ON THE PRINCIPLES OF LOGIC UNDER UNCERTAINTY

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1_ABSTRACT

Logic mainly refers to the way we reason [hence, we 'understand' things], which in turn relates to the way we can 'know' the world around us. But during 20th century, several advances in relation to our understanding of reality as well as to our limits to 'fully knowing' it, have forced us to adapt/re-estate some of the principles on which logic was built; mainly those usually designated as 'classical logic'.

While some issues are -more or less- widely accepted to have been solved by new approaches to logic [non-classical logics], other still show some controversy, which we try to shed some light from a new perspective based on Communication Theory [Shannon, 1948].

To help us more clearly explain the issue, we build on an easy example such as a bottle, which will allow us to easily explain/understand each of the reviewed issues.

2_THE PRINCIPLES OF LOGIC IN THE FRAMEWORK OF COMPLEXITY AND UNCERTAINTY

THE FOUR PRINCIPLES OF CLASSICAL LOGIC

Classical logic is usually understood as a binary logic built mainly on the ideas of Aristotle's, which reviewed qualities that can only be truth or false referred to objects.

Figure 01. Is it a bottle?. Classical Logic dealt mainly with issues *which can only be answered in a binary form. There are only two possible answers; yes or no; true or false.*

From classical logic, there are only two possibilities [two possible states of things] leading to **four principles or axioms** usually summarized as:

01 $$Identity \rightarrow \forall I; I \equiv I \quad (1)$$

02 $$Bivalence\ [two\ truth\ values] \rightarrow |x| = 1 \vee |x| = 0 \quad (2)$$

03 $$Non\ Contradiction \rightarrow \forall x; |x| \neq |\neg x| \quad (3)$$

04 $$Excluded\ Middle \rightarrow \nexists y; y = x \wedge y = \neg x \quad (4)$$

FUZZINESS, VAGUENESS AND TIME

Classical logic has been widely accepted as a general framework for producing knowledge [and more specifically, scientific knowledge], but some important changes will appear along the 20th century.

As we advance in our knowledge of the world, we try to know more aspects of things; we start dealing with qualities that can be in objects in a *fuzzy* manner, that can *change* in time, and that many times incorporate some *ambiguity*.

These three ideas are progressively incorporated into logical statements and analysis, leading to substantial modifications in the above principles, which we briefly review next.

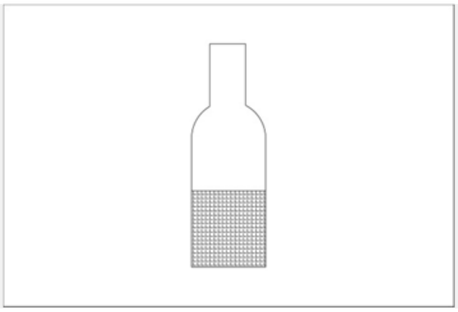

Figure 02. Is the bottle full or empty?

From classical logic, the question cannot be answered, leading to the need of a review that allows us to describe many aspects of reality that are not binary; that accept many possible states…

Fuzziness refers mainly to *contingency and imprecision;* to the fact that bottles are not necessarily full or empty, but can also be full or empty to an 'extent', which usually cannot be accurately measured [some states can hardly be distinguished one from another]. There are infinite possibilities/states of things, which equate saying that most statements may have infinite values of truth. This leads to **Fuzzy Logic** [built on Fuzzy Sets Theory, Zadeh 1965]

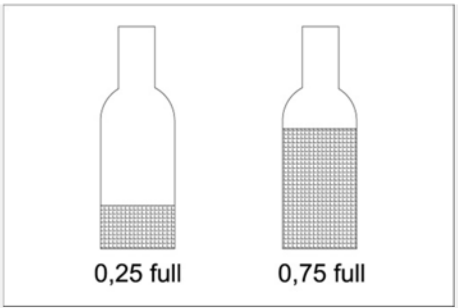

Figure 03. A bottle can be full or empty to an extent, which implies that the statement 'the bottle is full' may be truth to such extent. It has a 'degree of truth'

It allows understanding the 'sorites' paradox. If we pour a drop of water in an empty bottle, the bottle is still empty. If we pour another drop, it is a bit 'less empty'. If we keep on pouring drops of water, at some point the bottle will be full[1].

Time variation leads to **Temporal Logic** which appears to model the fact that tomorrow the bottle can be full to a different extent that it is full today; almost any truth value can change over time.

[1] Though it is not the aim of this article to review in detail the 'sorites' paradox, in our view it combines two issues. The first can be solved by 'Fuzzy Logic', while the second relates to computing the truth value of a concept [a heap] from an independent variable [the grain number]. The extent to which a number of grains form a 'heap' [i.e., a 'sorites' in greek] does not depend of the number of grains but of their shape. If we put infinite grains in a row [or drawing a spiral] they do not form a heap. It is therefore a paradox created partly by the choice of an inadequate variable to measure the concept. In a similar way, the truth value of the assertion 'the bottle is full' does not depend on the number of drops poured, but of the water volume/bottle capacity ratio.

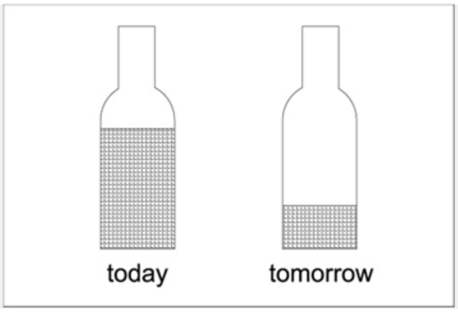

Figure 04. Today the bottle is full to a different extent than it may be tomorrow. Almost any statements' truth degree is linked to a particular time moment.

And linguistic terms **Vagueness** [ambiguity] mainly relates to two issues:

- First, terms are intentional summaries of objects' information, therefore, they [terms] only provide incomplete [vague] information about objects[2], with two goals:
  - *Being able to designate an almost infinite number of objects with a reduced number of terms*. Therefore, each term refers to a class of objects. Otherwise we would have to create a different term for each object[3].
  - *Being able to designate objects that change in time.* Any term referring to an object whose identity supports changing, refers to its *general form* and necessarily 'ignores' certain transformations of the object.
- And second, a linguistic term may have more than one meaning, yet their truth value in some statement may be different.

In general, vagueness does not lead to the development of a specific logic, but to the need of understating statements into a context [physical, temporal and semantic] and in the sense they are stated.

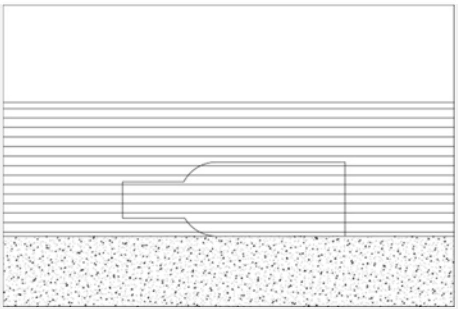

Figure 05. Is the bottle full or empty? We use the term 'full' in general to refer to the fact that we have filled it on purpose with some kind of fluid, which is different to that of its environment [e.g. a 'diver' considers his 'bottle' to be full when it contains air]. If 'full' referred only to the state of containing a fluid, then any bottle would always be full, as it is always full of liquid and/or air.

Truth values need to be assessed into a context/environment, and related to the sense in which statements are made.

The above issues have been developed by different authors, leading to re-enunciating/complementing the principles/axioms of classical logic. *Fuzziness, temporal variation and language vagueness shall be incorporated by fuzzy logic rules [Zadeh, 1965] and considering the*

[2] Therefore, they refer to common informational patterns [it may be to different objects –classes or different moments in time of the same object]. But summarizing information [selecting only some part of objects' information] means that we are not fully describing the object; some information is excluded from the description and the excluded information may sometimes be the key to fully understand statements' meaning.

[3] While conceptual objects may be identical, real objects never are, so if we do not summarize objects' information, we would need an almost infinite number of terms.

contexts [physical, temporal and semantic] and the sense in which the assertions are made, re-stating the above principles as:

- The *Law of Identity* shall be reformulated as there *'cannot be an object that is not identical to itself at the same point in time, context and in the same sense':*

01.1 $$Identity \rightarrow \forall I; I \equiv I \quad (5)$$

- *Truth Values* referring to qualities that can emerge in a continuous degree in objects may take any value in the range 0-1 [e.g., the bottle can be full/empty any value in the range 0-1]:

02.1 $$Infinite\ Truth\ values \rightarrow \forall x; |x| \rightarrow [0,1] \quad (6)$$

- The *Law of non-contradiction* in fuzzy terms shall be reformulated as *'there can be no concept whose truth and falsity relating an object are -at the same moment in time, context and in the same sense- different from one'* [e.g., the amount the bottle is full and empty must add up to one]:

03.1 $$Non - contradiction \rightarrow \forall x; |x| = 1 - |\neg x| \quad (7)$$

- The *Law of excluded middle* remains valid but it requires *interpreting the sign '¬' as 'absolutely false' not as the 'no' of natural language* [e.g., the bottle cannot be absolutely full and absolutely empty at the same time and in the same sense]:

04.1 $$Excluded\ middle \rightarrow \nexists y; y = x \wedge y = \neg x \quad (8)$$

UNCERTAINTY AND LOGIC

Imprecision, Ambiguity and Change seem to be sufficiently incorporated into the Principles of Logic, yet there is still one quality which along 20th century has proven to be a 'necessary' quality of knowledge: **Uncertainty[4].**

Any statement regarding any object is done by an observer, and when the observer has some lack of certainty/knowledge in relation to such object [it may be a conceptual object as a statement], uncertainty appears. It is therefore a quality of an object in relation to an observer when the observer does not fully know the state of such object.

The **observer** comes into the statements[5], introducing a 'point of view' and generating a wide range of situations difficult to model/understand … *What if the bottle has a label which prevents us from being able to see the extent to which it is full or empty?*

[4] Actually, the 'uncertainty' of any knowledge is implicit in the statement 'I know one thing: that I know no-thing', attributed to Socrates, but its full acceptance comes along the 20th century, mainly due to Gödel's Theorem, Chaos unpredictability, and Complexity Sciences.

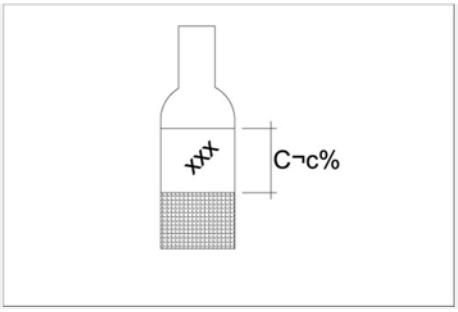

Figure 06. What happens behind the label?

The existence of an observer/subject implies a certain point of view [i.e., an 'observation point' as subjectivity], which in turn implies the possibility of 'hidden areas' to the observer … Behind the label the bottle can be full or empty; totally or to a certain extent, but we cannot fully know it, therefore we cannot state it as an absolute truth.

Uncertainty relates to our inability [as observers] of stating how the object is in some aspect, leading to our impossibility of stating either the total truth or falsity of any issue relating said object's aspect. *Uncertainty prevents us from deciding the truth value of the statements in every possible situation.*

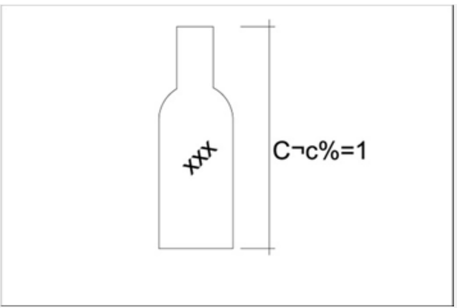

Figure 07: Is the bottle full or empty?

As bottles start to have 'labels', we no longer are able to state with complete certainty to which extent a bottle is full or empty, in a degree which relates to the size of the label. And if a bottle had a label totally covering it, then our degree of uncertainty in relation to such bottle would be complete, approaching us to the paradox of the 'Hooded man' [Eubulides of Miletus, IV BCE]

Uncertainty introduces some major changes we can summarize as:

- *Our statements always imply a Degree of Uncertainty* $C_{\neg c}[I,x]_{\%}$, which reduces the truth/falsity values we can state, that not always will add up to one.

  $$Non-contradiction \rightarrow \forall I,x: |x|' = 1 - C_{\neg c}[I,x]_{\%} - |\neg x|' \qquad (9)$$

  Being |x|' and |¬x|' the degree of truth/falsity we can assert, and $C_{\neg c}[I,x]_{\%}$ our uncertainty degree on relation to the degree of truth of x referred to I.

  It is important to highlight that the Uncertainty Degree is not an invariant property of an object, but of the object in relation to the observer relating to a particular concept x[6]. Since we are reviewing truth values, we use the term 'Uncertainty' referred to concepts truth/falsity.

- Only when our uncertainty when referring a concept x to an object I disappears, the truth degree we can state |x|' matches the truth degree |x|. As a consequence, *when referred to the degrees of truth/falsity that we can state* the last two principles lose their universality

[5] While previous logics mainly refer to objects' characteristics [deals with objectivity] Uncertainty mainly refers to the relation observer-object; to the subjectivity inherent to any statement we can make.

[6] For instance; we are total uncertain whether the Hooded man is or is not our brother [Uncertainty degree equals one], but we are totally certain that he is 'hooded' [Uncertainty degree equals zero]. But also, observation points are infinite. If the hood does not cover the whole face, maybe there is another observation point from which other observer could have no uncertainty in relation to the above question.

condition, becoming *rules that allow certain exceptions*[7]. If our uncertainty relating an aspect [or any aspect] of an object is complete [i.e., the object belongs entirely to Uncertainty Class], our statements relating it not necessarily comply with the last two principles.

This statement is false

Figure 08. The liar paradox [Eubulides of Miletus 4th century BCE] is built on a statement whose truth value belongs completely to Undecidability class We cannot state neither its truth nor its falsity; both its truth |x| as well as its falsity |¬x| are undecidable, therefore:

$$|x| = |\neg x| = Undecidable \in \neg C$$

Statements truth values' that entirely belong to Uncertainty classes do not comply neither with the Law of non-contradiction nor with the Law of Excluded Middle.

- And third, unpredictability of reality Chaos and systems' decision making ability implies *our uncertainty degree increases as it does the prediction time.*

Unpredictability and Undecidability[8] appear as synonym concepts to Uncertainty; hence their truth value referred to any object is approximately the same.

Uncertainty prevents us from asserting the complete truth or falsity of every statement, and as almost every statement implies a degree of uncertainty, apparently this leads us most of the times to the impossibility of stating truth/falsity values adding up to one.

However, as synonym concepts are logically equivalent, the uncertainty degree will increase the truth value of concepts synonym to 'uncertain' [or concepts with a high membership degree to Uncertainty class], while it will decrease the truth value of concepts antonym to uncertain [synonym to 'certain' or concepts with a high membership degree to Certainty class].

Our statements will always imply some uncertainty, which can be sometimes reduced if we account for the sense of the concepts which truth values' we need to measure. Uncertainty adds up to the truth/falsity value depending on the meaning of the measured concept, allowing us to reformulate the above equation as:

- If a concept involves [has a high membership degree to class] *Certainty [C]:*

$$x \in C \ \rightarrow \forall I, x\colon |x| \sim C_c[I]_{\%} \tag{10}$$

- If a concept involves [has a high membership degree to class] *Uncertainty [¬C]:*

$$\neg x \in \neg C \rightarrow \forall I, \neg x\colon |\neg x| \sim C_{\neg c}[I]_{\%} \tag{11}$$

[7] Something that Lukasiewicz suggests in relation to indeterminacy.

[8] It is important to highlight that we refer to Undecidability relating objects' membership to Truth/Falsity classes. Objects' which truth values are uncertain/undecidable may be fully decidable in other aspects [e.g., their membership to Uncertainty/Certainty classes].

The above has great importance that we can review it from two complementary points of view:

From a **semantic point of view,** we can find equivalent concepts as:

- *Certainty, Organization and Order.* Our Certainty in relation to objects' requires that their information is arranged according to some 'known' patterns [it is 'organized' or 'ordered'].
- *Uncertainty, Disorganization and Disorder.* Our Uncertainty in relation to objects' because their information is not arranged following any known pattern [it is disorganized or disordered].

From an **information point of view,** we can relate it to *Entropy/Neguentropy as measures of objects information's disorder [uncertainty]/order [certainty].* The **Logic of Uncertainty is largely equivalent to Logic of Entropy, approaching truth values and thermo-dynamical properties/states.**

And the Second Law of Thermodynamics states that in the absence of the unusual, Entropy increases in time. It equates saying that in the absence of the unusual, the label of the bottle [our uncertainty regarding the bottle] increases over time...

Figure 07. Uncertainty increases with the prediction interval. In the absence of an external input [an ordering or neguentropy increase], if we want to describe the bottle's state in the future, the size of the label will be increased.

$$\Delta C_{\neg c}[x] \geq 0$$

Uncertainty forces us to complement/re-enunciate the principles of logic. When objects have a degree of membership to class Uncertainty [when bottles have labels], we shall complement the last three principles stated above[9].

There are the classes of Uncertainty [¬C] which allow objects/statements[10] to not comply neither with the Law of non-contradiction nor with the Law of Excluded Middle, and ill usually require complementing the truth values of our statements:

- Truth values shall be supplemented by *Certainty values* which may take any value in the range 0-1.

2.2 $$Infinite\ Certainty\ Values \ \rightarrow \forall I; C_c[I]_{\%} \rightarrow [0,1] \quad (12)$$

[9] DRAE [2014] proposes that an object 'is anything that we can get to know'. Hence the identity principle is not compromised by uncertainty. If it is compromised [if we cannot assert that something is equal to itself at a moment in time and in the same sense] then it will not be an object.

[10] It is important to differentiate between objects and statements [conceptual objects]. While most objects still comply with the principles of logic, it is our statements which may no longer comply with them since the entry of uncertainty. The bottle is full or empty, but we cannot assert to what extent. The 'hooded man' is or is not our brother, but we cannot assert either of them.

- The *Law of non-contradiction* shall be complemented in terms of uncertainty as *'there can be no concept about which -when referred to an object- our certainty and uncertainty degree at one time, context and in one sense are different from one'*

3.2.a $$Non-contradiction \rightarrow \forall I; C_c[I]_{\%} = 1 - C_{\neg c}[I]_{\%} \quad (13)$$

And the Uncertainty Degree $C_{\neg c}[I]_{\%}$ reduces the truth degree we can state, so apparently …

3.2.b $$Non-contradiction \rightarrow \forall I, x: |x|' = 1 - C_{\neg c}[I]_{\%} - |\neg x|' \quad (14)$$

Being |x|' and |¬x|' the degree of truth/falsity we can assert.

But, when the measured concept has a high membership degree to Certainty or Uncertainty classes, we can reformulate it as:

If a concept involves [has a high membership degree to class] *Certainty [C]:*

3.2.c $$x\sim \in C \rightarrow \forall I, x: |x| \sim C_c[I]_{\%} \sim 1 - C_{\neg c}[I]_{\%} \quad (15)$$

If a concept involves [has a high membership degree to class] *Uncertainty [¬C]:*

3.2.d $$\neg x \in \neg C \rightarrow \forall I, \neg x: |\neg x| \sim C_{\neg c}[I]_{\%} \sim 1 - C_c[I]_{\%} \quad (16)$$

- The validity of the *Law of Excluded Middle* requires interpreting that this principle shall be met except by those objects that belong entirely to *Uncertainty* classes.

4.2.a $$\forall x; [x = A \ \wedge x = \neg A] \leftrightarrow [C_{\neg c}[A]_{\%} = 1: C_{\neg c}[A]_{\%} = C_{\neg c}[\neg A]_{\%}] \quad (17)$$

The above statements are 'formal' statements which apparently do not constitute particularly relevant reasoning. Then… which is their interest? To fully appreciate it, we have to review a particular type of concepts; those whose truth value we need to assess from the interaction of other concept's truth values.

3_ TRUTH VALUES, UNCERTAINTY AND COMPLEXITY

The term 'Complexity' has acquired an increasing importance nowadays. While it may be used in at least three main senses [Alvira, 2014c], let us use it in its etymological sense; as 'which is woven together', hence cannot be totally decomposed.

This is quite important because we currently use an increasing amount of concepts which truth value depends on the truth values of other sub concepts that interact to define its overall truth value. From an information perspective, *they constitute a non-additive set of information; the overall meaning/truth value most of the times is not the sum of the individual meanings.*

And building on Communication Theory, we can relate this non-additivity to uncertainty measuring and concepts membership to uncertainty classes. Information aggregation usually implies introducing uncertainty into the aggregated parameter, which can be approximately, measured used Entropy formulas.

And when calculating the overall truth value of the concept, this uncertainty needs to be added or subtracted depending on the meaning of the measured concept, what we review next.

First let us review a couple of concepts / formulas from Communication Theory, that we adapt considering that I is an object [factual or conceptual] and X is a concept [or conceptual object].

Communication Theory states that we can measure the amount of information provided by a description of each symbol 's' emitted by a source I using **Entropy's** formula [Shannon, 1948]:

$$H[s] = -K\sum_{i=1}^{n} p_i * log_2 p_i \qquad (18)$$

Being H[s] _ symbol's entropy, p_i_ the probability of each of the possible values for that symbol and K_ a constant depending on the chosen measuring unit.

The Entropy formula is an approach to measuring the information provided by the reception of a message from the maximum uncertainty or ignorance it is possible to have regarding its contents[11]. It builds on *Classical Logic* and *Probability Theory*, considering two possible values for each symbol *[true or false, hence base '2' of the logarithm]* and the probability of occurrence of each of them.

And to measure the amount of shared information between two objects, Communication Theory proposes the **Mutual Information** which can be conceptualized as "the average reduction of uncertainty of [a source] due to knowledge of another" [Crutchfield & Feldman, 2001, p. 5] and it is:

Mutual Information

$$I[I;X] = H[X] - H_X[I] \qquad (19)$$

H[X] is the maximum Entropy of X; i.e.: the maximum amount of ignorance that we may have about X [which therefore coincides with the maximum amount of information that we can acquire in relation to X] and its value depends on the range of possible symbols in X information/description.

[11] "Shannon's H [entropy] measures the degree of ignorance of the communication engineer when he designs the technical equipment in the channel" [Jaynes 1978, p. 25], and this conceptualization of *Entropy as Ignorance or Uncertainty* advances us that it also will allow us to measure *Certainty* as complementary value.

$H_x[I]$ is the **Conditional Entropy** of I which is a measure of "how uncertain we are on I when we know X" [Shannon 1948, p. 12], and its formula is:

Conditional Entropy

$$H_X[I] = -\sum_{i,j} p(i,j) * log_2\, p_i(j) \quad (20)$$

The three above formulas and concepts, allow us to pose our formulation of Certainty/Uncertainty degrees that we review next.

We want to propose a formulation on the degree of certainty/uncertainty we can have on the degree that a meaning [or concept] X is true referred to an object I[12], which requires reviewing two issues:

- *Measuring the 'mutual information'* between concept X and Object I
- Measuring the *meaning transformation* introduced by aggregating the truth values of concepts x_i.

And to calculate it, we are going to measure both issues for two special concepts:

- X = 'certainty' [C], which allows us calculating our *Certainty Degree* $C_c[I]_{\%}$ related to I.
- ¬X = 'uncertainty' [¬C], which allows us calculating our *Uncertainty Degree* $C_{\neg c}[I]_{\%}$ related to I.

Both measures will report the *Certainty/Uncertainty Degree* that, known the global state of an object I [truth value of X referred to object I], we have in relation to its *detailed status* [truth values of concepts x_i referred to I] and we use the following codes:

- $|x_i|$_ truth value when X=certainty
- $|\neg x_i|$ _ truth value when ¬X=uncertainty

First we review the formulation for the *Uncertainty Degree:*

MODELING FOR ¬X = 'UNCERTAINTY'

Let us assume that the description is complete so X's truth value can be fully determined from the truth values of some x_i sub-concepts, by means of a formulation that requires four steps:

STEP 1: MEASURING MUTUAL 'KNOWLEDGE' BETWEEN I AND ¬C

We can define the degree of truth of a concept X referred to an object I, as the extent to which the rules that define membership to class X are satisfied by such objet, and we can model it for ¬C based on the formula of the *Mutual Information*, i.e.:

$$I[I;\neg c] = H[\neg c] - H_{\neg c}[I]^{13} \quad (21)$$

[12] It is equivalent to calculating its Membership Grade to classes Certainty and Uncertainty.

[13] It is the maximum Certainty we can have relating I, less the Uncertainty we have about I once ¬C is known.

However, it is necessary to make some clarifications in relation to the formulation:

H[¬c] is the maximum possible truth value of concept ¬C [which for now we equate to its information content], and we can state two issues in relation to the term:

- it is the maximum value for $H_{\neg c}[I]_{max}$ to be reached when every $|\neg x_i|$ reaches value '1'

$$\forall i: |\neg x_i| = 1 \ \rightarrow H[\neg c]_{max} = -\sum_{i=1}^{n} P_i * log_m P_i \tag{22}$$

- it must match $H_c[I]_{max}$[14]:

$$H_{\neg c}[I]_{max} = H_c[I]_{max} = -\sum_{i=1}^{n} P_i * log_m P_i \tag{23}$$

And $H_I[\neg c]$ is the Conditional Entropy which incorporates a fundamental issue. **It comes to measuring the ignorance or lack of knowledge in terms of the measured concept and therefore the formula does not incorporate the truth values related to $\neg x_i$ sub-concepts $|\neg x_i|$ but their complementary $|x_i|$, i.e.:**

$$H_I[\neg c] = -\sum_{i=1}^{n} [1 - |\neg x_i|] * P_i * log_m P_i = -\sum_{i=1}^{n} |x_i| * P_i * log_m P_i \tag{24}$$

$|x_i|$_ are the values of the membership functions to the opposite concept to ¬c, i.e. to 'c'[15] and p_i_ is assigned by the probabilities structure of the *logical decomposition* of ¬c.

And by replacing the two terms in the formula of the Mutual Information we have:

$$I[I, \neg c] = -\sum_{i=1}^{n} P_i * log_m P_i + \sum_{i=1}^{n} |x_i| * P_i * log_m P_i \tag{25}$$

However, *adding the values of the description involves introducing uncertainty in the obtained value.* Known the truth value of X referred to I, our *uncertainty* related to object 'I' is greater than if we know the truth values of concepts x_i; i.e., a truth value of X may be describing different combinations of truth values of x_i.

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[14] The maximum amount of knowledge that may involve a concept is the same as its opposite concept may imply, since any relevant variable for C will also be relevant for ¬C. Thus,$H_{\neg c}[I]_{max}$= $H_c[I]_{max}$.

[15] That is, the membership functions of I to concept 'x=certainty'. The reason is that the truth values express knowledge [certainty] in relation to the revised concept, but what we want to measure is its lack of knowledge [ignorance], and thus in the formula we use the complementary values.

The *real meaning* of the added value is more 'uncertain' than that of the individual values. By adding the truth values of concepts x_i our uncertainty regarding the object increases[16].

It is necessary to review the meaning of the two terms in the above equation, because when comparing an object I with any concept X we find some relevant differences:

- The first term refers to 'certainty'; to the maximum amount of *knowledge* [or *ignorance*] that X [or ¬X] and 'I' may share, and **its value is determined by the logical decomposition of X / ¬X** [it does not depend on 'I' but of the concept X].
- The second term refers to 'uncertainty'; to the amount of *knowledge* that 'I' and ¬X do not share, and **its value is determined by x_i sub-concepts' truth values; therefore it depends on object I's features.**

However, as they refer to the concept ¬X='uncertainty', the meaning of the two terms is going to change:

- The first implies *certainty* regarding concept 'uncertainty', i.e., an assertion of a negation, and therefore *uncertainty [or denial].*
- The second implies *uncertainty* regarding the concept 'uncertainty', i.e., a double negation, and therefore *certainty [or affirmation].*

And this means that **by aggregating the truth values, the second term shall decrease its value on a percentage depending on the aggregated values,** which we calculate below.

STEP 2: CALCULATION OF UNCERTAINTY INCREASE DUE TO TRUTH VALUES AGGREGATION

We know that, in most situations, aggregating truth values involves loss of 'certainty'. We usually have greater certainty about the state of an object when we also know the state of such object in relation to x_i concepts [their truth values], that when we only know the truth value of X, and there are two limiting cases:

- A situation of *zero uncertainty increase,* which happens if the *truth value of all sub-concepts x_i is equal,* and therefore X's truth value is equal to every one of them.

$$\forall i \in I, x\colon |x_i| = k \rightarrow \forall i \in I, x\colon |x_i| = |X| = C_c[I]_{\%} \tag{26}$$

- A situation of *maximum uncertainty increase,* which occurs if half of the truth values take the minimum value, and the other half takes the maximum value, reaching the added value the highest possible differentiation with each truth value.

$$\begin{gathered} \forall i \in I, c\colon |x_i| = min\left[|x_i|_{i=1}^{n}\right] \;\wedge\; |x_{i+1}| = max\left[|x_i|_{i=1}^{n}\right] \\ \vee \\ \forall i \in I, c\colon |x_i| = max\left[|x_i|_{i=1}^{n}\right] \;\wedge\; |x_{i+1}| = min\left[|x_i|_{i=1}^{n}\right] \end{gathered} \tag{27}$$

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[16] The truth degree of concept 'uncertainty' increases and truth degree of concept 'certainty' is reduced in relation to I.

And in the latter case, the greatest loss of certainty is reached when the *minimum* and *maximum* value of the truth values are 0 and 1 respectively.

$$\forall i \in I, c: |x_i| = 0 \vee 1 \quad \wedge \quad |x_{i+1}| = 1 - |x_i| \tag{28}$$

Since we know the maximum added value, we can calculate the uncertainty introduced by any distribution of x_i sub-concepts' truth values as a measure of the Mutual Information for each truth value related to the added value [I], namely[17]:

$$I_c[I, |x_i|] = -\sum_{i=1}^{n} p_i * |x_i| * log_2 |x_i|^{18} \tag{29}$$

Being $I_c[I, |x_i|]$ the *'common knowledge'* to I and $|x_i|$.

Once 'I' is known, the maximum certainty as to the value of '$|x_i|$ ' is reached when every $|x_i|$ is equal to the arithmetic mean of the sub-concepts x_i truth values weighted by their probabilities, i.e.:

$$\forall i \in I: |x_i| = \overline{|x_\iota| * p_I} \leftrightarrow I_c[I, |x_i|]_{max} = \overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i} \tag{30}$$

This means that aggregating object's truth values, introduces a reduction of certainty that can be deducted from the distribution of their values, and that is for each truth value:

$$I_c[I, |x_i|]_\% = \frac{I_c[I, |x_i|]}{I_c[I, |x_i|]_{max}} = \frac{p_i * |x_i| * log_2 |x_i|}{\overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i}} \tag{31}$$

And for the set of all the sub-concepts:

$$I_c[I]_\% = \sum_{I=1}^{n} I_c[I, |x_i|]_\% = \sum_{i=1}^{n} \frac{p_i * |x_i| * log_2 |x_i|}{\overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i}} \tag{32}$$

STEP 3: CALCULATING I'S UNCERTAINTY/ UNCERTAINTY DEGREE

The formula for the Uncertainty can be derived from the above formulas:

$$C_{\neg c}[I] = H_c[I]_{max} * I_c[I]_{max} - H_I[\neg c] * I_c[I]_\% \tag{33}$$

[17] Before we calculate the 'common information' between concept X and object I. Now we want to calculate the common information between x_i sub-concepts' truth values and X's truth value, which inform us of the uncertainty introduced by adding them.

[18] Though we have departed from classical logic, we maintain 2 as the base for the logarithms for greater similarity to the Communication Theory since the result will be independent of such base.

$I_c[I]_{max}$ corresponds to the situation where all truth values are 1 and consequently its value is equal to 1, and therefore the uncertainty is[19]:

$$C_{\neg c}[I] = H_c[I]_{max} - H_I[\neg c] * I_c[I]_{\%} \tag{34}$$

This confirms that there is a limit to the maximum value of the uncertainty related to I, which is reached when the second term is zero, i.e.:

$$C_{\neg c}[I]_{max} = C[\neg c] = C[c] = H_c[I]_{max} \tag{35}$$

The calculation of the *Uncertainty Degree* $C_{\neg c}[I]_{\%}$ can also be deducted from previous formulas:

$$C_{\neg c}[I]_{\%} = 1 - \frac{H_I[\neg c]}{H_c[I]_{max}} * I_c[I]_{\%} \tag{36}$$

And we can express it as:

$$C_{\neg c}[I]_{\%} = 1 - H_I[\neg c]_{\%} * I_c[I]_{\%} \tag{37}$$

And because…

$$H_I[\neg c]_{\%} = \frac{\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i}{\sum_{i=1}^{n} P_i * log_2 P_i} \tag{38}$$

The Uncertainty Degree is:

$$C_{\neg c}[I]_{\%} = 1 - \frac{\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i}{\sum_{i=1}^{n} P_i * log_2 P_i} * \sum_{i=1}^{n} \frac{p_i * |x_i| * log_2 |x_i|}{\overline{p_\iota} * \overline{|x_\iota| * p_\iota} * log_2 \overline{|x_\iota| * p_\iota}} \tag{39}$$

However, usually the truth values of sub-concepts x_i have equal relevance for the truth value of X, which equates considering them to be equally likely, allowing a considerable simplification of the above formulation:

STEP 4: SIMPLIFICATION FOR EQUALLY LIKELY [RELEVANT] SUB-CONCEPTS' TRUTH VALUES

In most situations we can understand that *truth values are equally likely*[20] and this allows us to considerably simplify the above formulas, because:

$$\forall i \in I : p_i = \overline{p_\iota} \tag{40}$$

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[19] By referring to ¬C, the first term is 'uncertainty' and it does not reduce with the aggregation, while the second term is 'certainty' and therefore reduces when adding the information

[20] A truth value of X may be generated by many different combinations of truth values of concepts x_i, in which the range of influence of each sub-concept' truth value is the same.

Hence, the *certainty* produced by truth values' distribution is:

$$H_I[\neg c]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} |x_i| = \overline{|x_i|} \tag{41}$$

While the certainty reduction due to their aggregation is:

$$I_c[I]_{\%} = \sum_{I=1}^{N} I_c[|x_i|, I]_{\%} = \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i|} * log_2 \overline{|x_i|}} \tag{42}$$

And therefore, the *Uncertainty Degree* is:

$$C_{\neg c}[I]_{\%} = 1 - \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i|} * log_2 \overline{|x_i|}} \tag{43}$$

MODELING FOR X='CERTAINTY'

Let us now calculate the Certainty Degree as the opposite case to the above, i.e., when x='certainty', which allows us to highlight some interesting differences, starting from the formula of Common Information:

$$I[I; c] = H[I] - H_I[c] \tag{44}$$

And the difference appears when incorporating the complementary values into the Conditional Entropy formula, being:

$$H_I[c] = \sum_{i=1}^{n} [1 - |x_i|] * P_i * log_2 P_i \tag{45}$$

We can develop as:

$$H_I[c] = -\sum_{i=1}^{n} P_i * log_2 P_i + \sum_{i=1}^{n} |x_i| * P_i * log_2 P_i \tag{46}$$

And by substituting into the formula of the Mutual Information, we obtain:

$$I[I, c] = -\sum_{i=1}^{n} P_i * log_m P_i + \sum_{i=1}^{n} P_i * log_m P_i - \sum_{i=1}^{n} |x_i| * P_i * log_m P_i \tag{47}$$

Being annulled the first and second terms, and thus…

$$I[I, c] = -\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i \tag{48}$$

We see, therefore, that I[I;c] is the complementary value to that obtained for x='uncertainty':

$$I[I;c] = H_x[I]_{max} - I[I;\neg c] \tag{49}$$

That gives us the following formula expressed as a percentage:

$$I[I;c]_{\%} = 1 - I[I;\neg c]_{\%} \tag{50}$$

The value of the term $I_c[I]_{\%}$ *Uncertainty increase in the aggregation* refers to the same term, hence does not change its calculation procedure, and the overall *certainty* is:

$$C_c[I] = I[I;c] * I_c[I]_{\%} \tag{51}$$

And the obtained formula is again the complement of that proposed above for 'Uncertainty'; i.e., when x='uncertainty':

$$C_c[I] = C_c[I]_{max} - C_{\neg c}[I] \tag{52}$$

Being $C_c[I]$_ our certainty about I's microscopic state once we know its global status; $C_c[I]_{max}$_ the maximum certainty we may have; $C_{\neg c}[I]$ _ the uncertainty we have.

And the *Certainty Degree* $C_c[I]_{\%}$ is:

$$C_c[I]_{\%} = I[I;c]_{\%} * I_c[I]_{\%} \tag{53}$$

That is also the complementary value of the *Uncertainty Degree*, i.e.:

$$C_c[I]_{\%} = 1 - -C_{\neg c}[I]_{\%} \tag{54}$$

If we consider the situation of *equally likely truth values* then:

$$I[I,c]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} |x_i| = \overline{|x_i|} \tag{55}$$

Therefore the *Certainty Degree* is:

$$C_c[I]_{\%} = \overline{f_c[i]} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i|} * log_2 \overline{|x_i|}} \tag{56}$$

4_ CONCLUSIONS

The importance of understanding how our uncertainty affects statements' truth values is significant. Epistemology tells us that **almost every statement we can make implies some uncertainty** [e.g., currently 'almost every bottle has a label'], and we have to understand how it relates to the truth values we can assign to our statements relating objects.

And in this text we have reviewed two interesting issues:

The first is that *we can relate Certainty/Uncertainty to the truth value of statements if we consider the meaning of the measured concept.* Generally when we pose different formulations for measuring the truth degree of concepts, we are implicitly considering the effects of uncertainty.

The second is that building on Shannon's Entropy we have proven that *uncertainty is not only implicit in our measurements, but also in most information aggregations.* If we review it for concepts whose truth value depends on the aggregation of other concepts x_i, then we differentiate two situations:

- When a concept X shares some meaning with concept 'Certainty' [i.e.; it involves Certainty], its truth value will be between the arithmetic mean of truth value of concepts x_i and the *Certainty Degree* $C_c[I]_{\%}$, approaching $C_c[I]_{\%}$ the higher the degree of shared meaning.

$$C_c[I]_{\%} \sim |X| \leq \overline{|x_i|} \qquad (57)$$

- When a concept ¬X shares some meaning with concept 'uncertainty' [i.e.; it involves Uncertainty], its truth value will be between the arithmetic mean of truth value of concepts $\neg x_i$ and the *Uncertainty Degree* $C_{\neg c}[I]_{\%}$, approaching $C_{\neg c}[I]_{\%}$ the higher the degree of shared meaning.

$$\overline{|\neg x_i|} \leq |\neg X| \sim C_{\neg c}[I]_{\%} \qquad (58)$$

And this enables us to propose the following equations which can be applied to any possible X related to any I[21]:

Certainty
$$|X| \sim C_c[I]_{\%} = \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i| * log_2 |x_i|}} \qquad (59)$$

Uncertainty
$$|\neg X| \sim C_{\neg c}[I]_{\%} = 1 - \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i| * log_2 |x_i|}} \qquad (60)$$

However, the above formulations pose some indeterminacy in the limiting cases, so the formulations to be used for the general case are slightly different[22]:

Certainty
$$|X| \sim C_c[I]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} \left[|x_i| * \left[1 + \overline{|x_i|}_{i=1}^{n} - |x_i| \right] \right] \qquad (61)$$

Uncertainty
$$|\neg X| \sim C_{\neg c}[I]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} \left[|x_i| * \left[1 - \overline{|x_i|}_{i=1}^{n} + |x_i| \right] \right] \qquad (62)$$

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[21] Some applications of this issue are provided in Alvira 2014a, 2014b and 2014 c.

[22] The indeterminacy of the formulation based on Shannon's Entropy as well as the rationale of these formulations is explained in Alvira, 2014a and Alvira 2014c. It is noteworthy that these formulations are only valid in the case no sub-concept x_i exists such that its total falsity implies the total falsity of concept X. In the other case, formulations are provided also in the referred texts.

We see that an asymmetry appears between two types of concepts which derives from that existing between the concepts *Certainty and Uncertainty* and the concepts of *Truth and Falsity:*

- The *total certainty* about the truth [or falsity] of a proposition implies our belief in its *full truth [or falsity] and therefore on the complete falsity [or truth] of the contrary proposition.*
- The *total uncertainty* about the truth of a proposition *does not imply our belief in its complete falsity or in the complete truth of the contrary proposition;* it is only a measure of ignorance about its truth or falsity[23].

And this uncertainty, which entry into science becomes more visible along the 20th century, comes actually from the very conceptualization of objects in terms of *identity, difference and duality:*

The very *Law of Non-contradiction* implies this *asymmetry* because the prefix '¬' means 'no', which *refers to something other than the denied quality or concept but does not specify 'what it is'. When an object does not have a quality we do not know what qualities it has.* The non-quality involves denial of a known quality, but does not assert another quality in its place[24].

In reality the number of possible qualities or statements about objects is usually so high that to deny one usually does not mean a substantial uncertainty reduction[25]. **To reduce uncertainty we need to be able to assert some quality.**

The asymmetry between concepts involving Certainty/Uncertainty is of great importance, because its link with Entropy allows us to interpret this issue from the Second Law of Thermodynamics. If there is no input of outside negentropy, the truth degree of concepts involving Uncertainty tends to increase in objects over time.

This makes necessary to review the *Principle of Indifference[26].* If no neguentropy [ordering] is added, concepts implying Uncertainty become more probable over time; we will have more 'reasons' to believe that their 'emergence' in objects [their truth value] increases over time.

The former is of the utmost important for concepts' truth degree measuring related to real systems, because usually concepts' degree of truth relate to different thermodynamical states, hence any variation of truth degrees implies a variation on Entropy. These concepts include some of the most widely used and important concepts nowadays, such as Sustainability, Economic Stability, Depression, Talent, Happyness, etc…

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[23] This approaches us to Gödel's Undecidability.

[24] "The word 'no' is primarily of practical importance, since it allows to refer to something other without detailing what is that other" [Morris 1971, p. 64]

[25] In the context of a statement when possible messages are limited, denying something can provide an amount of information about the object that becomes noticeable. An example would be 'I spy game'; we go on denying claims until we can identify the object. In the context of 'all possible messages' denying something does not increase our knowledge.

[26] The *Principle of indifference or insufficient reason* [Laplace 1814] tells us that when we have a number of possibilities without a relevant difference between them, we must consider them to be equally likely.

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