About the replacement of metric tensor

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ABSTRACT
In the general relativity theory, study the replacement of the metric tensor in the Einstein gravity field equation.

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1. Introduction

In the general relativity, we study the replacement of the metric tensor.

The gravity field equation is

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \]  

(1)

In this time, we study the replacement of the cosmological term \( \Lambda g_{\mu\nu} \).

We look the co-invariant differentiation of scalars \( g_{\mu\nu} \cdot h_\alpha \).

In this time, \( g \neq \det(g_{\mu\nu}) \), \( g \cdot h \) are scalars.

2. The replacement of the metric tensor

\[ g_{\mu\nu} = \frac{\partial g}{\partial x^\mu} \cdot h_\alpha = \frac{\partial h}{\partial x^\nu} \]  

(2)

The co-invariant differentiation of the metric tensor \( g_{\mu\nu} \) is

\[ g_{\mu\nu\lambda\delta} = 0 \]  

(3)

Co-invariant differentiation of scalars \( g_{\mu\nu} \cdot h_\alpha \) is

\[ (g_{\mu\nu} \cdot h_\alpha)_\beta = g_{\mu\alpha} \cdot h_\nu + g_{\beta\mu} \cdot h_{\alpha\nu} \]

\[ = \left[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial g}{\partial x^\alpha} \right) - \Gamma^\lambda_{\alpha\mu} \left( \frac{\partial g}{\partial x^\lambda} \right) \right] \frac{\partial h}{\partial x^\nu} + \left[ \frac{\partial}{\partial x^\alpha} \left( \frac{\partial h}{\partial x^\mu} \right) - \Gamma^\lambda_{\alpha\nu} \left( \frac{\partial h}{\partial x^\lambda} \right) \right] \frac{\partial g}{\partial x^\nu} \]

\[ = \frac{\partial}{\partial x^\mu} \left( \frac{\partial g}{\partial x^\alpha} \right) \frac{\partial h}{\partial x^\nu} - \Gamma^\lambda_{\alpha\mu} \frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\nu} = \Gamma^\lambda_{\alpha\nu} \frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\nu} - \Gamma^\lambda_{\alpha\nu} \frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\nu} \]  

(4)

\[ g_{\mu\nu\alpha\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \Gamma^\beta_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta_{\alpha\nu} g_{\beta\mu} \]  

(5)

In this time, if suppose \( g_{\mu\nu} \cdot h_\alpha = g_{\mu\nu} \cdot g^\mu \cdot h^\nu = g^{\mu\nu} \)

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\mu\epsilon} \left( \frac{\partial g_{\mu\epsilon}}{\partial x^\nu} + \frac{\partial g_{\nu\epsilon}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon} \right) \]

\[ = \frac{1}{2} \frac{\partial g}{\partial x^\lambda} \left( \frac{\partial h}{\partial x^\nu} \right) + \frac{\partial g}{\partial x^\mu} \left( \frac{\partial h}{\partial x^\nu} \right) - \frac{\partial g}{\partial x^\nu} \left( \frac{\partial h}{\partial x^\mu} \right) \]  

(7)

Therefore, Eq(4) is
\[ g_{\mu\alpha} = (g_{\mu} \cdot h_{\alpha})_{,\alpha} = g_{\mu\alpha} \cdot h_{\alpha} + g_{\mu} \cdot h_{\alpha \alpha} \]

\[ = \frac{\partial}{\partial x^\alpha} \left( \frac{\partial g}{\partial x^\mu} \cdot \frac{\partial h}{\partial x^\nu} \right) - \Gamma_{\alpha \mu}^\nu \frac{\partial g}{\partial x^\xi} \frac{\partial h}{\partial x^\alpha} - \Gamma_{\alpha \nu}^\mu \frac{\partial g}{\partial x^\xi} \frac{\partial h}{\partial x^\nu} \]

\[ = \frac{\partial}{\partial x^\alpha} \left( \frac{\partial g}{\partial x^\mu} \cdot \frac{\partial h}{\partial x^\nu} \right) - \frac{1}{2} g^{\alpha \beta} \left( \frac{\partial g_{\alpha \mu}}{\partial x^\nu} + \frac{\partial g_{\alpha \nu}}{\partial x^\mu} - \frac{\partial g_{\nu \mu}}{\partial x^\alpha} \right) \frac{\partial g}{\partial x^\beta} \frac{\partial h}{\partial x^\alpha} \]

\[ - \frac{1}{2} g^{\alpha \beta} \left( \frac{\partial g_{\beta \nu}}{\partial x^\mu} + \frac{\partial g_{\beta \mu}}{\partial x^\nu} - \frac{\partial g_{\nu \mu}}{\partial x^\beta} \right) \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} \]

\[ = \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \]

\[ - \frac{1}{2} \left( \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} \right) \left( \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\beta} \right) \left( \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \right) \]

\[ + \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} - \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} - \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \]

\[ - \frac{1}{2} \left( \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} \right) \left( \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\beta} \right) \left( \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \right) \]

\[ + \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} - \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} - \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \]

\[ ] (8) \]

In this time,

\[ g_{\mu\nu} = \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} = g_{\nu\mu} = \frac{\partial h}{\partial x^\nu} \frac{\partial g}{\partial x^\mu} \rightarrow g \leftrightarrow h \] (According to \( \mu, \nu \))

\[ g^{\mu\nu} g_{\mu\nu} = \left( \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} \right) \left( \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} \right) = 1 \] (9)

Hence, if we calculates Eq(8),

\[ g_{\mu\alpha} = (g_{\mu} \cdot h_{\alpha})_{,\alpha} = g_{\mu\alpha} \cdot h_{\alpha} + g_{\mu} \cdot h_{\alpha \alpha} \]
\[
\begin{align*}
= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\alpha} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} \\
- \frac{1}{2} \left( \frac{\partial g}{\partial x_\alpha} \right) \left( \frac{\partial h}{\partial x_\nu} \right) \left[ 2 \frac{\partial g}{\partial x_\beta} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} \right] \\
- \frac{1}{2} \left( \frac{\partial g}{\partial x_\alpha} \right) \left( \frac{\partial g}{\partial x_\nu} \right) \left[ 2 \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} \right] \\
= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\alpha} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} \\
- \left( \frac{\partial g}{\partial x_\alpha} \right) \left( \frac{\partial h}{\partial x_\nu} \right) \frac{\partial h}{\partial x^\nu} \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \\
- \left( \frac{\partial h}{\partial x_\alpha} \right) \left( \frac{\partial g}{\partial x_\nu} \right) \frac{\partial^2 h}{\partial x^\alpha \partial x^\nu} \\
= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\nu} + \frac{\partial g}{\partial x^\alpha} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial h}{\partial x^\alpha} \frac{\partial^2 g}{\partial x^\mu \partial x^\alpha} - \frac{\partial^2 h}{\partial x^\mu \partial x^\alpha} \frac{\partial g}{\partial x^\alpha} = 0 \quad (10)
\end{align*}
\]

3. The replacement of the metric tensor in the Minkowski spacetime

\[
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} dx^\mu dx^\nu \\
= \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} \frac{\partial x^\nu}{\partial x^\alpha} \frac{\partial x^\mu}{\partial x^\beta} \eta_{\alpha\beta} dx^\alpha dx^\beta \\
= \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} dx^\alpha dx^\beta = \eta g_{\alpha\beta} dx^\alpha dx^\beta \quad (11)
\]

If we act the Lorentz transformation in Eq(11),

\[
c dt = \gamma \left( c dt + \frac{V}{c} dx^t \right), \quad dx = \gamma \left( dx^t + \frac{V}{c} dt \right), \quad dy = dy^t, \quad dz = dz^t \\
\frac{1}{c} \frac{\partial}{\partial t} = \gamma \left( - \frac{1}{c} \frac{\partial}{\partial t} - \frac{V}{c} \frac{\partial}{\partial x^t} \right), \quad \frac{\partial}{\partial x^t} = \gamma \left( \frac{\partial}{\partial x^t} - \frac{v}{c} \frac{\partial}{\partial t} \right), \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y^t}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z^t} \\
\gamma = 1/\sqrt{1-V^2/c^2}
\]

\[
ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{00} c^2 dt^2 + \eta_{11} dx^2 + \eta_{22} dy^2 + \eta_{33} dz^2
\]
\[
\begin{align*}
&= \frac{\partial g}{\partial t} \frac{\partial h}{\partial t} c^2 dt^2 + \frac{\partial g}{\partial x} \frac{\partial h}{\partial x} dx^2 + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} dy^2 + \frac{\partial g}{\partial z} \frac{\partial h}{\partial z} dz^2 \\
&= \gamma^2 \left( \frac{\partial g}{\partial t} \frac{\partial h}{\partial t} c^2 dt + \frac{\partial g}{\partial x} \frac{\partial h}{\partial x} dx + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} dy + \frac{\partial g}{\partial z} \frac{\partial h}{\partial z} dz \right)^2 \\
&= \gamma^2 \left( \frac{\partial g}{\partial x} \frac{\partial h}{\partial x} c^2 \frac{\partial h}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} c^2 \frac{\partial h}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial h}{\partial z} c^2 \frac{\partial h}{\partial z} \right) \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 \\
&= \gamma^2 \left( -1 + \frac{V^2}{c^2} \right) \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 + \gamma^2 \left( 1 - \frac{V^2}{c^2} \right) \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 + \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 \\
&= -\gamma^2 \left( c^2 dt^2 + 2dt^t \nu dx + \frac{V^2}{c^2} dx^2 \right) + \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 + \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 \\
&= -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{00} c^2 dt^2 + \eta_{11} dx^2 + \eta_{22} dy^2 + \eta_{33} dz^2
\end{align*}
\]

In this time,

\[
ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{00} c^2 dt^2 + \eta_{11} dx^2 + \eta_{22} dy^2 + \eta_{33} dz^2
\]

Eq(12) is

\[
ds^2 = \gamma^2 \left( \frac{\partial g}{\partial t} \frac{\partial h}{\partial t} c^2 dt + \frac{\partial g}{\partial x} \frac{\partial h}{\partial x} dx + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} dy + \frac{\partial g}{\partial z} \frac{\partial h}{\partial z} dz \right)^2 \\
= \gamma^2 \left( \frac{\partial g}{\partial x} \frac{\partial h}{\partial x} c^2 \frac{\partial h}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial h}{\partial y} c^2 \frac{\partial h}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial h}{\partial z} c^2 \frac{\partial h}{\partial z} \right) \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 \\
= \gamma^2 \left( -1 + \frac{V^2}{c^2} \right) \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 + \gamma^2 \left( 1 - \frac{V^2}{c^2} \right) \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 + \gamma^2 \left( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} \right)^2 \\
= -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{00} c^2 dt^2 + \eta_{11} dx^2 + \eta_{22} dy^2 + \eta_{33} dz^2
\]

4. Conclusion

Therefore, the gravity field equation is

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} h_\omega = -\frac{8\pi G}{c^4} T_{\mu\nu}
\]

\[
g_{\mu\nu} \cdot h_\omega = g_{\mu\nu} \cdot g^{\mu\nu} = g^{\mu\nu}
\]

For example, the metric tensor of Schwarzschild solution is

\[
g_{00} = -1 + \frac{2GM}{rc^2} = \frac{\partial g}{\partial t} \frac{\partial h}{\partial t} \\
g_{11} = \frac{1}{1 - \frac{2GM}{rc^2}} = \frac{\partial g}{\partial r} \frac{\partial h}{\partial r}
\]
\[ g_{22} = r^2 = \frac{\partial g}{\partial \theta} \frac{\partial h}{\partial \theta} \quad , \quad g_{33} = r^2 \sin^2 \theta = \frac{\partial g}{\partial \phi} \frac{\partial h}{\partial \phi} \] (16)

*Important caution

\[ g_{01} = 0 = \frac{\partial g}{\partial \theta} \frac{\partial h}{\partial r} \rightarrow \left( \frac{\partial g}{\partial \theta} \neq 0, \frac{\partial h}{\partial r} \neq 0 \rightarrow g_{00} \neq 0, g_{11} \neq 0 \right) \]

\[ g_{10} = 0 = \frac{\partial g}{\partial r} \frac{\partial h}{\partial \theta} \rightarrow \left( \frac{\partial g}{\partial r} \neq 0, \frac{\partial h}{\partial \theta} \neq 0 \rightarrow g_{00} \neq 0, g_{11} \neq 0 \right) \] (17)

The important point is

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} dx^\mu dx^\nu, ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} dx^\alpha dx^\beta \]

\[ ds^4 = \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} dx^\mu dx^\nu \cdot \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} dx^\alpha dx^\beta = g_{\mu \nu} g_{\alpha \beta} dx^\mu dx^\nu dx^\alpha dx^\beta \]

\[ = \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\beta} dx^\mu dx^\beta \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\nu} dx^\alpha dx^\nu = g_{\mu \beta} g_{\alpha \nu} dx^\mu dx^\beta dx^\alpha dx^\nu \]

\[ = \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\nu} dx^\mu dx^\nu \frac{\partial g}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta} dx^\mu dx^\beta = g_{\alpha \mu} g_{\nu \beta} dx^\alpha dx^\nu dx^\mu dx^\beta \] (18)

Hence,

\[ g_{\mu \nu} g_{\alpha \beta} = g_{\mu \beta} g_{\alpha \nu} = g_{\alpha \mu} g_{\beta \nu} \] (19)

Reference