A Computational Study of Sofas and Cars

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Abstract: There is a class of geometric problem that seeks to find the shape of largest area that can pass down a corridor of given form or turn round inside a given shape. A popular example is the moving sofa problem for a shape that can be moved round an L-shaped corner in a corridor of width one. This problem has a conjectured solution proposed by Gerver in 1992. We investigate some of these problems numerically giving strong empirical evidence that Gerver was right and that a similar solution can be constructed for the related Conway car problem.

The Sofa Problem

In 1966 Moser posed his moving sofa problem [1]. What is the shape of largest area that can be manoeuvred round a right angle turn in a corridor of width one? A shape that obviously works is a semi-circle of radius one, but Hammersley [2] extended the shape using a rectangular mid-section with a semi-circle cut out of one side. By Thales' theorem the semi-circle can slide round the corner touching the inner side in three places while the remaining arcs of the original semi-circle each touch the outer sides of the corridor in two further places. The maximum area of such a shape is given when the radius of the removed semi-circle is equal to $\frac{2}{\pi}$ making the area

$$A_H = \frac{\pi}{2} + \frac{2}{\pi} = 2.207416099 \ldots$$

However, this shape known as Hammersley's sofa is not the optimal solution because it can be reasoned that the corners at the end of the semi-circle can be rounded off allowing more area to be added on the outer arcs. This leaves open the question as to what the ultimate solution is and in the 1970s there were some attempts to improve on the solution numerically by Maruyama in 1973 [5] and Wagner in 1976 [6].

Further progress had to wait until 1991 when Gerver [3] defined a shape of larger area $A_G = 2.21953166777197 \ldots$ [4] which he proved can move round the corner as required. Gerver did not prove that his shape is the optimal solution but he showed that it satisfies an optimisation condition that the optimal solution must satisfy and conjectured that it is indeed the best possible.

In this study a new computational approach is used and is found to agree with Gerver's area to about eight significant figures. This provides good empirical evidence that Gerver is right.

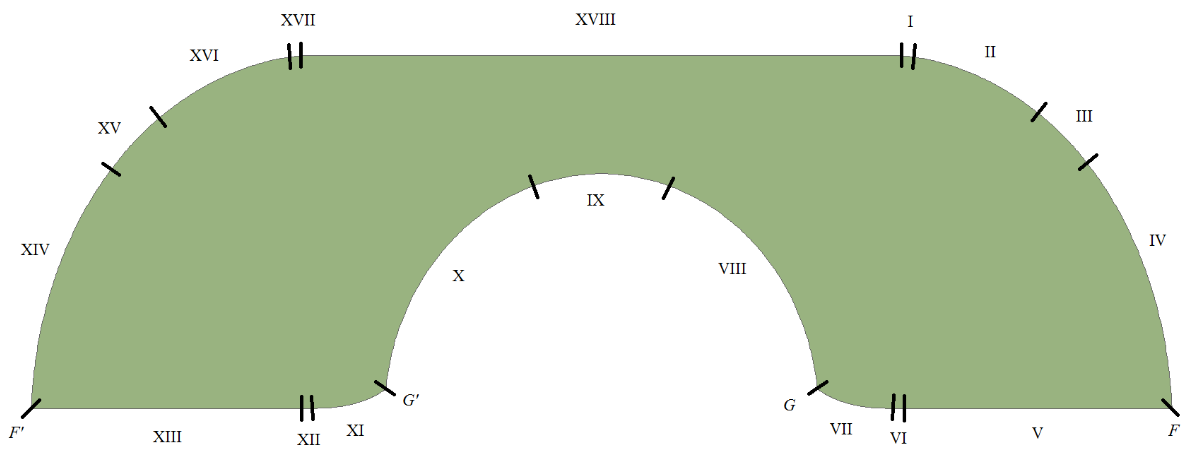

Figure 1 Gerver's Sofa

Gerver's sofa is bounded by 18 smooth sections as shown in figure 1. Each section is given an analytic definition such that V, XIII and XVIII are straight lines, I, VI, XII and XVII are circular arcs of radius a half, II, III, VII, XI, XV and XVI are involutes of circles (i.e. shapes formed by the ends of a cord winding round a circle), and IV and XIV are involutes of involutes of circles.

The motion round the corner during which the orientation of the sofa changes from an angle α of 0 through to 90° proceeds in five stages. At the beginning of this journey it will be touching the walls of the corridor along the lengths of the straight sections XVIII and XIII and can slide to the right until it touches at the point *F*. In the first stage of rotation for $0 < \alpha \leq \varphi = 2.24469765 \ldots^{\circ}$ the sofa touches the corridor in the arcs XII and XVII. For a fixed angle in this range it is free to move horizontally by a small amount until it touches the corridor either at the corner *F* or in the section VII. At the end of this stage when $\alpha = \varphi$ it will touch the corner of the corridor at *G* and the outer wall tangentially at *F*. In the second stage for $0 < \alpha \leq \theta = 39.03570106 \ldots^{\circ}$ the sofa touches the corridor in four places in sections XI, VIII, IV and XVI. The point G where the shape has a slope discontinuity does not touch the wall during this stage but is separated from it by a very small distance of less than 0.0012. At the end of stage two point *G'* is touching the corridor. In the third stage of rotation from $\theta < \alpha < 90^{o} - \theta$ the sofa touches the wall in only three places in sections XV, III and IX. Stage four is a mirror image of stage two in reverse and stage five is the mirror image of stage one in reverse.

Our goal is to attack the problem numerically and compare with Gerver's solution. A direct trial and error method might consist of constructing different shapes numerically and an algorithm to try and move them round the corner. In practice this would be very difficult to implement. Fortunately a much easier method can be found using a simple change of perspective. Instead of moving the sofa round the corner just imagine moving the corridor while the position of the sofa remains fixed. The sofa must be inside the corridor at all times which means it must be contained in the intersection of the corridor shape at each step of the journey. Indeed, for any given path of motion the corridor can take around the sofa the intersection of the set of corridor positions is a shape at least as large as any sofa it could have passed.

During its passage the corridor shape L will be turned through the angle α, for $0 \leq \alpha \leq 90^{o}$ to and the inside corner point of the corridor follows a path $P(\alpha) = (x(\alpha), y(\alpha))$ to form a parameterise set of shapes $\{L_P(\alpha), 0 \leq \alpha \leq 90^o\}$. The shape of a sofa is given by the maximum area over all possible continuous paths P of the intersection of this set

$$S = sup_P \bigcap_{0 \leq \alpha \leq 90^o} L_P(\alpha)$$

It is worth noting that in this form the moving sofa problem can be seen to fall into a general class of problems that require finding the maximum intersection or minimum (convex) union of a set of shapes under some allowed sets of transformation. For example Lebsgue's universal covering problem seeks the minimum convex union of all shapes of diameter one allowing them to be rotated, translated or reflected.

To construct the sofa numerically we can divide the path into N angular steps of size $\frac{1}{N}90^o$ and form the intersection of N shapes where the first and last shape is replaced with just a single long straight corridor whose position can be taken as fixed. The $N-1$ remaining positions are then varied iteratively to find the maximum area of the intersection shape. N must then be taken sufficiently large to give the shape to the required accuracy.

It should be noted that a number of assumptions are being made to conclude that this method gives the required solution including the following:

- That the optimal shape must actually turn through a right angle while going round the corner (This is not a trivial assumption since for example a unit square can get round the corner without turning at all.)
- That the limit of the discrete path as $N \to \infty$ is continuous.
- That the shape of maximum area formed from the intersection in connected.
- That the solution is unique.

To make the computation faster we make the additional unnecessary assumption that the path and therefore also the shape have a bilateral reflection symmetry

No attempt will be made to prove these assumptions but the result of the computation does support them.

Sofa Implementation

The computation was implemented in Java using the math.geom2d classes from sourceforge to perform the polygon transformations, intersections and area calculations.

It was found that the area converged only very slowly to a maximum if points in the path $P(\alpha)$ were varied individually. To improve convergence it was found much better to also vary Fourier components of the path, i.e. to make changes in the path of the form

$$\Delta P(\alpha) = (\mathrm{a} \cos K\alpha, b \sin K\alpha), K = 1,3,5, \ldots$$

To find the maximum while varying each variable it was sufficient to calculate the area for three suitably chosen values of the variable and fit a parabola to the results, then select the minimum as the value for the next iteration.

The error term in the convergence as a function of the number of steps along the path N is of order $\frac{1}{N}$. It can be observed that the largest source of this error comes from the concave boundary of the shape where it touches the inner corner of the corridor. This error can be reduced by smoothing this part of the boundary leaving residual error of order $\frac{1}{N^2}$. The coefficient of this error term appears to be stable so that it can be removed by extrapolation given the calculated area for two large values of N. In this way the error is reduced to provide a much more rapid convergence.

Sofa Results

The figures below show some of the optimal shapes formed for low values of N with and without the smoothing. The corridors have also been included to illustrate how it moves. The final shape is indistinguishable by eye from Gerver's solution.

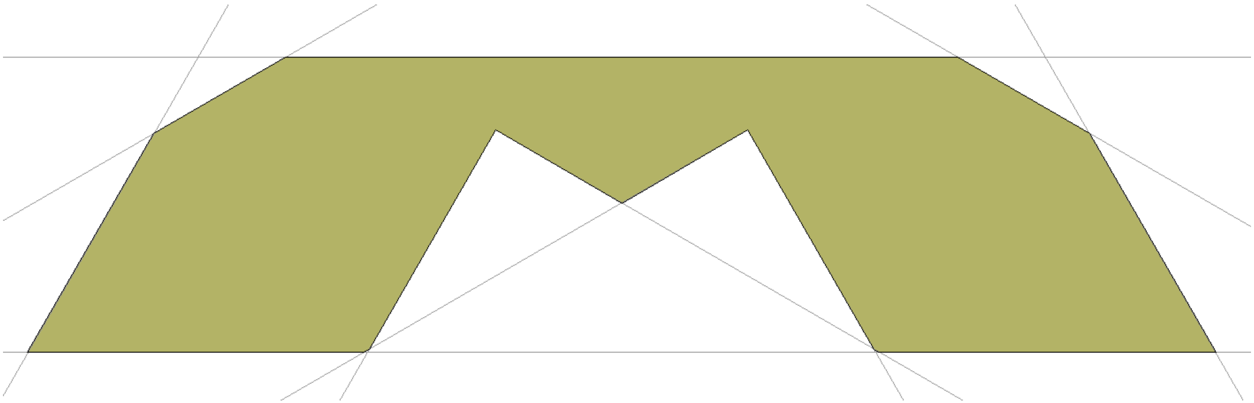

Figure 2 maximal intersection for N = 3

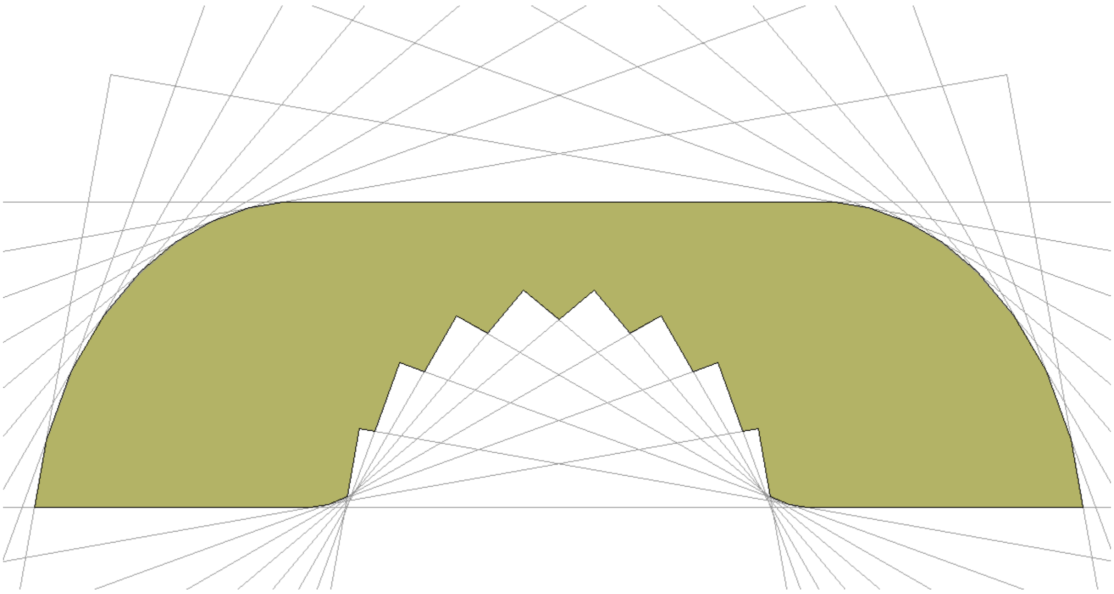

Figure 3 N = 9

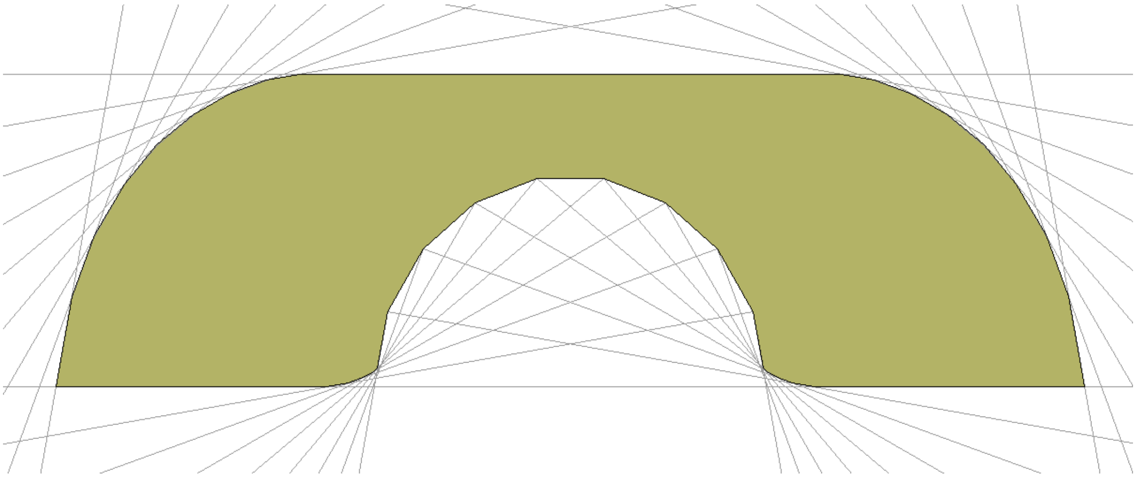

Figure 4 N = 9 with smoothing

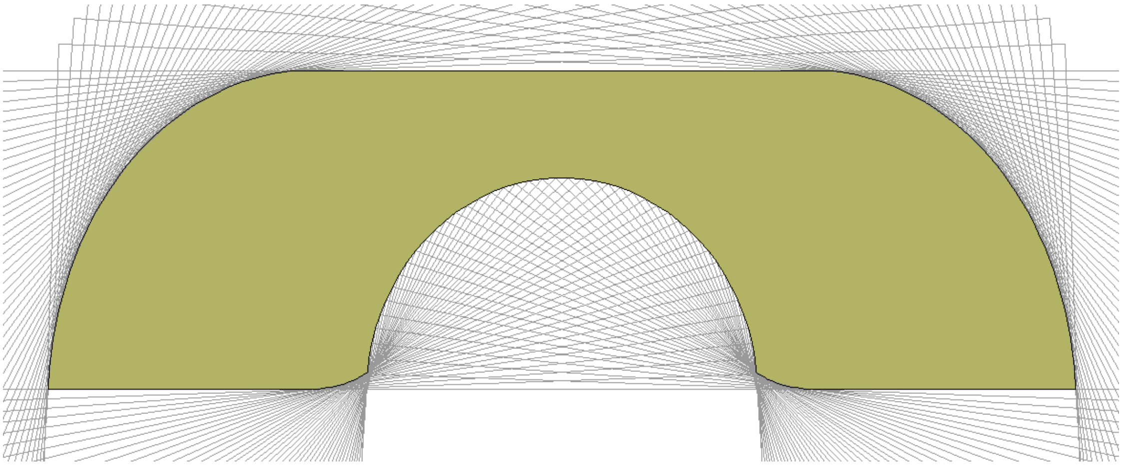

Figure 5 N = 45 smoothed

For $N \gtrsim 45$ the form of the sofa becomes visibly indistinguishable but we can continue to calculate the area as shown in table 1

N	Area of smoothed shape	Extrapolated area
25	2.221574880	
45	2.220276858	2.219697384
101	2.219661926	2.219509622
225	2.219557518	2.219531171
501	2.219537021	2.219531843

Table 1

The best numerical estimate of the area therefore agrees with the computed area for Gerver's sofa to nearly eight significant figures.

Given that the computed solution is so close to that of Gerver it seems very likely that Gerver's conjecture of optimality is indeed correct.

In his analysis Gerver also worked from the point of view where the sofa is fixed and the corridor moves round the shape in small steps. It is not hard to show that the shape of the intersection with maximal area for fixed N is a balanced polygon, i.e. a polygon shape such all edges parallel to a given direction fall on two lines whose total lengths on each line are equal. Gerver proved that the optimal solution to the sofa problem must be the limit of a sequence of such shapes as $N \to \infty$ and that his shape is such a limit. This does not quite prove that his shape is the optimal solution since there are other shapes which satisfy this condition. For example a regular polygons with an even number of sides are also balanced polygons which converge to a circle of diameter one.

The Conway Car Problem

At a Copenhagen meeting of a group of mathematicians in the late 1960s the moving sofa problem and a number of different variations were discussed. The group probably included Moser, Conway, Shephard and four others (see e.g. [7]) Two of the variations were to find the shape of largest area that can go round both left and right handed corners and another shape that could use a T-junction to turn round. It is not hard to see that a shape which can turn both left and right can also turn in the T-junction, but it is not so obvious that the reverse is true. A car that turns into a T-junction might make use of some of the extra space in the continuation of the road, in which case it would not be able to turn round an L-shaped corner. The computations reported here do support the conjecture that the same shape is the best result for both problems. Nothing was published from the meeting but these variations have been associated with Conway and the unknown solution is referred to as the Conway car.

Before proceeding to an investigation of the Conway Car it is worthwhile to consider the most restrictive variant of the corridor problem. What is the shape of largest area that can pass through any corridor of width at most one? To be precise the shape must be able to pass between two continuous curves such that the minimum distance of any two points on both curves is at least one. It is not hard to prove that one shape which fits is a boat shaped area bounded by two arcs of radius one and angle 120 degrees. This has an area $A = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} = 1.2283698 \ldots$ However, this is not the optimal solution. Another smaller shape that fits is a circle of radius one but this can always be followed by a second circle of radius one that touches it. The shape consisting of two touching circles is therefore a solution with area $A = \frac{\pi}{2}$. This does appear to be the best possible solution. Indeed if we only require that the shape can pass round the sharpest possible 180 degree U-turn in both directions then the touching circles shape is probably the best possible. This also tells us that $\frac{\pi}{2}$ is a lower bound for the optimal area for the solution of any problem of this type including the Conway car.

The only published solution to the Conway car problem appears to the numerical result of Maruyama from 1973 [5] This is reproduced in figure 6.

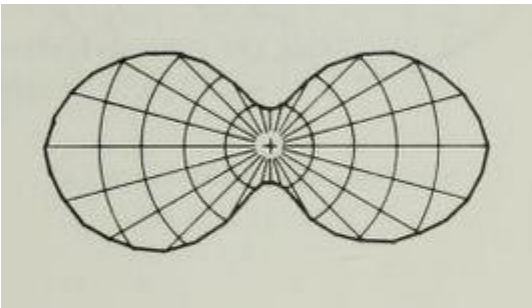

Figure 6: Maruyama's car solution

Car implementation

The computation for the Conway Car problem is very similar to that of the Sofa problem, except that the intersection of corridors turning the opposite way must be included. Equivalently, the intersection includes L-shaped corridors rotated through angles α for

$$0 \leq \alpha \leq 90^o \text{ and } 270^o \leq \alpha \leq 360^0$$

The car problem has a four-fold symmetry combining left-right and forward-backward bilateral symmetries. This does not mean that the solution form must automatically have these symmetries because the symmetry could be spontaneously broken. In the computation for the car this symmetry was therefore not assumed but it was nevertheless found to be realised in the results.

The optimal shape thus found (also known as “the bikini”) is shown in Figure 7

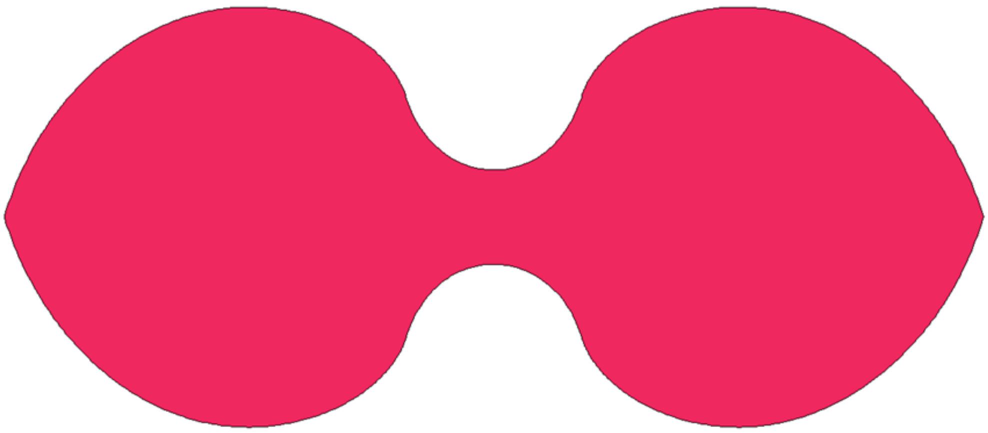

Figure 7: The Bikini

The area has been calculated using 200 stepping points to be $A = 1.64495$

It is likely that the curves of this shape can be described using the same equations that form Gerver’s Sofa, but with different values for the angular constants. This is left as an exercise for the reader.

References

[1] Moser, L., 'Problem 66-11: Moving furniture through a hallway', SIAM Rev. (1966), 381.

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[3] Gever, J. L., 'On Moving a Sofa Around a Corner', Geometriae Dedicata 42: 267-283, 1992

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[6] Wagner, N. R., “The Sofa Problem”, American Math Monthly 83 (1876) 188-189.

[7] Stewart, I. “Another Fine Math You Got Me Into” Dover 1992