Quaternions and Hilbert spaces

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Abstract

This is a compilation of quaternionic number systems, quaternionic function theory, quaternionic Hilbert spaces and Gelfand triples.

Contents

1 Introduction

It is not generally known that separable Hilbert spaces can only handle number systems that form division rings. This was inescapably proven by Maria Pia Solèr in the sixties of the last century.

Only three suitable division rings exist: the real numbers, the complex numbers and the quaternions. The first two are contained in the last one. Thus the most elaborate separable Hilbert space is a quaternionic Hilbert space.

See: "Division algebras and quantum theory" by John Baez.http://arxiv.org/abs/1101.5690

According to my experience hardly any scientist knows that quaternionic number systems, and continuous quaternionic functions exist in 16 versions that only differ in their discrete symmetry.

Also most scientist do not notice what separable stands for. It means that eigenspaces of operators can only contain a countable number of eigenvalues. For example operators whose eigenspaces contain all rational numbers may exist, but operators whose eigenspaces contain all (or a closed set of) real numbers can only exist in a non-separable Hilbert space, such as a Gelfand triple.

By the way, each infinite dimensional separable Hilbert space owns a Gelfand triple.

Great resemblance exist between Maxwell-like equations and quaternionic differential equations. However, also significant differences exist. This paper indicates what these differences are.

2 Quaternion geometry and arithmetic

Quaternions and quaternionic functions offer the advantage of a very compact notation of items that belong together.

Quaternions can be considered as the combination of a real scalar and a 3D vector that has real coefficients. This vector forms the imaginary part of the quaternion. Quaternionic number systems are division rings. Other division rings are real numbers and complex numbers. Octonions do not form a division ring.

Bi-quaternions exist whose parts exist of a complex scalar and a 3D vector that has complex coefficients. Bi-quaternions do not form division rings. This paper does not use them.

2.1 Notation

We indicate the real part of quaternion a by the suffix a_0.

We indicate the imaginary part of quaternion a by bold face $\boldsymbol{a}$.

$$a = a_0 + \boldsymbol{a} \quad (1)$$

2.2 Sum

$$c = c_0 + \boldsymbol{c} = a + b \quad (1)$$

$$c_0 = a_0 + b_0 \quad (2)$$

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} \tag{3}$$

2.3 Product

$$f = f_0 + \boldsymbol{f} = d\, e \tag{1}$$

$$f_0 = d_0 e_0 - \langle \boldsymbol{d}, \boldsymbol{e} \rangle \tag{2}$$

$$\boldsymbol{f} = d_0 \boldsymbol{e} + e_0 \boldsymbol{d} \pm \boldsymbol{d} \times \boldsymbol{e} \tag{3}$$

The $\pm$ sign indicates the influence of right or left handedness of the number system[1].

$\langle \boldsymbol{d}, \boldsymbol{e} \rangle$ is the inner product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

$\boldsymbol{d} \times \boldsymbol{e}$ is the outer product of $\boldsymbol{d}$ and $\boldsymbol{e}$.

2.4 Norm

$$|a| = \sqrt{a_0 a_0 + \langle \boldsymbol{a}, \boldsymbol{a} \rangle} \tag{1}$$

2.5 Rotation

Quaternions are often used to represent rotations.

$$c = ab/a \tag{1}$$

rotates the imaginary part of b that is perpendicular to the imaginary part of a[2].

[1] Quaternionic number systems exist in 16 symmetry flavors. Within a coherent set all elements belong to the same symmetry flavor.

[2] See Q-FORMULÆ

3 The separable Hilbert space Ң

We will specify the characteristics of a generalized quaternionic infinite dimensional separable Hilbert space. The adjective “quaternionic” indicates that the inner products of vectors and the eigenvalues of operators are taken from the number system of the quaternions. Separable Hilbert spaces can be using real numbers, complex numbers or quaternions. These three number systems are division rings.

3.1 Notations and naming conventions

$\{f_x\}_x$ means ordered set of f_x . It is a way to define functions.

The use of bras and kets differs slightly from the way Dirac uses them.

|f> is a ket vector, f> is the same ket

<f| is a bra vector, <f is the same bra

A is an operator.

|A is the same operator

A† is the adjoint operator of operator A.

A| is the same operator as A†

| on its own, is a nil operator

|A| is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$ means: sum over all items with index k.

$\int_x$ means: integral over all items with parameter x.

3.2 Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements |f>, |g> and |h> and numbers *a* and *b*:

3.2.1 Ket vectors

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \tag{1}$$

$$(|f> + |g>) + |h> = |f> + (|g> + |h>) \qquad (2)$$

$$|(a + b)\, f > = |f>\cdot a + |f>\cdot b \qquad (3)$$

$$(|f> + |g>)\cdot a = |f>\cdot a + |g>\cdot a \qquad (4)$$

$$|f>\cdot 0 = |0> \qquad (5)$$

$$|f>\cdot 1 = |f> \qquad (6)$$

Depending on the number field that the Hilbert space supports, *a* and *b* can be real numbers, complex numbers or (real) quaternions.

3.2.2 Bra vectors

The **bra** vectors form the dual Hilbert space **H**† of **H** .

$$<f| + <g| = <g| + <f| = |g + f> \qquad (1)$$

$$(<f| + <g|) + <h| = <f| + (<g| + <h|) \qquad (2)$$

$$<f\,(a + b)> = <f|\cdot a + <f|\cdot b = a^*\cdot <f| + b^*\cdot <f| \qquad (3)$$

$$(<f| + <g|)\cdot a = <f|\cdot a + <g|\cdot a = a^*\cdot <f| + a^*\cdot <g| \qquad (4)$$

$$0\cdot <f| = <0| \qquad (5)$$

$$1\cdot <f| = <f| \qquad (6)$$

3.2.3 Scalar product

The Hilbert space contains a **scalar product,** also called **inner product,** $<f|g>$ that combines **H** and **H**† in a direct product that we also indicate with **H**.

For Hilbert spaces the values of inner products are restricted to elements of a division ring.

The scalar product $<f|g>$ satisfies:

$$<f|g + h> = <f|g> + <f|h> \qquad (1)$$

$$<f|\{|g>\cdot a\}_g = \{<f|g>\}_g\cdot a \qquad (2)$$

With each ket vector $|g>$ in **H** belongs a bra vector $<g|$ in $\mathbf{H}^\dagger$ such that for all bra vectors $<f|$ in $\mathbf{H}^\dagger$

$$<f|g> = <g|f>^* \qquad (3)$$

$$<f|f> = 0 \text{ when } |f> = |0> \qquad (4)$$

$$<f|a\ g> = <f|g>\cdot a = <g|f>^*\cdot a = <g\ a|f>^* = (a^*\cdot <g|f>)^* = <f|g>\cdot a \qquad (5)$$

In general is $<f|a\ g> \neq <f\ a|g>$. However for real numbers r holds $<f|r\ g> = <f\ r|g>$

Remember that when the number field consists of quaternions, then also $<f|g>$ is a quaternion and a quaternion q and $<f|g>$ do in general not commute.

The scalar product defines a **norm**:

$$||f|| = \sqrt{(<f|f>)} \qquad (6)$$

And a **distance**:

$$D(f,g) = ||f - g|| \qquad (7)$$

The Hilbert space **H** is closed under its norm. Each converging row of elements of converges to an element of this space.

3.2.4 Separable

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space **H** is determined by its values on this countable dense subset.

3.2.5 Base vectors

The Hilbert space **H** is **separable**. That means that a countable row of elements $\{f_n>\}$ exists that **spans** the whole space.

If $<f_n|f_m> = \delta(m,n) = [1 \text{ when } n = m; 0 \text{ otherwise}]$
then $\{|f_n>\}$ forms an **orthonormal base** of the Hilbert space.

A ket base $\{|k>\}$of **H** is a minimal set of ket vectors $|k>$ that together span the Hilbert space **H**.

Any ket vector $|f>$ in **H** can be written as a linear combination of elements of $\{|k>\}$.

$$|f> = \sum_k (|k>\cdot<k|f>) \qquad (1)$$

A bra base $\{<b|\}$of $\mathbf{H}^\dagger$ is a minimal set of bra vectors $<b|$ that together span the Hilbert space $\mathbf{H}^\dagger$.

Any bra vector $<f|$ in $\mathbf{H}^\dagger$ can be written as a linear combination of elements of $\{<b|\}$.

$$<f| = \sum_b (<f|b>\cdot<b|) \qquad (2)$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an othonormal base.

3.2.6 Operators

Operators act on a subset of the elements of the Hilbert space.

3.2.6.1 Linear operators

An operator Q is linear when for all vectors $|f>$ and $|g>$ for which Q is defined and for all quaternionic numbers a and b:

$$|Q\cdot a\ f> + |Q\cdot b\ g> = |a\cdot Q\ f> + |b\cdot Q\ g> = |Q\ f>\cdot a + |Q\ g>\cdot b = \qquad (1)$$

$$Q\ (|f>\cdot a + |g>\cdot b) = Q\ (|a\ f> + |b\ g>) \qquad (2)$$

B is **colinear** when for all vectors $|f>$ for which B is defined and for all quaternionic numbers *a* there exists a quaternionic number *c* such that:

$$|B\cdot a\ f\rangle = |a\cdot B\ f\rangle = |B\ f\rangle\ c\cdot a\cdot c^{-1} \quad (3)$$

If |f> is an eigenvector of operator A with quaternionic eigenvalue *a*, then is |*b* f> an eigenvector of A with quaternionic eigenvalue $b\cdot a\cdot b^{-1}$.

A| = $A^\dagger$ is the **adjoint** of the **normal** operator A. |A is the same as A.

$$\langle f\ A|\ g\rangle = \langle fA^\dagger|g\rangle^* \quad (4)$$

$$A^{\dagger\dagger} = A \quad (5)$$

$$(A\cdot B)^\dagger = B^\dagger\cdot A^\dagger \quad (6)$$

|B| is a **self adjoint** operator.

| is a nil operator.

The construct |f><g| acts as a linear operator. |g><f| is its adjoint operator.

$$\sum_n \{|f_n\rangle\cdot a_n\cdot\langle f_n|\}, \quad (7)$$

where a n is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

$\{\langle q|f\rangle\}_q$ is a function f(q) of parameter *q*.

$\{\langle g|q\rangle\}_q$ is a function g(q) of parameter *q*.

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.

So, usually |q> is an eigenvector of a normal operator Q with eigenvalues *q*.

{q} is the set of eigenvalues of Q.

$\{q\}_q$ is the ordered field of eigenvalues of *q*.

{|q>}$_q$ is the ordered set of eigenvectors of Q.
{<q|f>}$_q$ is the **Q view** of |f>.

3.2.6.2 Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

If |a> is an eigenvector of normal operator A with eigenvalue *a* then

$$< a|A|a > = < a|a|a > = < a|a > \ a$$

indicates that the eigenvalues are taken from the same number system as the inner products.

A normal operator is a continuous linear operator.

A normal operator in **H** creates an image of **H** onto **H**. It transfers closed subspaces of **H** into closed subspaces of **H**.

Normal operators represent continuous quantum logical observables.

The normal operators N have the following property.

$$\text{N: } \mathbf{H} \Rightarrow \mathbf{H} \tag{1}$$

N commutes with its **(Hermitian) adjoint** $N^\dagger$

$$N \cdot N^\dagger = N^\dagger \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators: $U^\dagger = U^{-1}$, unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators): $N^\dagger = N$;
- **Anti-Hermitian** or anti-self-adjoint operators: $N^\dagger = -N$;

- **Anti-unitary** operators: $I^\dagger = -I = I^{-1}$, anti-unitary operators are bounded;
- **positive operators**: $N = MM^\dagger$
- **orthogonal projection** operators: $N = N^\dagger = N^2$

3.2.6.3 Spectral theorem

For every compact self-adjoint operator *T* on a real, complex or quaternionic Hilbert space **H**, there exists an orthonormal basis of **H** consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel (null space) of T admits, either a finite orthonormal basis of eigenvectors of T, or a countable infinite orthonormal basis {en} of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\} \subset R$, such that $\lambda_n \rightarrow 0$. Due to the fact that **H** is separable the set of eigenvectors of T can be extended with a base of the kernel in order to form a complete orthonormal base of **H**.

If *T* is compact on an infinite dimensional Hilbert space **H**, then *T* is not invertible, hence $\sigma(T)$, the spectrum of *T*, always contains 0. The spectral theorem shows that $\sigma(T)$ consists of the eigenvalues $\{\lambda_n\}$ of *T*, and of 0 (if 0 is not already an eigenvalue). The set $\sigma(T)$ is a compact subset of the real line, and the eigenvalues are dense in $\sigma(T)$.

A normal operator has a set of eigenvectors that spans the whole Hilbert space **H**.

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that **H** is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, in general the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on **H** has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

3.2.6.4 Eigenspace

The set of eigenvalues $\{q\}$ of the operator Q form the eigenspace of Q

3.2.6.5 Eigenvectors and eigenvalues

For the eigenvector |q> of normal operator Q holds

$$|Q\, q> = |q\, q> = |q>\cdot q \tag{1}$$

$$<q\, Q^\dagger| = <q\, q^*| = q^*\cdot<q| \tag{2}$$

$$\forall_{|f> \in H} [\{< f|Q\, q >\}_q = \{< f|q > q\}_q = \{< q\, Q^\dagger|f >^*\}_q = \{q^* < q|f >^*\}_q] \tag{3}$$

The eigenvalues of 2^n-on normal operator are 2^n-ons. For Hilbert spaces the eigenvalues are restricted to elements of a division ring.

$$Q = \sum_{j=0}^{n-1} \mathrm{I}_j Q_i \tag{4}$$

The Q_j are self-adjoint operators.

3.2.6.6 Generalized Trotter formula

For bounded operators $\{A_j\}$ hold:

$$\lim_{n\to\infty}\left(\prod_{j=1}^{p} e^{A_j/n}\right)^n = \exp\left(\sum_{j=1}^{p} A_j\right) = \lim_{n\to\infty}\left(1+\frac{\sum_{j=1}^{p} A_j}{n}\right)^n \tag{1}$$

In general

$$\exp\left(\sum_{j=1}^{p} A_j\right) \neq \prod_{j=1}^{p} e^{A_j} \tag{2}$$

In the realm of quaternionic notion the Trotter formula is confusing.

3.2.6.7 Unitary operators

For unitary operators holds:

$$U^{\dagger} = U^{-1} \tag{1}$$

Thus

$$U\cdot U^{\dagger} = U^{\dagger}\cdot U = 1 \tag{2}$$

Suppose *U* = *I* + *C* where *U* is unitary and *C* is compact. The equations *U U** = *U*U* = *I* and *C* = *U* − *I* show that *C* is normal. The spectrum of *C* contains 0, and possibly, a finite set or a sequence tending to 0. Since *U* = *I* + *C*, the spectrum of *U* is obtained by shifting the spectrum of *C* by 1.

The unitary transform can be expressed as:

$$U = \exp(\tilde{\underline{I}}\cdot\Phi/\hbar) \tag{3}$$

$$\hbar = h/(2\cdot\pi) \tag{4}$$

Φ is Hermitian. The constant h refers to the granularity of the eigenspace.

Unitary operators have eigenvalues that are located in the unity sphere of the 2^n-ons field.

The eigenvalues have the form:

$$u = \exp(\underline{\mathbf{i}}\cdot\phi/\hbar) \tag{5}$$

ϕ is real. $\underline{\mathbf{i}}$ is a unit length imaginary number in 2^n-on space. It represents a direction.

u spans a sphere in 2^n-on space. For constant $\underline{\mathbf{i}}$, u spans a circle in a complex subspace.

3.2.6.7.1 ***Polar decomposition***

Normal operators N can be split into a real operator A and a unitary operator U. U and A have the same set of eigenvectors as N.

$$N = ||N||\cdot U = A\cdot U \tag{1}$$

$$N = A\cdot U = U\cdot A \tag{2}$$

$$= A\cdot \exp(\underline{\tilde{\mathbf{I}}}\cdot\Phi)/\hbar)$$

$$= \exp\,(\Phi_r + \underline{\tilde{\mathbf{I}}}\cdot\Phi)/\hbar)$$

Φ_r is a positive normal operator.

3.2.6.8 Ladder operator

3.2.6.8.1 General formulation

Suppose that two operators X and N have the commutation relation:

$$[N, X] = c\cdot X \tag{1}$$

for some scalar *c*. If |n> is an eigenstate of *N* with eigenvalue equation,

$$|N\, n> = |n>\cdot n \tag{2}$$

then the operator *X* acts on |n> in such a way as to shift the eigenvalue by *c*:

$$|N\cdot X\, n> = |(X\cdot N + [N, X])\, n> \tag{3}$$

$$= |(X\cdot N + c\cdot X)\, n>$$

$$= |X\cdot N\, n> + |X\, n>\cdot c$$

= |*X* n>·n + |*X* n>·c

= |*X* n>·(n+c)

In other words, if |n> is an eigenstate of *N* with eigenvalue *n* then |*X* n> is an eigenstate of *N* with eigenvalue *n* + *c*.

The operator *X* is a *raising operator* for *N* if *c* is real and positive, and a *lowering operator* for *N* if *c* is real and negative.

If *N* is a Hermitian operator then *c* must be real and the Hermitian adjoint of *X* obeys the commutation relation:

$$[N, X^{\dagger}] = - c \cdot X^{\dagger} \quad (4)$$

In particular, if *X* is a lowering operator for *N* then $X^{\dagger}$ is a raising operator for *N* and vice-versa.

3.2.7 Unit sphere of Ⱨ

The ket vectors in **Ⱨ** that have their norm equal to one form together the **unit sphere** **Θ** of **Ⱨ**.

Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere **Θ**of **Ⱨ**.

3.2.8 Bra-ket in four dimensional space

The Bra-ket formulation can also be used in transformations of the four dimensional curved spaces.

The bra $\langle f$ is then a covariant vector and the ket $g\rangle$ is a contra-variant vector. The inner product acts as a metric.

$$s = \langle f|g\rangle \quad (1)$$

The effect of a linear transformation L is then given by

$$s_L = \langle f|Lg\rangle \quad (2)$$

The effect of a the transpose transformation $L^{\dagger}$ is then given by

$$\langle f L^{\dagger} |g\rangle = \langle f|Lg\rangle \quad (3)$$

For a unitary transformation U holds:

$$\langle Uf|Ug\rangle = \langle f|g\rangle \quad (4)$$

These definitions work for curved spaces with a Euclidian signature as well as for curved spaces with a Minkowski signature.

$$\langle \nabla f|\nabla g\rangle = \langle f|\nabla^2 \mathrm{g}\rangle = \langle f|\Box \mathrm{g}\rangle \quad (5)$$

3.2.9 Closure

The closure of **Ⱨ** means that converging rows of vectors converge to a vector of **Ⱨ**.

In general converging rows of eigenvalues of Q do not converge to an eigenvalue of Q.

Thus, the set of eigenvalues of Q is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of 2^n-ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of Q has holes.

The set of eigenvalues of operator Q includes 0. This means that Q does not have an inverse.

The rigged Hilbert space **Ħ** can offer a solution, but then the direct relation with quantum logic is lost.

3.2.10 Canonical conjugate operator P

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

Q has eigenvectors $\{|q>\}_q$ and eigenvalues *q*.

P has eigenvectors $\{|p>\}_p$ and eigenvalues *p*.

For each eigenvector |q> of Q we define an eigenvector |p> and eigenvalues *p* of P such that:

$$< q|p > = < p|q >^* = \; exp\,(\hat{\imath} \cdot p \cdot q/\hbar) \tag{1}$$

$\hbar \;=\; h/(2\pi)$ is a scaling factor. $< q|p >$ is a quaternion. **î** is a unit length imaginary quaternion.

3.2.11 Displacement generators

Variance of the scalar product gives:

$$\boldsymbol{i}\,\hbar\,\delta < q|p > = \; -p < q|p > \delta q \tag{1}$$

$$\boldsymbol{i}\,\hbar\,\delta < p|q > = \; -q < p|q > \delta p \tag{2}$$

In the rigged Hilbert space **Ħ** the variance can be replaced by differentiation.

Partial differentiation of the function <q|p> gives:

$$\boldsymbol{i}\,\hbar\,\partial/\partial q_s < q|p > = \; -p_s < q|p > \tag{3}$$

$$\boldsymbol{i}\,\hbar\frac{\partial}{\partial p_s} < p|q > = \; -q_s < p|q > \tag{4}$$

3.3 Quaternionic L² space

The space of quaternionic measurable functions is a separable quaternionic Hilbert space. For example quaternionic probability density distributions are measurable.[3]

This space is spanned by an orthonormal basis of quaternionic measurable functions. The shared affine-like versions of the parameter space of these functions is called **Palestra**[4]. When the Palestra is non-curved, then this base has a canonical conjugate, which is the quaternionic Fourier transform of the original base.

As soon as curvature of the Palestra arises, this relation is disturbed.

With other words: "In advance the Palestra has a virgin state."

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[3] http://en.wikipedia.org/wiki/Lp_space#Lp_spaces

[4] The name Palestra is suggested by Henning Dekant's wife Sarah. It is a name from Greek antiquity. It is a public place for training or exercise in wrestling or athletics

4 Gelfand triple

The separable Hilbert space only supports countable orthonormal bases and countable eigenspaces. The rigged Hilbert space Ħ that belongs to a separable Hilbert space Ң is a Gelfand triple. It supports non-countable orthonormal bases and continuum eigenspaces.

A rigged Hilbert space is a pair (Ң, Φ) with **Ң** a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map *i* is continuous. Its name is not correct, because it is not a Hilbert space.

Identifying **Ң** with its dual space **Ң***, the adjoint to *i* is the map

$$i^*: Ң = Ң^* \to \Phi^* \qquad (1)$$

The duality pairing between Φ and Φ^* has to be compatible with the inner product on **Ң**, in the sense that:

$$\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_{Ң} \qquad (2)$$

whenever $u \in \Phi \subset Ң$ and $v \in Ң = Ң^* \subset \Phi^*$.

The specific triple ($\Phi \subset Ң \subset \Phi^*$) is often named after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion *i* with its adjoint *i**

$$i^* i: \Phi \subset Ң = Ң^* \to \Phi^* \qquad (3)$$

4.1 Understanding the Gelfand triple

The Gelfand triple of a real separable Hilbert space can be understood via the enumeration model of the real separable Hilbert space. This enumeration is obtained by taking the set of eigenvectors of a normal operator that has rational numbers as its eigenvalues. Let the smallest enumeration value of the rational enumerators approach zero. Even when zero is reached, then still the set of enumerators is countable. Now add all limits of converging rows of rational enumerators to the enumeration set. After this operation the enumeration set has become a continuum and has the same cardinality as the set of the real numbers. This operation converts the Hilbert space into its Gelfand triple and it converts the normal operator in a new operator that has the real numbers as its eigenspace. It means that the orthonormal base of the Gelfand triple that is formed by the eigenvectors of the new normal operator has the cardinality of the real numbers. It also means that linear operators in this Gelfand triple have eigenspaces that are continuums and have the cardinality of the real numbers[5].

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[5] This story also applies to the complex and the quaternionic Hilbert spaces and their Gelfand triples.

The same reasoning holds for complex number based Hilbert spaces and quaternionic Hilbert spaces and their respective Gelfand triples.

5 Functions as Hilbert space operators

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits. By using bra-ket notation, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined starting from an orthogonal base of vectors.

Let $\{q_i\}$ be the set of rational quaternions and $\{|q_i\rangle\}$ be the set of corresponding base vectors.

$|q_i\rangle q_i \langle q_i|$ is the configuration parameter space operator.

Let $f(q)$ be a quaternionic function.

$|q_i\rangle f(q_i) \langle q_i|$ defines a new operator that is based on function $f(q)$.

In the Gelfand triple, the continuous function $f(q)$ can be defined between a continuum eigenspace that acts as target space and the eigenspace of the reference operator $|q\rangle q \langle q|$ *that* acts as parameter space. $|q\rangle f(q) \langle q|$ defines a curved continuum.

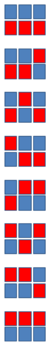
In the Gelfand triple the dimension of a subspace loses its significance. Thus a function that is derived from the representation of a coherent swarm in Hilbert space has a dimension in Hilbert space, but loses that characteristic in its representation in the Gelfand triple.

6 Quaternionic function symmetry flavors

Another fact that hardly anyone knows is that quaternionic number systems, coherent sets of quaternionic numbers and continuous quaternionic functions exits in 16 versions that only differ in their discrete symmetry sets. This is due to the four dimensions of quaternions. For example quaternionic number systems exist in left handed and right handed versions.

Quaternions can be mapped to Cartesian coordinates along the orthonormal base vectors 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; with $\boldsymbol{ij} = \boldsymbol{k}$

Continuous quaternionic functions do not switch to other symmetry flavors.

• If the real part is ignored, then still ***8*** symmetry flavors result • They are marked by special indices, for example $\boldsymbol{\psi}^{④}$ • $\boldsymbol{\psi}^{⓪}$is the reference symmetry flavor • They are also marked by colors $N, R, G, B, \bar{B}, \bar{G}, \bar{R}, \bar{N}$ • Half of them is right handed, **R** • The other half is left handed, **L**	
 $\boldsymbol{\psi}^{⓪} N$ ***R*** $\boldsymbol{\psi}^{①} R$ ***L*** $\boldsymbol{\psi}^{②} G$ ***L*** $\boldsymbol{\psi}^{③} B$ ***L*** $\boldsymbol{\psi}^{④} \bar{B}$ ***R*** $\boldsymbol{\psi}^{⑤} \bar{G}$ ***R*** $\boldsymbol{\psi}^{⑥} \bar{R}$ ***R*** $\boldsymbol{\psi}^{⑦} W$ ***L***	The colored rectangles reflect the directions of the axes

Also continuums feature a symmetry flavor. The reference symmetry flavor is the symmetry flavor of the parameter space of the function that describes the continuum. This parameter space is a flat continuum.

If the continuous quaternionic function describes the density distribution of a set of discrete objects, then this set can be attributed with the same symmetry flavor.

6.1.1 Symmetry flavor conversion tools

Quaternionic conjugation

$$(\psi^{x})^{*} = \psi^{(7-x)}; x = ⓪, ①, ②, ③, ④, ⑤, ⑥, ⑦$$

Via quaternionic rotation, the following normalized quaternions ϱ^{x} can shift the indices of symmetry flavors of coordinate mapped quaternions and for quaternionic functions:

$$\varrho^{①} = \frac{1+\boldsymbol{i}}{\sqrt{2}};\ \varrho^{②} = \frac{1+\boldsymbol{j}}{\sqrt{2}};\ \varrho^{③} = \frac{1+\boldsymbol{k}}{\sqrt{2}};\ \varrho^{④} = \frac{1-\boldsymbol{k}}{\sqrt{2}};\ \varrho^{⑤} = \frac{1-\boldsymbol{j}}{\sqrt{2}};\ \varrho^{⑥} = \frac{1-\boldsymbol{i}}{\sqrt{2}}$$

$$\boldsymbol{ij} = \boldsymbol{k};\ \ \boldsymbol{jk} = \boldsymbol{i};\ \ \boldsymbol{ki} = \boldsymbol{j}$$

$$\varrho^{⑥} = \left(\varrho^{①}\right)^*$$

For example

$$\psi^{③} = \varrho^{①}\psi^{②}/\varrho^{①}$$

$$\psi^{③}\varrho^{①} = \varrho^{①}\psi^{②}$$

$$\psi^{⓪} = \varrho^{x}\psi^{⓪}/\varrho^{x};\ \psi^{⑦} = \varrho^{x}\psi^{⑦}/\varrho^{x}$$

Also strings of symmetry flavor convertors change the index of symmetry flavor of the multiplied quaternion or quaternionic function. The convertors can act on each other.

For example:

$$\varrho^{①}\varrho^{②} = \varrho^{②}\varrho^{③} = \varrho^{③}\varrho^{①} = \frac{1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}}{2}$$

The result is an isotropic quaternion. This means:

$$\varrho^{①}\psi^{②}/\varrho^{x} = \varrho^{②}\psi^{③}/\varrho^{x} = \psi^{x}$$

7 Quaternionic functions

7.1 Norm

Square-integrable functions are normalizable. The norm is defined by:

$$\|\psi\|^2 = \int_V |\psi|^2\, dV \tag{1}$$

$$= \int_V \{|\psi_0|^2 + |\boldsymbol{\psi}|^2\}dV$$

$$= \|\psi_0\|^2 + \|\boldsymbol{\psi}\|^2$$

7.2 Differentiation

If g is differentiable then the quaternionic nabla ∇g of g exists.

The quaternionic nabla ∇ is a shorthand for $\nabla_0 + \boldsymbol{\nabla}$

$$\nabla_0 = \frac{\partial}{\partial\tau} \tag{3}$$

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \tag{4}$$

$$h = h_0 + \boldsymbol{h} = \nabla g \tag{4}$$

$$h_0 = \nabla_0 g_0 - \langle \boldsymbol{\nabla}, \boldsymbol{g} \rangle \tag{5}$$

$$\boldsymbol{h} = \nabla_0 \boldsymbol{g} + \boldsymbol{\nabla} g_0 \pm \boldsymbol{\nabla} \times \boldsymbol{g} \tag{6}$$

$$\phi = \nabla\psi \;\Rightarrow \phi^* = (\nabla\psi)^* \tag{7}$$

$$(\nabla\psi)^* = \nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle - \nabla_0 \boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0 \mp \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{8}$$

$$\nabla^*\psi^* = \nabla_0\psi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\psi} \rangle - \nabla_0 \boldsymbol{\psi} - \boldsymbol{\nabla}\psi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\psi} \tag{9}$$

Similarity of these equations with Maxwell equations is not accidental. In Maxwell equations several terms in the above equations have been given special names and special symbols. Similar equations occur in other branches of physics. Apart from these differential equations also integral equations exist.

7.2.1 Gauge transformation

For a function χ that obeys the ***quaternionic wave equation***[6]

$$\nabla^*\nabla\chi = \nabla_0\nabla_0\chi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\chi \rangle = 0 \tag{1}$$

[6] Be aware, this is the quaternionic wave equation. This is not the common form of the wave equation, which is complex number based.

the value of ϕ in

$$\phi = \nabla\psi \tag{2}$$

does not change after the gauge transformation[7]

$$\psi \to \psi + \xi = \psi + \nabla^*\chi \tag{3}$$

$$\nabla\xi = 0 \tag{4}$$

$$\chi = \chi_0 + \boldsymbol{\chi} \tag{5}$$

Thus in general:

$$\nabla^*\nabla\psi = \nabla_0\nabla_0\psi + \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\psi\rangle = \rho \neq 0 \qquad (6$$

ρ is a quaternionic function.

Its real part ρ_0 represents an object density distribution.

Its imaginary part $\boldsymbol{\rho} = \boldsymbol{v}\,\rho_0$ represents a current density distribution.

Equation (1) forms the basis of the generalized (quaternionic) Huygens principle[8].

$$\nabla^*\nabla\chi_0 = 0 \tag{7}$$

Equation (7) has 3D isotropic wave fronts as its solution. χ_0 is a scalar function. By changing to polar coordinates it can be deduced that a general solution is given by:

$$\chi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} \tag{8}$$

[7] The qualification gauge transformation is usually given to a transformation that leaves the Laplacian untouched. Here we use that qualification for transformations that leave the quaternionic differential untouched.

[8] The papers on Huygens principle use the complex number based wave equation, which differs from the quaternionic wave equation.

Where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

$$\nabla^* \nabla \boldsymbol{\chi} = 0 \tag{9}$$

Here $\boldsymbol{\chi}$ is a vector function.

Equation (9) has one dimensional wave fronts as solutions:

$$\boldsymbol{\chi}(z, \tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{10}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function.

The imaginary $\boldsymbol{i}$ represents the base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the wave front.

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\boldsymbol{f} = c\boldsymbol{f}' \\
&\frac{\partial^2 \boldsymbol{f}}{\partial \tau^2} = c\frac{\partial}{\partial \tau}\boldsymbol{f}' = c^2\boldsymbol{f}'' \\
&\frac{\partial \boldsymbol{f}}{\partial z} = \boldsymbol{i}\boldsymbol{f}' \\
&\frac{\partial^2 \boldsymbol{f}}{\partial z^2} = \boldsymbol{i}\frac{\partial}{\partial z}\boldsymbol{f}' = -\boldsymbol{f}'' \\
&\frac{\partial^2 \boldsymbol{f}}{\partial \tau^2} + \frac{\partial^2 \boldsymbol{f}}{\partial z^2} = (c^2 - 1)\boldsymbol{f}''
\end{aligned} \tag{11}$$

If $c = \pm 1$, then $\boldsymbol{f}$ is a solution of the quaternionic wave equation.

7.3 Displacement generator

The definition of the differential is

$$\Phi = \nabla \psi \tag{1}$$

In Fourier space the nabla becomes a displacement generator.

$$\tilde{\Phi} = \mathcal{M}\tilde{\psi} \tag{2}$$

$\mathcal{M}$ is the ***displacement generator***

A small displacement in configuration space becomes a multiplier in Fourier space.

In a paginated space-progression model the displacements are small and the displacement generators work incremental. The multipliers act as superposition coefficients.

7.4 The coupling equation

The coupling equation follows from peculiar properties of the differential equation. We start with two normalized functions ψ and φ and a normalizable function $\Phi = m\,\varphi$.

$$||\psi|| = ||\varphi|| = 1 \tag{1}$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla\psi = m\,\varphi \tag{2}$$

$\Phi = \nabla\psi$ defines the differential equation. (3)

$\nabla\psi = \Phi$ formulates a continuity equation. (4)

$\nabla\psi = m\,\varphi$ formulates the coupling equation. (5)

It couples ψ to φ. m is the coupling factor.

$$\nabla\psi = m_1\,\varphi \tag{6}$$

$$\nabla^*\varphi = m_2\,\zeta \tag{7}$$

$$\nabla^*\nabla\psi = m_1\,\nabla^*\varphi = m_1 m_2 \zeta = \rho \tag{8}$$

Each double differentiable quaternionic function corresponds to a normalized density distribution.

7.4.1 In Fourier space

The Fourier transform of the coupling equation is:

$$\mathcal{M}\tilde{\psi} = m\tilde{\varphi} \quad (1)$$

$\mathcal{M}$ is the ***displacement generator***

8 Difference with Maxwell-like equations

The difference between the Maxwell-Minkowski based approach and the Hamilton-Euclidean based approach will become clear when the difference between the coordinate time t and the proper time τ is investigated. This becomes difficult when space is curved, but for infinitesimal steps space can be considered flat. In that situation holds:

Coordinate time step vector = proper time step vector + spatial step vector

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

This influence is easily recognizable in the corresponding wave equations:

In Maxell-Minkowski format the wave equation uses coordinate time t. It runs as:

$$\partial^2\psi/\partial t^2 - \partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 - \partial^2\psi/\partial z^2 = 0$$

Papers on Huygens principle work with this formula or it uses the version with polar coordinates.

For 3D the general solution runs:

$\psi = f(r - ct)/r$, where $c = \pm 1$; f is real

For 1D the general solution runs:

$\psi = f(x - ct)$, where $c = \pm 1$; f is real

For the Hamilton-Euclidean version, which uses proper time τ, we use the quaternionic nabla ∇:

$$\nabla = \left\{\frac{\partial}{\partial\tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \nabla_0 + \boldsymbol{\nabla};$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla}$$

$$\nabla\psi = \nabla_0\,\psi_0 - (\boldsymbol{\nabla}, \boldsymbol{\psi}) + \nabla_0\,\boldsymbol{\psi} + \boldsymbol{\nabla}\,\psi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\psi}$$

The $\pm$ sign reflects the choice between right handed and left handed quaternions.

In this way the Hamilton-Euclidean format of the wave equation runs:

$$\nabla^*\nabla\psi = \nabla_0\nabla_0\psi + (\boldsymbol{\nabla}, \boldsymbol{\nabla})\psi = 0$$

$$\partial^2\psi/\partial\tau^2 + \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = 0$$

Where $\psi = \psi_0 + \boldsymbol{\psi}$

For the general solution holds: $f = f_0 + \boldsymbol{f}$

For the real part ψ_0 of ψ:

$\psi_0 = f_0\,(\boldsymbol{i}\,r - c\,\tau)/r$, where $c = \pm 1$ and $\boldsymbol{i}$ is an imaginary base vector in radial direction

For the imaginary part $\boldsymbol{\psi}$ of ψ:

$\boldsymbol{\psi} = \boldsymbol{f}(\boldsymbol{i}\,z - c\,\tau)$, where $c = \pm 1$ and $\boldsymbol{i} = \boldsymbol{i}(z)$ is an imaginary base vector in the x, y plane

The orientation $\theta(z)$ of $\boldsymbol{i}(z)$ in the x, y plane determines the polarization of the 1D wave front.

9 Integral continuity equations

The integral equations that describe cosmology are:

$$\int_V \nabla\,\rho\, dV = \int_V s\, dV \tag{1}$$

$$\int_V \nabla_0 \rho_0\, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle\, dV + \int_V s_0\, dV \tag{2}$$

$$\int_V \nabla_0\, \boldsymbol{\rho}\, dV = -\int_V \boldsymbol{\nabla} \rho_0\, dV - \int_V \boldsymbol{\nabla} \times \boldsymbol{\rho}\, dV + \int_V \boldsymbol{s}\, dV \tag{3}$$

$$\frac{d}{d\tau} \int_V \rho\, dV + \oint_S \widehat{\boldsymbol{n}} \rho\, dS = \int_V s\, dV \tag{4}$$

Here $\widehat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(\tau, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(\tau, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V*. If ρ_0 is stable then in the above formula ρ stands for

$$\rho = \rho_0 + \boldsymbol{\rho} = \rho_0 + \frac{\rho_0 \boldsymbol{v}}{c} \tag{4}$$

It is the flux (flow per unit of area and per unit of progression) of ρ_0 . τ stands for progression.

10 Formula compendium

10.1 Vectors

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \boldsymbol{a} \rangle = \delta_{ij} a_i b_j = |\boldsymbol{a}||\boldsymbol{b}| cos(\theta) \tag{1}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} = \epsilon_{ijk} \widehat{\boldsymbol{x}}_{\boldsymbol{i}} a_j b_k \tag{2}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle^2 + \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b} \rangle^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \tag{3}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \times \boldsymbol{c} \rangle = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \rangle \tag{4}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0} \tag{5}$$

$$\langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{c} \times \boldsymbol{d} \rangle = \langle \boldsymbol{a}, \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d}) \rangle = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \langle \boldsymbol{b}, \boldsymbol{d} \rangle - \langle \boldsymbol{a}, \boldsymbol{d} \rangle \langle \boldsymbol{b}, \boldsymbol{d} \rangle \tag{6}$$

10.2 Nabla

$$\boldsymbol{\nabla} a_0 = \widehat{\boldsymbol{x}}_{\boldsymbol{i}} \partial_i a_0 \tag{1}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle = \partial_i a_i \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \epsilon_{ijk} \widehat{\boldsymbol{x}}_{\boldsymbol{i}} \partial_j a_k \tag{3}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} a_0 \rangle = \boldsymbol{\nabla}^{\boldsymbol{2}} a_0 \tag{4}$$

$$\boldsymbol{\nabla}(a_0 b_0) = \ a_0 \boldsymbol{\nabla}(b_0) + b_0 \boldsymbol{\nabla}(a_0) \tag{5}$$

$$\langle \boldsymbol{\nabla}, a_0 \boldsymbol{a} \rangle = \langle \mathbf{a}, \boldsymbol{\nabla} a_0 \rangle + a_0 \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle \tag{6}$$

$$\langle \boldsymbol{\nabla} a_0, \boldsymbol{\nabla} b_0 \rangle = \langle \boldsymbol{\nabla}, a_0 \boldsymbol{\nabla} b_0 \rangle - a_0 \boldsymbol{\nabla}^{\boldsymbol{2}} b_0 \tag{7}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{a} \times \boldsymbol{b} \rangle = \langle \mathbf{b}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle - \langle \mathbf{a}, \boldsymbol{\nabla} \times \boldsymbol{b} \rangle \tag{8}$$

$$\langle \nabla a_0, \nabla \times \boldsymbol{a} \rangle = -\langle \nabla, \boldsymbol{a} \times \nabla a_0 \rangle \quad (9)$$

$$\langle \nabla \times \boldsymbol{a}, \nabla \times \boldsymbol{b} \rangle = \langle \boldsymbol{b}, \nabla \times (\nabla \times \boldsymbol{a}) \rangle - \langle \nabla, (\nabla \times \boldsymbol{a}) \times \boldsymbol{b} \rangle \quad (10)$$

$$\nabla \times (a_0 \boldsymbol{a}) = a_0 \nabla \times \boldsymbol{a} - \boldsymbol{a} \times \nabla a_0 \quad (11)$$

$$\nabla \times (a_0 \nabla a_0) = (\nabla a_0) \times \nabla b_0 \quad (12)$$

$$\langle \mathbf{a}, \nabla \times \boldsymbol{b} \rangle = \langle \mathbf{a} \times \nabla, \boldsymbol{b} \rangle \quad (13)$$

$$\langle \nabla, \nabla \times \boldsymbol{a} \rangle = 0 \quad (14)$$

$$\nabla \times \nabla a_0 = \mathbf{0} \quad (15)$$

$$\langle \nabla \times \nabla, \boldsymbol{a} \rangle = 0 \quad (16)$$