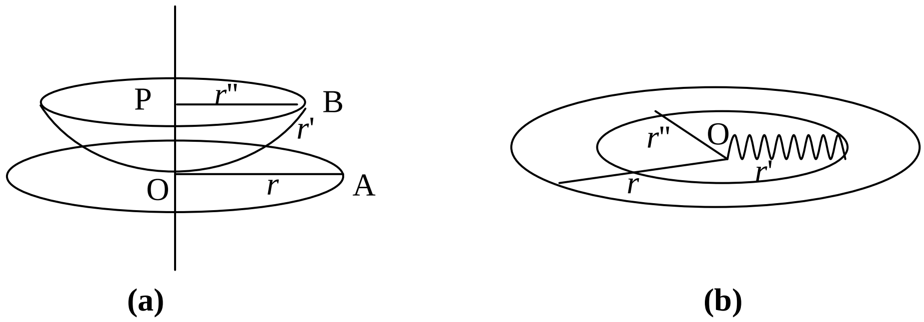


Figure 2 **(a)** The rotating disc is warped. The radius on the surface of the warped disc is r'. The straight line (Euclidean) radius is r''.The stationary radius is r. (**b)** The rotating disc forms ripples. The length of the radius on the rippled surface is r'. The straight line (Euclidean) radius of the disc is r''. The stationary radius is r.

As we will see in the discussion below, if w is finite but $r \to \infty$ then $r' \to \infty$, $r'' \to \frac{c}{w}$, $w' \to 0$, $\cos\varphi \to 0$. The lab observer O' will see the surface radius (r') tend very slowly (logarithmically) to infinity and $w'r' \to c$, while the straight (on the flat Euclidean surface) radius r'' tends to $\frac{c}{w}$.

Physically, r'' is the radius that an observer O'' on the laboratory frame will observe, who is located at a distance greater than $\frac{c}{w}$ from the axis of rotation of the disc. If O'' enters into the region of distance less than $\frac{c}{w}$ from the axis of rotation then his geometry ceases to be Euclidean. He becomes observer of type O'. He is now on the rippled surface (which he perceives as flat) his pi is now π' and the radius of the disc is now given by r', which is not limited by any boundary. A question, however, remains: How is the time rate of O' related to that of O''. We argue as follows: At first since signals bend and do not reach out of the radius $\frac{c}{w}$, there can be no communication between O' and O''. Suppose now that both O', O'' lie within the region of radius $\frac{c}{w}$. They can fix their clocks and indeed since both lie on the same non rotating frame and are not under the influence of acceleration, they find that their time rates are the same. If we now increase gradually the frequency of the disc rotation, their time rates will not be affected since they are stationary on the same non-rotating frame. If we continue to increase the frequency of rotation, O'' will eventually lie outside the region of radius $\frac{c}{w}$. Just before he crosses the boundary the clock of O'' will have a certain delay (due to their distance) but run at the same rate as that of O'. Since rotation of the disc does not affect the time rate of their clocks, because they do not participate in the rotation, we expect that the time rate for O' and O'' after O'' crosses the boundary to remain equal or that $\Delta t' = \Delta t'' = \Delta t_{stat}$, although they cannot exchange light signals. If we denote by double primes the quantities observed by O'', the above implies that

$$\Delta t'' = \Delta t' = \Delta \tilde{t} \frac{1}{\sqrt{1 - \frac{w'^2 r'^2}{c^2}}} \tag{30}$$

Regarding lengths observer O'' sees the lengths of observer O smaller by a factor of $\cos\varphi$. Hence,

$$\Delta L'' = \frac{r''}{r} \Delta L = \Delta L \cos\varphi \tag{31}$$

Further, epicenter angles are not affected,

$$\Delta\Theta = \Delta\Theta'' \tag{32}$$

Hence, there is agreement on frequency of revolution measurements, that is

$\nu = \nu' = \nu''$ because $\nu'' = \frac{\Delta\Theta''}{\Delta t_{stat}}$

Also from (14) and (29) $\pi' r' = \pi r''$. Since $\pi = \pi''$ (the pi on the flat contracted disc is the normal pi) it follows that $w'' = 2\pi''\nu'' = 2\pi\nu = w$. From these observations one easily deduces using (26) that $w'r' = w''r'' = wr''$. Substituting in (26) we find,

$$\cos\varphi = \sqrt{1 - \frac{w''^2 r''^2}{c^2}} = \sqrt{1 - \frac{w'^2 r'^2}{c^2}} \tag{33}$$

7. Discussion of Results

First we note that from (26), (29) and because $w'r' = w''r'' = wr''$

$$\lim_{wr \to \infty} w'r' = \lim_{wr \to \infty} w''r'' = c \tag{34}$$

$$\lim_{w \to \infty} r'' = \lim_{w \to \infty} r' = \lim_{w \to \infty} \left(\frac{r}{\sqrt{1 + w^2 t^2}}\right) = 0 \tag{35}$$

And similarly,

$$\lim_{w \to 0} r'' = \lim_{w \to 0} r' = \lim_{w \to 0} \left(\frac{r}{\sqrt{1 + w^2 t^2}}\right) = r \tag{36}$$

Also

$$\lim_{r \to \infty} w' = 0 \tag{37}$$

And

$$\lim_{w \to \infty} w' = \infty \tag{38}$$

1. As $wr \to \infty$; $wr'' = w'r' \to c$, and $\cos\varphi \to 0$ and $\upsilon_r' \to 0$ and $\upsilon_\theta' \to c$. This says that as the tangential velocity of the rotating disc becomes big, the rays at the circumference are almost tangential and their tangential velocity approaches the speed of light, while their radial velocity tends to zero. In other words, light signals emanating from the center, O, will bend and turn around in circles expanding very slowly as they will be bend almost entirely tangentially.
2. In particular if $w \to \infty$ while r (and hence t) remains finite, $w' \to \infty$, $\cos\varphi \to 0$, $r' \to 0$, $r'' \to 0$, $\upsilon_r' \to 0$ and $\upsilon_\theta' \to c$. In this case, when the angular velocity (w) becomes big, while the rest radius (r) remains finite, the radius (r' and r'') for the lab observers O' and O'' shrinks to become very small (but $wr'' = w'r' \to c$), the light signals staring from the center, O, bend to turn in circles ($\cos\varphi \to 0$) with tangential velocity $\upsilon_\theta' \to c$ and the radial velocity of the light signals tends to zero ($\upsilon_r' \to 0$)
3. If w is finite but $r \to \infty$ then $r' \to \infty$, $r'' \to \frac{c}{w}$, $w' \to 0$, $\cos\varphi \to 0$. Observer O' will see the surface radius (r') tend very slowly (logarithmically) to infinity (while $w'r' \to c$), while O'' will see the radius (r'') tend to $\frac{c}{w}$. The light signals bend and go around in tighter and tighter circles with tangential velocity that tends to c ($\upsilon_\theta' \to c$), while the radial velocity drops to zero.
4. It is straightforward that when there is no rotation $w = 0$ we end back in frame K with $r = r' = r''$, $\Delta L' = \Delta L = \Delta\tilde{L}$, $\upsilon_r' = \upsilon_r$ as expected.
5. To show that the light signals will spiral out in tighter and tighter circles we may examine

$$\frac{dr'}{dr} = \frac{1}{\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{1}{2}}} = \cos\varphi \tag{39}$$

which is increasing with diminishing rate as r increases. Similar observations hold for r'' where

$$\frac{dr''}{dr} = \frac{1}{\left(1 + \frac{w^2 r^2}{c^2}\right)^{\frac{3}{2}}} \tag{40}$$

6. Angle $\theta = \measuredangle \mathrm{AOB}$ in Figure 1 is traversed by the signal whilst it travels the length r, and is given by $\theta = wt$ where $t = \frac{r}{c}$ or $\theta = \frac{wr}{c} = \tan\varphi = \sinh\frac{wr'}{c}$. Angle θ can become very big and even be the result of many revolutions.
7. If the signal originates from the perimeter towards the center, then its path will be symmetric with respect to the radius OB' in Figure 1.
8. A plot of r' for increasing time will look like the following Figure 3

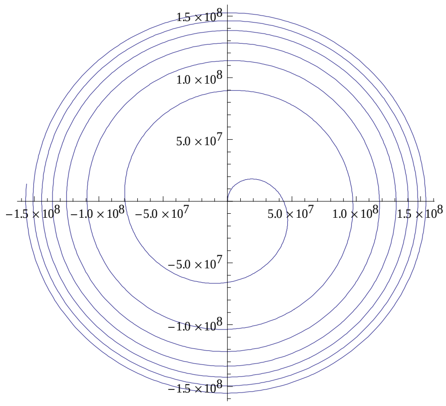


Figure 3 The path of a light signal originating from the center of a rotating frame as seen by the non rotating observer O'. Numbers on the axes are nonessential since scaling changes with w.

8. Generalization to three Dimensions

An observer O at the center O of a rotating frame K is rotating with the frame. He carries a rod (similar to the one we used in the 2 dimensional case above) pointing radially but with an angle ξ with respect to the z axis. The situation is depicted in Figure 4

z

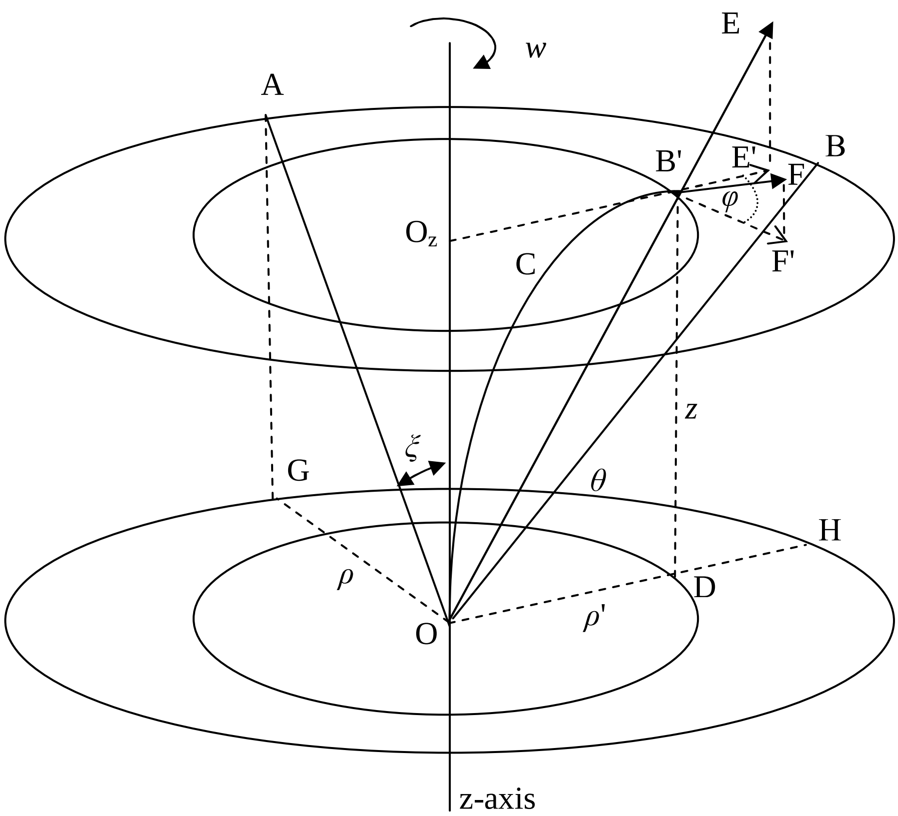


Figure 4 The signals originate from O and move along the rod OA for observer O, who rotates with the frame. The non-rotating observer O' sitting at O will see the signal travel a helical path OCB' while the rod travels from position OA to OB until the signal traverses the rod. The projection of the velocity vector B'F of the signal on the plane of rotation is B'F', as observer O'

perceives it, has magnitude $c\sin\xi$. The angle $\mathrm{E'\hat{B}'F'}=\varphi$ is the angle of deflection from the radius $\mathrm{O_z B'}=\mathrm{OD}=\rho'$ (in cylindrical coordinates). The angle traversed by the rod is $\mathrm{G\hat{O}H}=\theta$

Suppose a signal with velocity c originates from O with angle ξ with respect to the axis of rotation as seen by observer O and is directed towards A through the rod OA (see Figure 4). The radial (in cylindrical coordinates (ρ,θ,z)) velocity of the signals for observer O is $c\sin\xi = c\frac{\rho}{\sqrt{\rho^2+z^2}}$ and the z component is $c\cos\xi$. Suppose now that O rotates with the rod OA with frequency ν' as seen by another observer O' that sits on top of O but does not rotate with O. Let also ν be the frequency that O thinks his frame, K, rotates with respect to K' of observer O'. Because of Assumption 1, $\nu'=\nu$ and it has meaning to define the angular velocity of the frame K as $w\equiv 2\pi\nu$. Observer O' will see the signal travel the helical path OCB' in the same time that it takes to traverse the rod for observer O, while the rod moves from position OA to OB. For him light travels along the helical path with the same velocity c and the z component equals that of observer O ($c\cos\xi$). The velocity vector for observer O' is B'F and it makes an angle ξ with the z-axis. The projection B'F' of the velocity vector B'F (tangential to the helical path for observer O') on the plane of rotation is denoted as υ'_{proj} and

$$\upsilon'_{proj} = c\sin\xi \tag{41}$$

The angle between it and the radial (in cylindrical coordinates) B'E' for Observer O' is φ. This angle is called *the angle of deflection* of the velocity vector of the signal from the radial direction. Let us denote the velocity in the radial direction (in cylindrical coordinates) as observer O' sees it by υ'_ρ. Then $\upsilon'_\rho = \upsilon_{proj}\cos\varphi$ and therefore,

$$\upsilon'_\rho = c\sin\xi\cos\varphi \tag{42}$$

The tangential component of the light signal for observer O' on the plane of rotation will be perpendicular to υ'_ρ and will be given by

$$\upsilon'_\theta = c\sin\xi\sin\varphi \tag{43}$$

Also we want

$$\upsilon_\theta{}' = w'\rho' \tag{44}$$

As usual, the primed quantities $\upsilon_{proj}{}'$, υ'_ρ, υ'_θ, ρ', w' are as observer O' perceives them. For the same reasons (Lorentz contraction of perimeter) as in the two dimensional case we require that (14) holds. Namely,

$$2\pi'\rho' = 2\pi\rho\sqrt{1-\frac{w'^2\rho'^2}{c^2}} \tag{45}$$

Solving for π' and substituting in $w'=2\pi'\nu$ we find

$$w'^2 = \frac{w^2\rho^2c^2}{c^2\rho'^2+\rho^2w^2\rho'^2} \tag{46}$$

where $w=2\pi\nu$ and we note that (46) is the same as (16), as expected.
Finally,

$$\upsilon'_\rho = \frac{d\rho'}{dt} \tag{47}$$

Equations (42), (43), (44), (46), (47) are five equations in five unknowns: $\varphi, w', \rho', \upsilon'_\rho, \upsilon'_\theta$ given w, ρ, ξ.
From (43) and (44)

$$\sin\varphi = \frac{w'\rho'}{c\sin\xi} \tag{48}$$

and hence,

$$\cos\varphi = \sqrt{1 - \frac{w'^2\rho'^2}{c^2\sin^2\xi}} \tag{49}$$

with the condition $w'\rho' \le c\sin\xi$. And using (46)

$$\cos\varphi = \sqrt{1 - \frac{w^2\rho^2}{(c^2+\rho^2 w^2)\sin^2\xi}} \tag{50}$$

Since $\rho = ct\sin\xi$ and $z = ct\cos\xi$, we may write the above relation as,

$$\cos\varphi = \sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}} = \sqrt{\frac{c^2-w^2z^2}{c^2+w^2\rho^2}} \tag{51}$$

with the condition that $1-w^2t^2\cos^2\xi \ge 0$ (or $z \le \frac{c}{w}$) (or $w'\rho' \le c\sin\xi$)

(Note that $\cos\varphi = 0$ either when $z = \frac{c}{w}$ or when $w\rho$ goes to ∞)

Substituting in (42)

$$\upsilon'_\rho = c\sin\xi\cos\varphi = c\sin\xi\sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}} \tag{52}$$

and since $\upsilon'_\rho = \frac{d\rho'}{dt}\frac{dt}{dt'} = \frac{d\rho'}{dt}$,

$$\rho' = c\sin\xi\int_0^t \sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}}\,dt \tag{53}$$

It is convenient to represent the integral in the RHS of (53) as a function of ξ and t. So we define,

$$I(\xi,t) = \int_0^t \sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}}\,dt \tag{54}$$

Then we may rewrite (53) as

$$\rho' = c\sin\xi I(\xi,t) \tag{55}$$

Note that for $\xi = \frac{\pi}{2}$, (55) becomes

$\rho'\big]_{\xi=\frac{\pi}{2}} = c\int_0^t \frac{dt}{\sqrt{1+w^2t^2}} = \frac{c}{w}\operatorname{arcsinh} wt = \frac{c}{w}\operatorname{arcsinh}\frac{wr}{c}$, which is what we found for the two dimensional case.

Below we summarize the following equations that are useful for calculations,

$$\sin\varphi = \frac{w'\rho'}{c\sin\xi} = \frac{w\rho}{\sin\xi\sqrt{c^2+\rho^2w^2}} = \frac{wt}{\sqrt{1+w^2t^2\sin^2\xi}} \tag{56}$$

$$\cos\varphi = \sqrt{1-\frac{w'^2\rho'^2}{c^2\sin^2\xi}} = \sqrt{\frac{c^2-w^2z^2}{c^2+\rho^2w^2}} = \sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}} \tag{57}$$

$$\tan\varphi = \frac{wt}{\sqrt{1-w^2t^2\cos^2\xi}} = \frac{w\rho}{\sin\xi\sqrt{c^2-w^2z^2}} = \frac{w'\rho'}{\sqrt{c^2\sin^2\xi-w'^2\rho'^2}} \tag{58}$$

$$\frac{w'\rho'}{w\rho} = \frac{2\pi'\nu\rho'}{2\pi\nu\rho} = \frac{\pi'\rho'}{\pi\rho} = \frac{c}{\sqrt{c^2+w^2\rho^2}} = \cos\varphi\Big]_{z=0} = \cos\varphi\Big]_{\xi=\frac{\pi}{2}} \tag{59}$$

$$I(\xi,t) = \int_0^t \cos\varphi dt \tag{60}$$

It is also useful to keep in mind that $\sin\xi = \frac{\rho}{\sqrt{\rho^2+z^2}}$, $\cos\xi = \frac{z}{\sqrt{\rho^2+z^2}}$, $\tan\xi = \frac{\rho}{z}$,

$ct = \sqrt{\rho^2+z^2}$

In most of the cases below we will express formulas either using the pair of variables (ξ,t) or the pair (ρ,z) and using the above relations we will be able to transfer from one type of expression to the other.

As in the two dimensional case ρ'' is the (Euclidean) straight radius regardless of the value of z and must satisfy the Lorentz contraction of the perimeter

$$2\pi\rho'' = 2\pi\rho\sqrt{1-\frac{w^2\rho''^2}{c^2}} \tag{61}$$

Solve to find

$$\rho'' = \rho\frac{c}{\sqrt{c^2+w^2\rho^2}} = \cos\varphi\Big]_{z=0} \tag{62}$$

noting that from (57)

$$\cos\varphi\Big]_{z=0} = \frac{c}{\sqrt{c^2+w^2\rho^2}} \tag{63}$$

Now solving (62) for ρ we find that

$$\rho = \frac{\rho'' c}{\sqrt{c^2-w^2\rho''^2}} \tag{64}$$

From (64) we see that $\rho'' < \frac{c}{w}$ and as $\rho\to\infty$, $\rho''\to\frac{c}{w}$

We are led, therefore, to consider two types of laboratory observers as we did in the two dimensional case. One is O'', who is far enough so that he is outside the cylinder of radius $\frac{c}{w}$ and axis the axis of rotation. This observer lives in Euclidean space and he regards all light signals from O bounded by the above mentioned cylinder, which for him has radius ρ''. The other laboratory observer O' is near the body of rotation and in fact within the above cylinder, where he perceives ρ'. For the latter observer, as we will see, the signals are not bounded but cover the whole space and he does not see an outside region as O'' does.

9. Plot of Signals in three Dimensions

9.1. In the Rippled Space (radius ρ') : observer O'

Under the condition that $1-w^2t^2\cos^2\xi \ge 0$ (which is required for $\cos\varphi$ to a real number) and $w \ne 0$, and $0 < \xi \le \frac{\pi}{2}$ we want to calculate $I(\xi,t)$

Make the substitution

$$k = i\cot\xi \tag{65}$$

And

$$x = iwt\sin\xi \tag{66}$$

Where $\imath = \sqrt{-1}$ and then (54) becomes

$$I(\xi,t) = \frac{1}{iw\sin\xi}\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}dx \tag{67}$$

But the integral on the RHS is an incomplete Elliptic integral of the second kind denoted as $E(k,x)$. Therefore, (67) can be written as

$$I(\xi,t) = \frac{1}{iw\sin\xi}E(iwt\sin\xi, i\cot\xi) \tag{68}$$

which can be used for calculations.

If we take the definite integral of (53) for $0 \le t \le \frac{1}{w\cos\xi}$ (the limit allowed by the condition $1-w^2t^2\cos^2\xi \ge 0$) we find the max value that ρ' can take for each particular w and ξ. Namely,

$$\rho'_{\mathrm{m}}(\xi) = c\sin\xi\int_0^{\frac{1}{w\cos\xi}}\sqrt{\frac{1-w^2t^2\cos^2\xi}{1+w^2t^2\sin^2\xi}}dt = c\sin\xi I(\xi,\frac{1}{w\cos\xi}) \tag{69}$$

This says that for any fixed ξ, the signals in the radial direction are bounded (see Figure 5). The signals bend and rotate until they reach $z = \frac{c}{w}$ at $t_{\mathrm{m}} = \frac{1}{w\cos\xi}$, the angle of deflection becomes 90$^{\mathrm{o}}$ degrees, the radial velocity diminishes to zero and the radial distance of the signal becomes.

$$\rho'_{\mathrm{m}}(\xi) = c\sin\xi I_{\mathrm{m}}(\xi) \tag{70}$$

where we denote $I_{\mathrm{m}}(\xi) = I(\xi,\frac{1}{w\cos\xi})$

However, for $\xi = \frac{\pi}{2}$ we come back to the two dimensional case and the signal is not bounded. It expands slowly all the time logarithmically and again the radial velocity tends to zero.

We said that all signals when they reach $\rho'_{\mathrm{m}} = c\sin\xi I_{\mathrm{m}}(\xi)$, they are at $z_{\mathrm{m}} = \frac{c}{w}$. After that the signals have fixed radius ($\rho'_{\mathrm{m}}(\xi)$) but continue to spiral in the z direction according to $z = ct\cos\xi$ (see Figure 5 and 6).

Further, because of (48) $w'\rho' = c\sin\xi\sin\varphi \le c\sin\xi$ and for ρ'_{m}, where $\sin\varphi = 1$ we have

$$w'_{\mathrm{m}}\rho'_{\mathrm{m}} = c\sin\xi \tag{71}$$

and hence at ρ'_{m} the angular velocity of the signals w'_{m} for observer O' is

$$w'_{\mathrm{m}} = \frac{c}{\rho'_{\mathrm{m}}} \tag{72}$$

The effect of w is to scale down distances as it increases. The faster the body rotates, the faster the signals bend making tighter revolutions closer to the body.
The signals are not limited, but cover the whole space as angle ξ varies from 0 to 2π

It will be convenient to call "Inner Region" the part of space where $|z| \le \dfrac{c}{w}$ and "Outer Region" the space for $|z| > \dfrac{c}{w}$.

In Figure 5 we plot the path of the signals in the region $0 \le z \le \dfrac{c}{w}$ for several signals with different values of ξ.The paths are for the region $-\dfrac{c}{w} \le z \le 0$ are symmetric with respect to the plane of rotation.

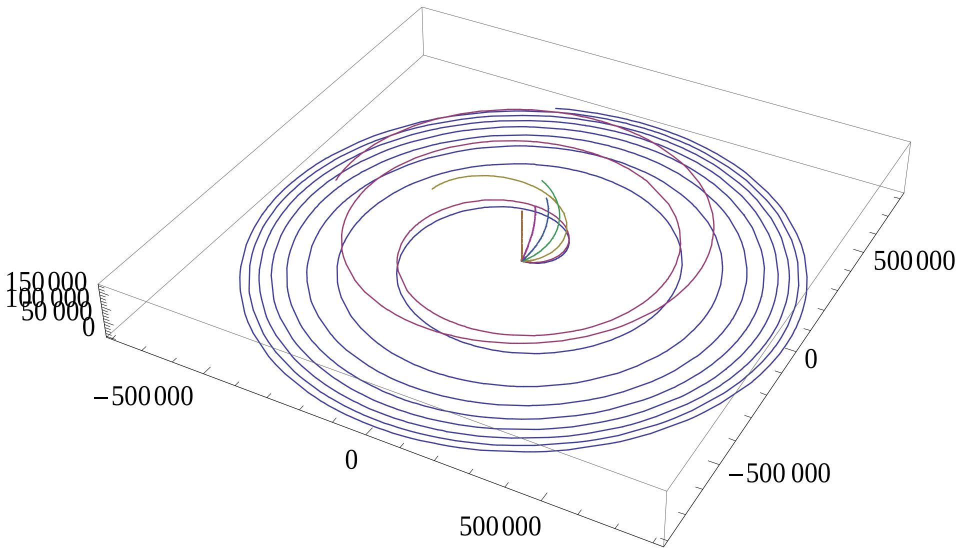


Figure 5 Plot of signals emanating from O for observer O' as they rotate with radius ρ' while advancing in the z direction for the region $0 \le z \le c/w$ for different values of ξ. The numbers on the axes are not essential since they depend on the value of w.

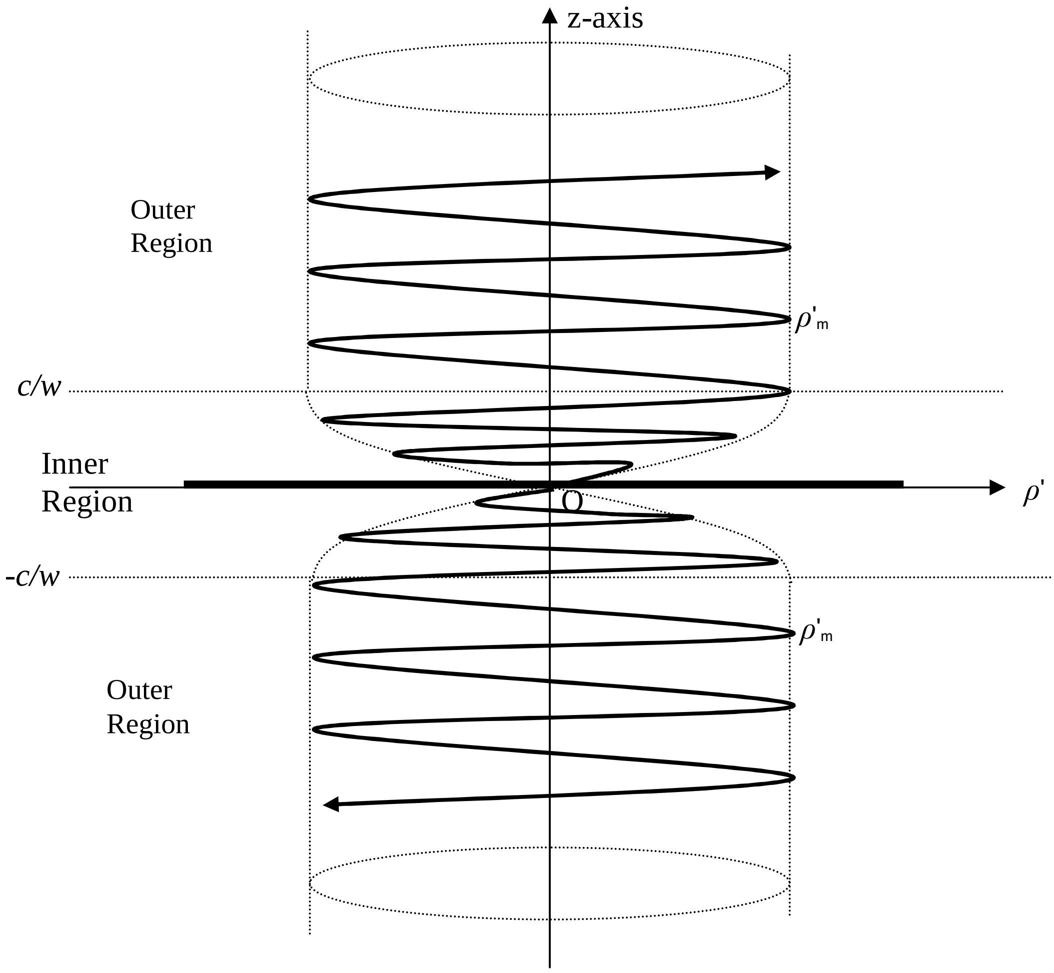


Figure 6 The path of a signal originating at O with cylindrical coordinates (z , ρ') as observer O' sees it. The signal has constant velocity in the z direction, while it circles around the z axis with radius ρ'. The radius ρ' is increasing in the region $|z| \leq c/w$ and at $|z| = c/w$ it reaches a maximum value ρ'_{m}. After that, for $|z| \geq c/w$, the signal rotates around the z axis with constant radius ρ'_{m}. Similar observations hold for the negative z semi axis. For $\xi = \pi/2$ the signal rotate in a disc at $z = 0$ and expands slowly (logarithmically) outward. As ξ varies the signals cover the whole space according to observer O'.

9.2. Outside the Rippled Space (straight radius ρ'') : observer O''

The signal has constant velocity in the z direction equal to $c\cos\xi$ as in the previous case and rotates around the z axis with radius (recall (62)), $\rho'' = \rho \dfrac{c}{\sqrt{c^2 + w^2\rho^2}}$.

Since $z = ct\cos\xi$ and $\rho = ct\sin\xi$,

$$\rho'' = \frac{ct\sin\xi}{\sqrt{1 + w^2t^2\sin^2\xi}} \tag{73}$$

As we discussed above, the radius of the signal increases while $|z| \le \frac{c}{w}$, which means that $0 \le t \le \frac{1}{w\cos\xi}$. The maximum radius is

$$\rho''_m = \frac{ct_m \sin\xi}{\sqrt{1+w^2 t_m^2 \sin^2\xi}} \tag{74}$$

where $t_m = \frac{1}{w\cos\xi}$, which corresponds to $\rho_m = ct_m \sin\xi = \frac{c}{w}\tan\xi$, and simplifying (74) we get,

$$\rho''_m = \frac{c}{w}\sin\xi \tag{75}$$

After it reaches ρ''_m, the signal keeps the radius of revolution constant and advances in the z direction with constant speed describing a helical path. In Figure 7**,** we plot the signal in the inner region. That is the region before it reaches ρ''_m at $z = \frac{c}{w}$. The paths are symmetrical with respect to the plane of rotation for the region $-\frac{c}{w} \le z \le 0$.

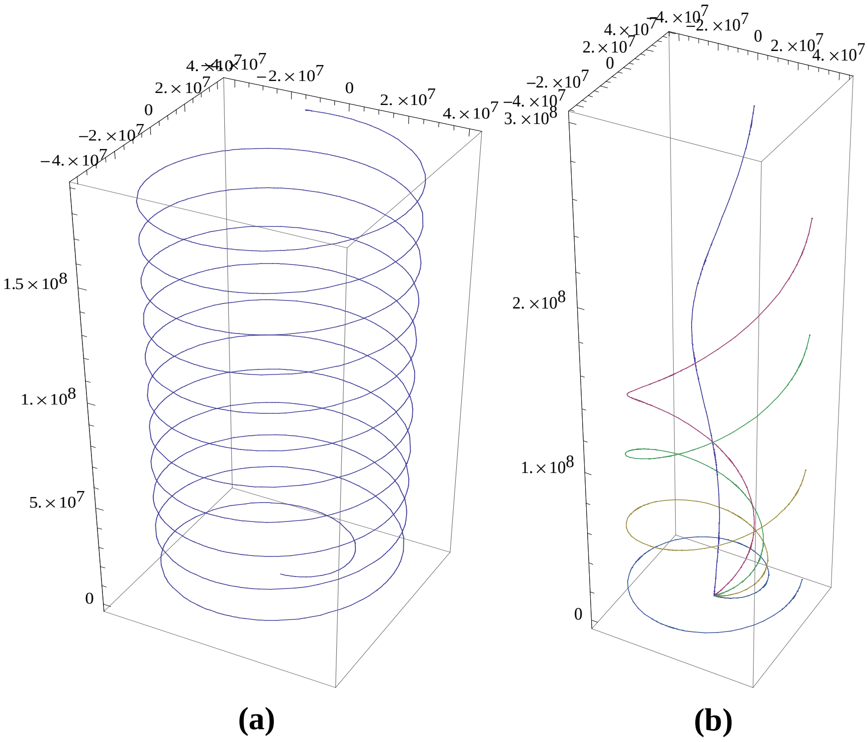


Figure 7 The path of signals for observer O'' in the region $0 \le z \le c/w$ **(a)** The path of a single signal **(b)** Paths of multiple signals with different values of ξ for the same time interval.

In Figure 8 we present a sketch of the path of a signal both in the inner ($|z| \le c/w$) and outer region ($|z| \ge c/w$) as seen by observer O''. Observe that $\rho''_m = \frac{c}{w}\sin\xi \le \frac{c}{w}$ and becomes equal to c/w for $\xi = \pi/2$. Therefore, all signals are bounded by the cylinder of radius c/w and axis the z-axis. We call the region outside the cylinder "External Region".

The path of the signal in the inner region was described above. For the outer region the signal having reached the radius $\rho''_m = \frac{c}{w}\sin\xi$, continues to move helically in the z direction (-z for the negative z axis) keeping their radius fixed at ρ''_m.

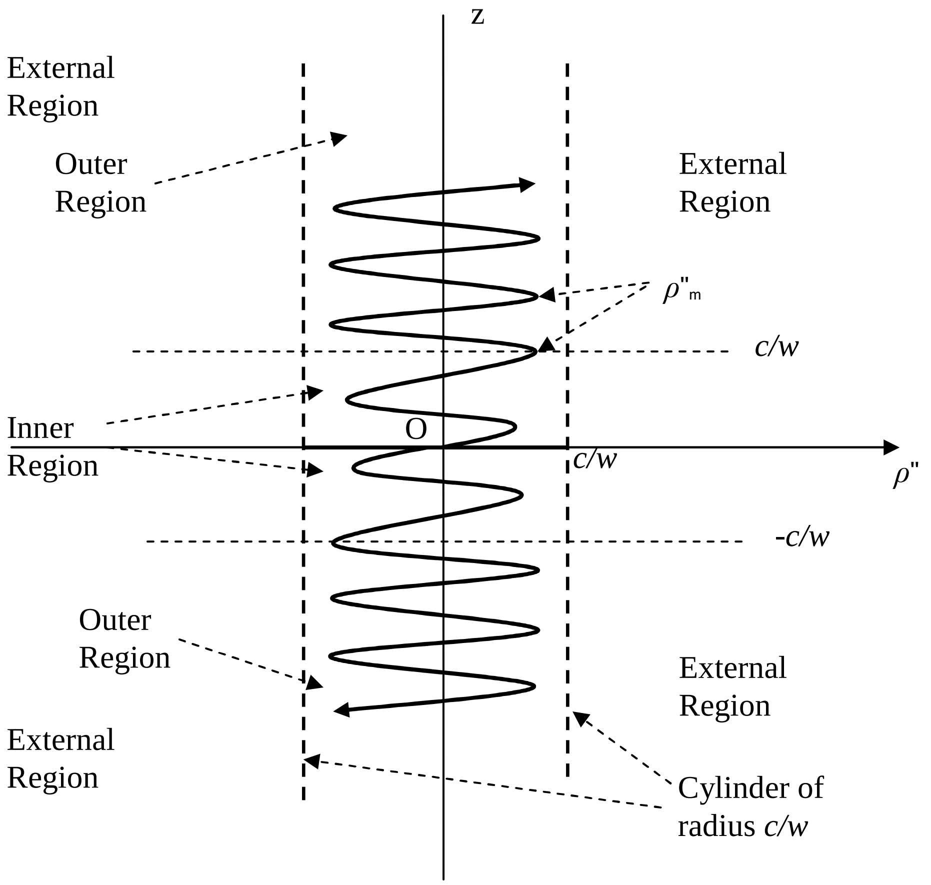


Figure 8 The path of a signal originating at O with cylindrical coordinates z and ρ''. The signal has a constant velocity in the z direction while it forms circles around the z axis of radius ρ''. The radius ρ'' increases and reaches a maximum $\rho''_m = \frac{c}{w}\sin\xi$, when $z = \frac{c}{w}$. Then the signal travels parallel to the z axis at $\rho''_m = \frac{c}{w}\sin\xi$. The same observations hold for the negative z semi axis as well. For $\xi = \pi / 2$ the signals stay on a disc at $z = 0$ and cycle with increasing radius that tends asymptotically to c / w. All signals are bounded by the cylinder of radius c / w and axis z.

10. Rotating Disc with Slippage

10.1. The two-Dimensional Case

If we allow the angular velocity w to vary as a function of the radius it is like having a disc that consists of rings with small width that slide one after the other. Let for example

$$w = w_0 f(r) \tag{76}$$

where $0 < f(r) < 1$ and non-increasing in r

In this setup there is a multitude of O type observers, who stand at the origin O, one for each particular value of the radius r, who rotate with the angular velocity of that particular ring. Observer O' is standing on the center on top of O but is not rotating with the disc. The clocks of the two types of observers will run at the same rate. Again equations (11) to (18) continue to hold. Substituting (76) into (18) we find,

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 f(r)^2}} = c\cos\varphi \tag{77}$$

and

$$\cos\varphi = \frac{c}{\sqrt{c^2 + w_0^2 r^2 f(r)^2}} = \frac{1}{\sqrt{1 + w_0^2 t^2 f(ct)^2}} \tag{78}$$

If we let $f(r) = e^{-\lambda r}$ we find

$$\frac{dr'}{dt} = \frac{c}{\sqrt{1 + w_0^2 t^2 e^{-2\lambda r}}} \tag{79}$$

or

$$r' = c\int \frac{dt}{\sqrt{1 + w_0^2 t^2 e^{-2\lambda r}}} + const \tag{80}$$

To see how r' behaves take the derivative of $\cos\varphi$

$$\frac{d}{dr}\cos\varphi = \frac{1}{c}\frac{d}{dt}\cos\varphi = -\frac{w_0^2 t e^{-2\lambda ct}(1 - \lambda ct)}{c(1 + w_0^2 t^2 e^{-2\lambda ct})^{\frac{3}{2}}} \tag{81}$$

At $t = 0$ the derivative is negative then as t increases the derivative increases and at $t = \dfrac{1}{\lambda c}$ it becomes zero and then positive and finally for big t it tends to zero. A plot of $\cos\varphi$ appears in Figure 9

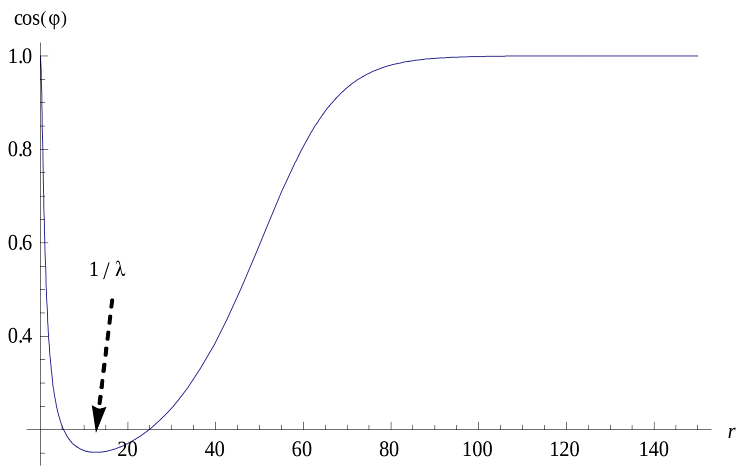


Figure 9 The cosine of deflection angle φ versus *r*. The deflection angle starts at zero (cos0°=1). Then it increases reaching almost 90° degrees (for big enough w_0) at $r = \frac{1}{\lambda}$. Then it falls again to zero (cos90°=1) asymptotically.

The deflection angle initially at 0° increases to almost 90° degrees (closer to 90° for higher w_0) and then drops fast to zero. This behavior is similar to the behavior we have examined for the rotation without slippage. Namely, as the angle of deflection increases the signals start rotating in tighter circles until they reach $r = \frac{1}{\lambda}$. Then the signals rotate in less and less tight circles until they are directed asymptotically radially outward ($\varphi = 0$).

In this setup although space is not divided, we may talk about an observer O' within the circle $r \le \frac{1}{\lambda}$ or close to it if outside, and an observer O'' at infinity (the far away observer) so that the space of the second observer is not affected by the rotation and is Euclidean in his vicinity. This observer will extend his perception of space as Euclidean to cover the rotating body too. What he will see is the projection r'' of r' in his space as we described in Figure 2(a) and Figure 2(b). Since both observers of type O and O'' have the same *π* the contraction of the perimeter due to normal Lorentz contraction requires that,

$2\pi r'' = 2\pi r\sqrt{1 - \frac{w^2 r''^2}{c^2}}$ and solving we get $r'' = \frac{rc}{\sqrt{c^2 + w^2 r^2}} = r\cos\varphi$ where $w = w_0 f(r)$.

The choice of slippage according to the rule $f(r) = e^{-\lambda r}$ can be justified by the assumption that for a disc consisting of slipping rings each rings slips with respect to the previous by the same proportion in angular velocity. Consider for example a width r of *n* layers. The first has velocity w_0, the second $w_0\alpha^{\frac{r}{n}}$, the third $w_0(\alpha^{\frac{r}{n}})^2$ the nth $w_0\beta^n$ where $\beta = \alpha^{\frac{r}{n}}$. Each layer slips with respect to the previous by proportion β forming a geometric series. Therefore, letting $n \to \infty$ the ring at radius r will have angular velocity $w_0\alpha^r$. For the case that $0 < \alpha < 1$, where we are interested, we may substitute $\alpha = e^{-\lambda}$ where $\lambda > 0$ and then $w_0\alpha^r = w_0 e^{-\lambda r}$ or $f(r) = e^{-\lambda r}$.

10.2. The three-Dimensional Case

For the three dimensional case we assume that there is slippage both in the radial and the z direction. To achieve this we assume that

$$w = w_0 e^{-\lambda\rho - \mu z} = w_0 e^{-ct(\lambda\sin\xi + \mu\cos\xi)} \tag{82}$$

where $\lambda \ge 0$, $\mu \ge 0$ and usually $\lambda < \mu$, since we expect slippage to be less in the radial direction than in the z direction for a rotating disk. Relations (42) to (47) continue to hold with where $w = w_0 e^{-\lambda\rho - \mu z} = w_0 e^{-ct(\lambda\sin\xi + \mu\cos\xi)}$

A For observer O'

$$\frac{d\rho'}{dt} = c\sin\xi\cos\varphi = c\sin\xi\sqrt{\frac{1-e^{-2tc(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\cos^2\xi}{1+e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\sin^2\xi}} \tag{83}$$

where

$$\cos\varphi = \sqrt{\frac{c^2 - e^{-2(\lambda\rho+\mu z)}w_0^2z^2}{c^2 + e^{-2(\lambda\rho+\mu z)}w_0^2\rho^2}} = \sqrt{\frac{1-e^{-2tc(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\cos^2\xi}{1+e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\sin^2\xi}} \tag{84}$$

because $z = ct\cos\xi$ and $\rho = ct\sin\xi$. Therefore,

$$\rho' = c\sin\xi I(\xi,t,\lambda) \tag{85}$$

where

$$I(\xi,t,\lambda,\mu) = \int_0^t \sqrt{\frac{1-e^{-2tc(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\cos^2\xi}{1+e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\sin^2\xi}}dt \tag{86}$$

For $\xi = \frac{\pi}{2}$ we have,

$$I(\frac{\pi}{2},t,\lambda,\mu) = \int_0^t \sqrt{\frac{1}{1+w_0^2t^2e^{-2ct\lambda}}}dt \tag{87}$$

Observe that (84) is the same as (57), where $w = w_0e^{-\lambda\rho-\mu z}$. The same is true for (56) (58), (59).

Looking now at $\cos\varphi$ we take the derivative to find its minimums. After some straight forward manipulation we find,

$$\frac{d}{dt}\cos\varphi = \begin{cases} -\dfrac{w_0^2te^{-2ct(\lambda\sin\xi+\mu\cos\xi)}(1-ct(\lambda\sin\xi+\mu\cos\xi))}{\sqrt{1+e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\sin^2\xi}} \\ \left(\dfrac{\cos^2\xi}{\sqrt{1-e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\cos^2\xi}} + \dfrac{\sin^2\xi\sqrt{1-e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\cos^2\xi}}{\sqrt{1+e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}w_0^2t^2\sin^2\xi}}\right) \end{cases} \tag{88}$$

The second factor in parentheses is positive. Therefore, looking at the first factor we see that it starts negative for $t = 0$ and then changes sign at

$$t_{\min} = \frac{1}{c(\lambda\sin\xi+\mu\cos\xi)} \tag{89}$$

This corresponds to

$$\rho_{\min} = ct_{\min}\sin\xi = \frac{\sin\xi}{\lambda\sin\xi+\mu\cos\xi} \tag{90}$$

and

$$z_{\min} = ct_{\min}\cos\xi = \frac{\cos\xi}{\lambda\sin\xi+\mu\cos\xi} \tag{91}$$

The value of $\cos\varphi$ at the minimum is

$$\cos\varphi\Big|_{t=t_{\min}} = \begin{cases} \sqrt{\dfrac{(\lambda\sin\xi+\mu\cos\xi)^2c^2 - w_0^2e^{-2}\cos^2\xi}{(\lambda\sin\xi+\mu\cos\xi)^2c^2 + w_0^2e^{-2}\sin^2\xi}} & \text{for } (\lambda\sin\xi+\mu\cos\xi)^2c^2 - w_0^2e^{-2}\cos^2\xi \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

(92)

The condition $(\lambda\sin\xi+\mu\cos\xi)^2c^2 - w_0^2e^{-2}\cos^2\xi \geq 0$ can be given the form

$$\frac{\cos\xi}{\lambda\sin\xi+\mu\cos\xi}\le\frac{ce}{w_0} \tag{93}$$

On account of (91) this condition becomes

$$z_{\min}\le\frac{ce}{w_0} \tag{94}$$

The parametric plot of $z_{\min}$ versus $\rho_{\min}$ as ξ varies from 0 to $\frac{\pi}{2}$ is a straight line as can be easily verified:
First we observe that

$$\tan\xi=\frac{\rho}{z} \tag{95}$$

But

$$z=\frac{\cos\xi}{\lambda\sin\xi+\mu\cos\xi}=\frac{1}{\lambda\tan\xi+\mu} \tag{96}$$

and substituting (95) in (96),

$$z=\frac{1}{\lambda\frac{\rho}{z}+\mu} \tag{97}$$

and solving for z

$$z=-\lambda\rho+\frac{1}{\mu} \tag{98}$$

which is the equation of a straight line (z as a function of ρ) as required to be verified.
The plot appears in Figure 10.

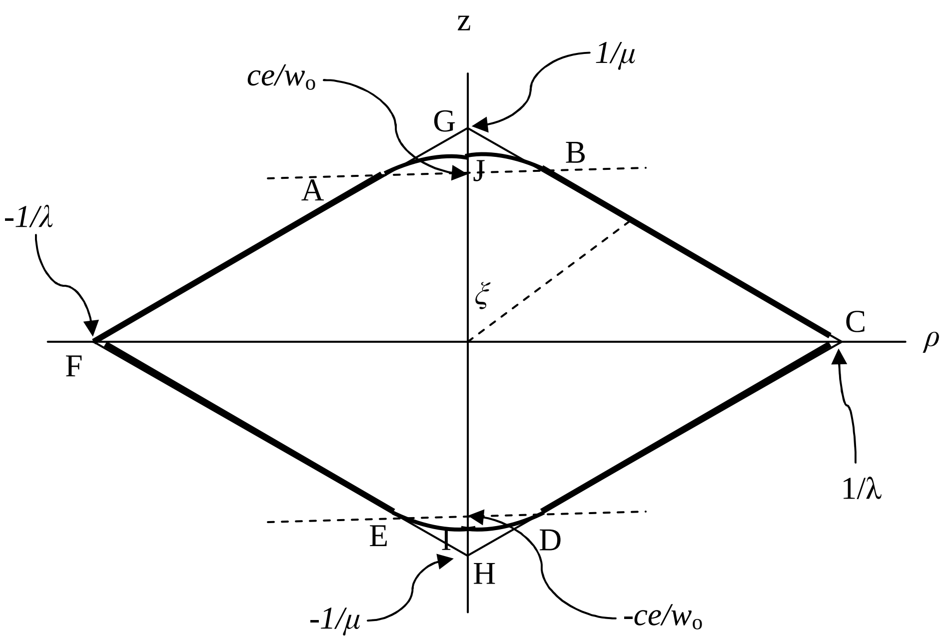


Figure 10 The diagram in bold lines, AJBCDIEF, describes the locus of points where $\cos\varphi$ is minimum. The limits ED and AB have been drawn so that $\frac{ce}{w_0} < \frac{1}{\mu}$. In the opposite case the diagram is given by the rhombus GCHF. The segments AJB and EID have been drawn arbitrarily as they are the result of the condition $c^2 - e^{-2(\lambda\rho+\mu z)} w_0^2 z^2 \ge 0$

The diagram ABCDEF by revolution around the z axis is the locus of points where $\cos\varphi$ is minimum. The cut off lines AB and ED are drawn assuming that $\frac{ce}{w_0} < \frac{1}{\mu}$. If this is not true, then the locus is given by the rhombus by revolution around the z-axis, GCHF. A signal starting at the origin of the axis with inclination angle ξ will move helically around the z axis in the z direction with increasing radius. By the time it crosses the above described locus, $\cos\varphi$ will have become minimum meaning that the signal points sideways with the maximum angle and thus advance slower in the radial direction and circle around in tighter circles. After that, $\cos\varphi$ increases and the direction of the signal returns asymptotically back to the radial direction. When $\frac{ce}{w_0} < \frac{1}{\mu}$, the segment AJB and EID will be given by the condition $c^2 - e^{-2(\lambda\rho+\mu z)} w_0^2 z^2 \ge 0$ For those signals that cross AJB or EID, they have $\cos\varphi = 0$ at the time of crossing. After that slippage will decrease the angle of deflection returning it gradually to zero. In any case it is only the radial direction that is hindered to advance due to the increase in the angle of deflection, while the z direction does not experience such hindrance.

B For Observer O''

Observer O'' is the far away observer that is not influenced by the rotation. For him the space is flat and the perimeter he sees is contracted by the relativistic factor $\sqrt{1-\frac{w^2\rho''^2}{c^2}}$ at any z. That is

$$2\pi\rho'' = 2\pi\rho\sqrt{1-\frac{w^2\rho''^2}{c^2}} \tag{99}$$

or solving ,

$$\rho'' = \frac{\rho c}{\sqrt{c^2 + w^2\rho^2}} = \frac{\rho c}{\sqrt{c^2 + w_0^2\rho^2 e^{-2(\lambda\rho+\mu z)}}} = \frac{ct\sin\xi}{\sqrt{1+w_0^2 t^2 \sin^2\xi e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}}} \tag{100}$$

We may denote the angle of deflection in the horizontal plane as

$$\cos\varphi'' = \frac{1}{\sqrt{1+w_0^2 t^2 \sin^2\xi e^{-2ct(\lambda\sin\xi+\mu\cos\xi)}}} = \frac{c}{\sqrt{c^2 + w_0^2\rho^2 e^{-2(\lambda\rho+\mu z)}}} \tag{101}$$

(which corresponds to $\cos\varphi'' = \cos\varphi'\rbrack_{z=0}$)

and then (100) becomes

$$\rho'' = \rho\cos\varphi'' \tag{102}$$

Taking $\frac{d\rho''}{d\rho}$ we see that it is positive

$$\frac{d\rho''}{d\rho}=c\frac{c^2+\lambda\rho^3w_0^2e^{-2(\lambda\rho+\mu z)}}{(c^2+\rho^2w_0^2e^{-2(\lambda\rho+\mu z)})^{\frac{3}{2}}} \tag{103}$$

and hence $\lim_{\rho\to\infty}\rho''=\infty$ unless $\lambda=0$ in which case $\lim_{\rho\to\infty}\rho''=\frac{c}{w_0}$, which agrees with the result for the no slippage case. Further, for t big, $\cos\varphi''\to 1$ and because of (102) $\rho''\to\rho$.

Looking at the derivative of $\cos\varphi''$

$$\frac{d}{dt}\cos\varphi''=-\frac{w_0^2te^{-2ct(\lambda\sin\xi+\mu\cos\xi)}\cos^2\xi(1-ct(\lambda\sin\xi+\mu\cos\xi))}{(1+w_0^2t^2\sin^2\xi e^{-2ct(\lambda\sin\xi+\mu\cos\xi)})^{\frac{3}{2}}} \tag{104}$$

It starts at $t=0$ with value equal to 0 and immediately becomes negative. Then increases to change sign at the minimum of $\cos\varphi''$ at,

$$t_{\min}=\frac{1}{c(\lambda\sin\xi+\mu\cos\xi)} \tag{105}$$

Hence

$$\rho_{\min}=ct_{\min}\sin\xi=\frac{\sin\xi}{\lambda\sin\xi+\mu\cos\xi} \tag{106}$$

The value of $\cos\varphi''$ at the minimum is

$$\cos\varphi''_{\min}=\frac{1}{\sqrt{1+\frac{w_0^2e^{-2}\sin^2\xi}{(\lambda\sin\xi+\mu\cos\xi)^2}}} \tag{107}$$

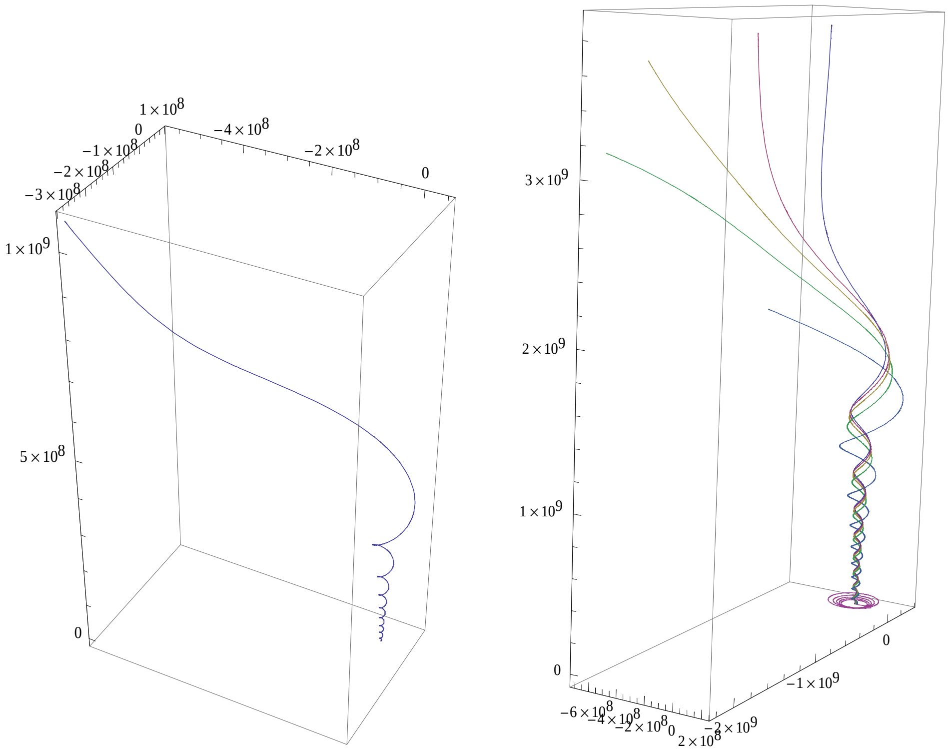
And using (102) the value of ρ'' at the min is

$$\rho''_{\min}=\rho_{\min}\cos\varphi''_{\min}=\frac{\sin\xi}{(\lambda\sin\xi+\mu\cos\xi)}\frac{1}{\sqrt{1+\frac{w_0^2e^{-2}\sin^2\xi}{(\lambda\sin\xi+\mu\cos\xi)^2}}} \tag{108}$$

while for z'' at the minimum we have

$$z''_{\min}=z_{\min}=ct_{\min}\cos\xi=\frac{\cos\xi}{\lambda\sin\xi+\mu\cos\xi} \tag{109}$$

The plot of the signal as it is given by (100), while z advances according to $z=ct\cos\xi$ appears in Figure 11



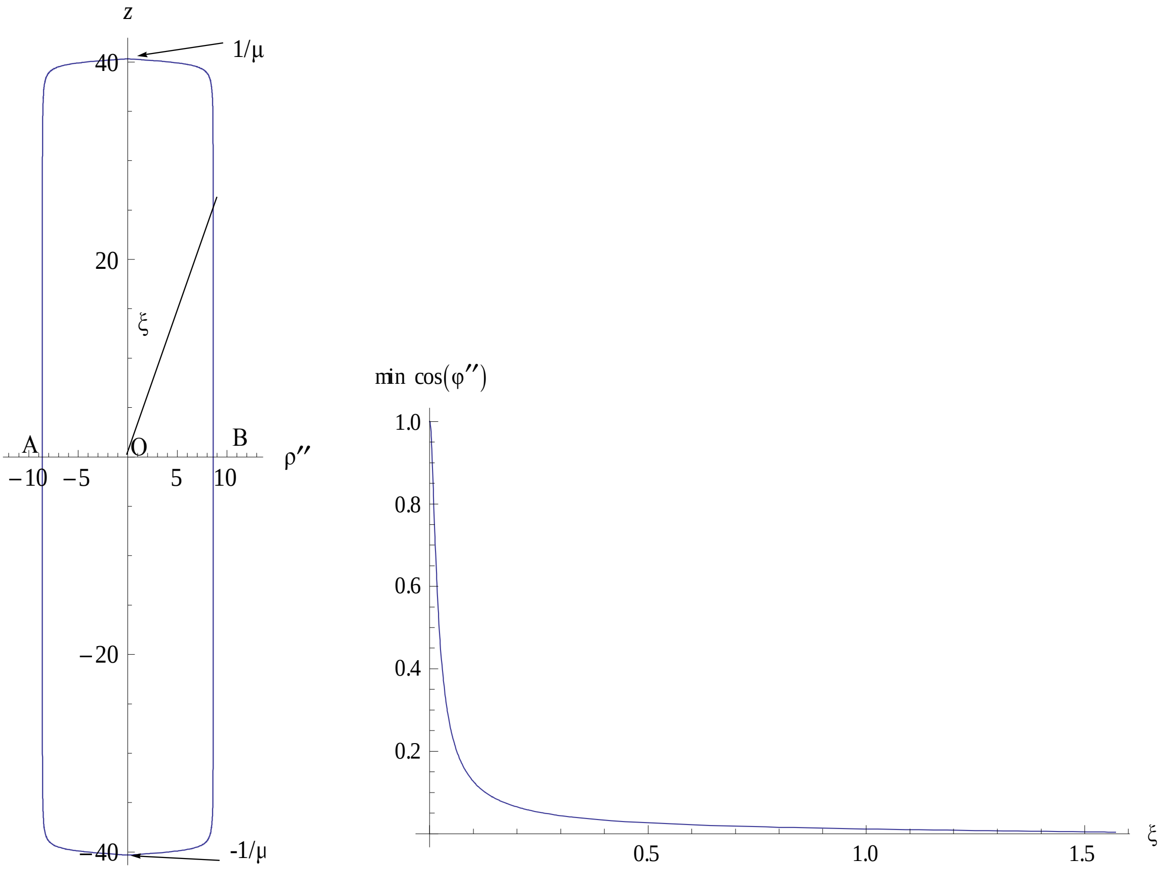
(a) `**(b)**

Figure 11 The signal path as it advances in time upward in the z direction, while rotating at increasing in time radial distance ρ''. **(a)** A single signal path. **(b)** Many signal paths for the same time interval with different ξ. The signals paths towards the positive z semi axis only are drawn. To complete the picture one must imagine the same jet of signals towards the negative z direction.

We see how most of the signal except those for which $\xi \approx 90^0$ travel tight to each other in the z direction until they break up to return asymptotically to the radial direction. The jet like formation is symmetric with respect to the plane of rotation and another jet emanates towards the negative z direction. To draw Figure 11 one must note that the angle θ that the rotating signal traverse (see Figure 1) is given by,

$$\theta = \int_0^t w_0 e^{-ct(\lambda \sin\xi + \mu \cos\xi)} dt = \frac{w_0}{c(\lambda \sin\xi + \mu \cos\xi)}\left(1 - e^{-ct(\lambda \sin\xi + \mu \cos\xi)}\right) \qquad (110)$$

We may also plot the line or locus of points ($\rho''_{\min}$, $z_{\min}$), where $\cos\varphi''$ attains its minimum as a function of ξ. This plot appears in Figure 12(a) and it forms a cigar like locus but it may be wider close to a rectangle with rounded corners depending on the values of λ and μ.The width of the locus decreases with increasing w_0. Observe that $\text{OA}=\text{OB} = \dfrac{1}{\sqrt{\lambda^2 + w_0^2 e^{-2}}}$ and that the locus crosses the z axis at $\dfrac{1}{\mu}$ and at $-\dfrac{1}{\mu}$. The value of $\cos\varphi''$ (Figure 12(b)), which measures the deflection from the radial direction is 1 for $\xi = 0$. This says that for the locus points that are close to the z axis the signals have no deflection. But as ξ increases to $\dfrac{\pi}{2}$, $\cos\varphi''$ rapidly decreases to zero thus making a 90° degrees angle to the radial direction. The deflection of the signals at the locus of maximum deflection of φ'', forms a sideways "barrier" for the signals. After that the signals gradually regain their radial direction. This “barrier is the reason that signals are kept together in a jet like formation as in Figure 11(b).



(a) **(b)**

Figure 12 In **(a)** is the locus of points (ρ'', z), where $\cos\varphi''$ is minimum (parameter ξ varies from 0 to $\pi/2$). The width narrows dramatically as w increases. The shape and roundness also varies with parameters λ and μ. For $\xi = \frac{\pi}{2}$ the distance $\mathrm{OA{=}OB} = \frac{1}{\sqrt{\lambda^2 + w_0^2 e^{-2}}}$. In **(b)** the value of $\cos\varphi''$ at the previous locus points is shown as a function of ξ. The value which starts at 1 when $\xi = 0$ drops very fast to almost zero and remains there. This means that at the locus points described by **(a)** form a sideways deflection “barrier” except for points close to the z axis.

11. Conclusion

Starting from the assumption that two observers rotating with respect to the other around a common axis will agree on epicenter angles as fractions of a circle but not necessarily on the value of π, we find the length of the radius of a rotating disc as seen by the non rotating observer. The radius will be contracted but not with the same factor as the perimeter because we allow the value of π to change for the non rotating observer with regards to his measurements on the rotating disc. We argued that we have to consider two types of non rotating observers. One within the radius *c/w* and one outside. For the non rotating observer within *c/w*, a light signal starting radially from the origin of the rotating disc (frame) that rotates with the frame will gradually turn sideways forming tighter circles until it asymptotically reaches 90º degrees deflection from the radial as the radius tends to

infinity. His space is distorted and π is different. For the outside observer light rays behave as above but they are confined to the radius *c/w*, while his space remains Euclidean. In three dimensions the light rays follow helical paths and we have a cylinder of radius *c/w* to distinguish between observers.
If we allow rotation with slippage (rotation is not uniform but decreases exponentially as the radius increases), the space is not divided by a cylinder of radius *c/w*, but still there is contraction of the radial distances. The light rays originating from the origin with radial direction will again turn sideways until they reach a maximum deflection from the radial direction and then asymptotically turn back to the radial. The locus of points, where the maximum deflection occurs is determined. For an observer within this locus it (the locus) looks like rhombus by revolution, while for an observer far away it looks like a cigar.
Although more space in this paper was devoted to the no slippage case the physical importance of the slippage case is perhaps bigger, since it is related to the notion of frame dragging and does not impose a horizon.
For the case of observer O'', it is worth noting the jet like formations in the direction of the positive and negative z axis of the signals that emanate from the origin of the rotating frame.

12. References

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