Bohr-like model for black holes

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Abstract

It is an intuitive but general conviction that black holes (BHs) result in highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in quantum gravity. Here we show that such an intuitive picture is more than a picture, discussing a model of quantum BH somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913. Our model has important implications on the BH information puzzle and on the non-strictly random character of Hawking radiation. It is also in perfect agreement with existing results in the literature, starting from the famous result of Bekenstein on the area quantization.

This paper improves, clarifies and finalizes some recent results that, also together with collaborators, we published in various peer reviewed journals.

Preliminary results on the model in this paper have been recently discussed in an Invited Lecture at the 12th International Conference of Numerical Analysis and Applied Mathematics.

1 Introduction

Realizing a complete theory of quantum gravity, which will unify general relativity and quantum mechanics, is unanimously considered one of the most important tasks in the framework of theoretical physics. In fact, such a fundamental result will permit to obtain a better understanding of the universe. A key point of this issue is to realize a definitive model of quantum BH as BHs are generally considered theoretical laboratories for ideas in quantum gravity. It is indeed an intuitive but general conviction that, in some respects, BHs are the fundamental bricks of quantum gravity in the same way that atoms are the fundamental bricks of quantum mechanics. This analogy suggests that the BH mass should have a discrete spectrum. In this paper, we show that the such an intuitive picture is more than a picture. Considering the natural correspondence between Hawking radiation [1] and BH quasi-normal modes (QNMs) [2–4], we show that QNMs can be really interpreted in terms of BH quantum levels discussing a BH model somewhat similar to the semi-classical Bohr model of the structure of a hydrogen atom [5, 6]. This issue has important consequences on the BH information puzzle [37] and on the non-strictly random character of Hawking radiation.. In fact, showing BHs in terms of well defined quantum mechanical systems, having an ordered, discrete quantum spectrum, looks consistent with the unitarity of the underlying theory of quantum gravity and with the idea that information should come out in BH evaporation. A fundamental feature of the Bohr-like model that we are going to analyse is the discreteness of the BH horizon area as the function of the QNMs principal quantum number, which is consistent with various models of quantum gravity where the spacetime is fundamentally discrete [43, 44].

Preliminary results on the Bohr-like model for BHs have been recently discussed in an Invited Lecture at the 12th International Conference of Numerical Analysis and Applied Mathematics [42].

2 Tunnelling mechanism, non-strictly black body spectrum and effective temperature

We start recalling that an elegant and widely used explanation for Hawking radiation [1] is today the tunnelling mechanism [10–16]. Let us consider an object that is classically stable. If it becomes unstable from a quantum mechanical point of view, tunnelling is naturally suspected. Thus, the mechanism of particle creation by BHs [1] can be described as tunnelling arising from vacuum fluctuations near the BH horizon [10–16]. If a virtual particle pair is created just inside the horizon, the virtual particle with positive energy can tunnel out and materialize outside the BH as a real particle. In the same way, if a virtual particle pair is created just outside the horizon, the particle with negative energy can tunnel inwards. In both situations, the particle with negative energy is absorbed by the BH. Thus, the BH mass decreases and the particle with positive energy propagates towards infinity. The consequent emission of quanta appears as Hawking radiation. Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), in strictly thermal approximation the probability of emission is [1, 10, 11]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{1}$$

where ω is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ is the

Hawking temperature. Taking into account the energy conservation, i.e. the BH contraction which enables a varying geometry, one gets the fundamental correction of Parikh and Wilczek [10, 11]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})], \tag{2}$$

where the additional term $\frac{\omega}{2M}$ is present. We have recently finalized the tunnelling picture of Parikh and Wilczek showing that the probability of emission (2) is indeed associated to the two distributions [16]

$$\begin{aligned} < n >_{boson} = \tfrac{1}{\exp[4\pi(2M-\omega)\omega]-1} \\ < n >_{fermion} = \tfrac{1}{\exp[4\pi(2M-\omega)\omega]+1}, \end{aligned} \tag{3}$$

for bosons and fermions respectively, which are *non*-strictly thermal. By introducing the *effective temperature* [2–4, 16]

$$T_E(\omega) \equiv \frac{2M}{2M-\omega}T_H = \frac{1}{4\pi(2M-\omega)}, \tag{4}$$

one rewrites eq. (4) in a Boltzmann-like form similar to eq. (1)

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{5}$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor,* with $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$. Thus, the effective temperature replaces the Hawking temperature in the equation of the probability of emission [2–4, 16]. We recall that there are various fields of science where one takes into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature which represents the temperature of a black body that would emit the same total amount of radiation. We introduced the concept of effective temperature in BH physics in [3, 4] and used it in [2–4, 16, 17] and, together with collaborators, in [18, 19]. The effective temperature depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the BH radiation spectrum from the strictly thermal feature [2–4, 16]. The introduction of the effective temperature permits the introduction of other *effective quantities*. Considering the initial BH mass *before* the emission, M, and the final BH mass *after* the emission, $M-\omega$, one introduces the *BH effective mass* and the *BH effective horizon* [2–4, 16] as

$$M_E \equiv M - \frac{\omega}{2},\ r_E \equiv 2M_E, \tag{6}$$

during the BH contraction, i.e. *during* the emission of the particle [2–4, 16]. Such effective quantities are average quantities [2–4, 16]. In fact, r_E is the average of the initial and final horizons while M_E is the average of the initial and final masses [2–4, 16]. The effective temperature T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission $T_{H\text{ initial}} = \frac{1}{8\pi M}$, *after* the emission $T_{H\text{ final}} = \frac{1}{8\pi(M-\omega)}$)[2–4, 16].

3 Quasi-normal modes as black hole quantum levels

One considers Dirac delta perturbations [2–4, 16] which represent subsequent absorptions of particles having negative energies which are associated to emissions of Hawking quanta in the above discussed mechanism of particle pair creation. BH response to perturbations are QNMs [2–4, 7, 17–20], which are frequencies of radial spin-j perturbations obeying a time independent Schröedinger-like equation [2–4, 20]. They are the BH modes of energy dissipation which frequency is allowed to be complex [2–4, 20]. The intriguing idea to model the quantum BH in terms of BH QNMs arises from a remarkable paper by York [21]. For large values of the quantum "overtone" number n, where $n = 1, 2, ...$, QNMs become independent of both the spin and the angular momentum quantum numbers [2–4, 7, 20, 23, 24], in perfect agreement with *Bohr's Correspondence Principle* [22], which states that "transition frequencies at large quantum numbers should equal classical oscillation frequencies". In other words, Bohr's Correspondence Principle enables an accurate semi-classical analysis for large values of the principal quantum number n, i.e, for excited BHs. By using that principle, Hod has shown that QNMs release information about the area quantization as QNMs are associated to absorption of particles [23, 24]. Hod's work was refined by Maggiore [7] who solved some important problems. On the other hand, as QNMs are *countable* frequencies, ideas on the *continuous* character of Hawking radiation did not agree with attempts to interpret QNMs in terms of emitted quanta, preventing to associate QNMs to Hawking radiation [20]. Recently, the authors of [25–28] and ourselves and collaborators [2–4, 17–19] observed that the non-thermal spectrum of Parikh and Wilczek [10, 11] also implies the countable character of subsequent emissions of Hawking quanta. This issue enables a natural correspondence between QNMs and Hawking radiation, permitting to interpret QNMs also in terms of emitted energies [2–4, 17–19]. In fact, Dirac delta perturbations due to discrete subsequent absorptions of particles having negative energies, which are associated to emissions of Hawking quanta in the mechanism of particle pair creation by quantum fluctuations, generates BH QNMs [2–4, 17–19]. On the other hand, the correspondence between emitted radiation and proper oscillation of the emitting body is a fundamental behavior of every radiation process in science. Based on such a natural correspondence between Hawking radiation and BH QNMs, one can consider QNMs in terms of quantum levels also for emitted energies [2–4, 17–19].

Let us discuss the model. For large values of the principal quantum number n, i.e, for excited BHs, and independently of the angular momentum quantum number, the QNMs expression of the Schwarzschild BH which takes into account the non-strictly thermal behavior of the radiation spectrum is obtained replacing the Hawking temperature with the effective temperature in the standard, strictly thermal, equation for the quasi-normal frequencies as [2–4]

$$\omega_n = a + ib + 2\pi i n \times T_E(|\omega_n|)$$
$$\simeq 2\pi i n \times T_E(|\omega_n|) = \tfrac{in}{4M-2|\omega_n|}, \tag{7}$$

where a and b are real numbers with $a = (\ln 3)\times T_E(|\omega_n|),\ b = \pi \times T_E(|\omega_n|)$ for $j = 0, 2$ (scalar and gravitational perturbations), $a = 0,\ b = 0$ for $j = 1$ (vector perturbations) and $a = 0,\ b = \pi \times T_E(|\omega_n|)$ for half-integer values of j. An intuitive derivation of eq. (7) can be found in [3, 4]. We *rigorously* derived such an equation in the Appendix of [2]. In any case, it is better to further clarify this fundamental point. The Schwarzschild line element is [16, 39]

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{8}$$

Historical notes to this notion can be find in [39]. The Schwarzschild radius (event horizon) is given by $r_H = 2M$ [16, 39] and $\frac{1}{4M}$ is the BH surface gravity. We note that, due to the non strictly black body behavior of the spectrum, the Hawking temperature *shows a discrete behavior in time*. Therefore, the introduction of the effective temperature does not degrade the importance of the Hawking temperature [2–4, 16–19]. This is why, as the Hawking temperature changes with a discrete character in time, the effective temperature represents, in a certain sense, the value of the Hawking temperature *during* the emission of the particle [2–4, 16–19]. In other words, the introduced effective temperature takes into account the non-strictly black body character of the radiation spectrum and the non-strictly continuous character of subsequent emissions of particles.

Let us introduce [16]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)} = \beta_H\left(1-\frac{\omega}{2M}\right), \tag{9}$$

where $\beta_H \equiv \frac{1}{T_H}$. One uses Hawking's periodicity arguments [16, 40, 41], to write down the euclidean form of the metric as [16]

$$ds_E^2 = x^2\left[\frac{d\tau}{4M\left(1-\frac{\omega}{2M}\right)}\right]^2 + \left(\frac{r}{r_E}\right)^2 dx^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{10}$$

The line element (10) is regular at $x = 0$ and $r = r_E$. τ is treated as an angular variable having period $\beta_E(\omega)$ [16, 40, 41]. One replaces the quantity $\sum_i \beta_i \frac{\hbar^i}{M^{2i}}$ in [40] with the quantity $-\frac{\omega}{2M}$ [16]. Following the analysis in [40] in detail the *effective Schwarzschild line element* is obtained as [16]

$$ds_E^2 \equiv -(1-\frac{2M_E}{r})dt^2 + \frac{dr^2}{1-\frac{2M_E}{r}} + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{11}$$

and we can also easily show that r_E in eq. (10) is the same as in eq. (6) [16]. The *effective surface gravity* is in turn defined as $\frac{1}{4M_E}$. Thus, the BH *dynamical* geometry during the emission of the particle is taken into account by the effective line element (11) [16]. Although this does not mean that one can immediately replace $T_H(M)$ with $T_H(M-\frac{\omega}{2})$ in eq. (7), but the effective line element (11) permits to introduce the *effective equations* [2–4]

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)-\omega^2\right)\phi, \tag{12}$$

$$V(x)=V\left[x(r)\right]=\left(1-\frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2}+2\frac{(1-j^2)M_E}{r^3}\right) \tag{13}$$

and

$$\begin{array}{c} x=r+2M_E\ln\left(\frac{r}{2M_E}-1\right) \\ \frac{\partial}{\partial x}=\left(1-\frac{2M_E}{r}\right)\frac{\partial}{\partial r}. \end{array} \tag{14}$$

In order to simplify the following equations, here we also set

$$2M_E=r_E\equiv 1 \quad \textit{and} \quad m\equiv n+1. \tag{15}$$

We stress that the *Planck mass* m_p is equal to 1 in Planck units. Then, one rewrites (7) as

$$\frac{\omega_m}{m_p^2}=\frac{\ln 3}{4\pi}+\frac{i}{2}(m-\frac{1}{2})+\mathcal{O}(m^{-\frac{1}{2}}), \quad \textit{for}\, m\gg 1, \tag{16}$$

where now $m_p\neq 1$. Setting

$$\tilde{\omega}_m\equiv\frac{\omega_m}{m_p^2}, \tag{17}$$

eqs. (7), (12), (13) and (14) read

$$\tilde{\omega}_m=\frac{\ln 3}{4\pi}+\frac{i}{2}(m-\frac{1}{2})+\mathcal{O}(m^{-\frac{1}{2}}), \quad \textit{for}\, m\gg 1, \tag{18}$$

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)-\tilde{\omega}^2\right)\phi, \tag{19}$$

$$V(x)=V\left[x(r)\right]=\left(1-\frac{1}{r}\right)\left(\frac{l(l+1)}{r^2}-\frac{3(1-j^2)}{r^3}\right) \tag{20}$$

and

$$\begin{array}{c} x=r+\ln\left(r-1\right) \\ \frac{\partial}{\partial x}=\left(1-\frac{1}{r}\right)\frac{\partial}{\partial r} \end{array} \tag{21}$$

respectively.

Now, if one replies the same rigorous analytical calculation in the Appendix of [2] or the analogous calculation by Motl in [20] but starting from eqs. (19), (20) and (21) and satisfying purely outgoing boundary conditions both at the effective horizon ($r_E=2M_E$) and in the asymptotic region ($r=\infty$), the final result will be, obviously and rigorously, eq. (7). A key point has to be clarified. One could take position against the above analysis claiming that M_E and r_E (and consequently the tortoise coordinate and the Regge-Wheeler potential) are frequency dependent. But we note that eq. (15) translates such a frequency dependence into a continually rescaled mass unit in the discussion in the Appendix of [2]. It is simple to show that such a rescaling is extremely slow and always included within a factor 2. Thus, it does not influence the analysis in the Appendix of [2]. In fact, we note that, although $\tilde{\omega}$ in the analysis in the Appendix of [2] can be very large because of definition (17), ω must instead be always minor than the BH initial mass as BHs cannot emit more energy than their total mass. Inserting this constrain in eq. (6) one gets the range of permitted values of $M_E(|\omega_n|)$ as

$$\frac{M}{2} \leq M_E(|\omega_n|) \leq M. \tag{22}$$

Thus, setting $2M_E(|\omega_n|) = r_E(|\omega_n|) \equiv 1(|\omega_n|)$ one sees that the range of permitted values of the continually rescaled mass unit is always included within a factor 2. On the other hand, the countable sequence of QNMs is very large, see the discussion below and [2, 3]. This implies that the mass unit's rescaling is extremely slow. Therefore, the reader can easily check, by reviewing the discussion in the Appendix of [2] step by step, that the continually rescaled mass unit did not influence the analysis.

Let us discuss another argument which emphasizes the correctness of the analysis in the Appendix of [2]. We can choose to consider M_E as being constant within the range (22). In that case, we show that such an approximation is very good. Eq. (22) implies indeed that the range of permitted values of $T_E(|\omega_n|)$ is

$$T_H = T_E(0) \leq T_E(|\omega_n|) \leq 2T_H = T_E(|\omega_{n_{max}}|), \tag{23}$$

where T_H is the initial BH Hawking temperature. Then, if we fix $M_E = \frac{M}{2}$ in the analysis, the approximate result is

$$\omega_n \simeq 2\pi i n \times 2T_H. \tag{24}$$

On the other hand, if one fixes $M_E = M$ as in thermal approximation, the approximate result is

$$\omega_n \simeq 2\pi i n \times T_H. \tag{25}$$

We see that both the approximate results in correspondence of the extreme values in the range (22) have the same order of magnitude. Thus, fixing $2M_E = r_E \equiv 1$ does not change the order of magnitude of the final (approximated) result with respect to the exact result. In particular, setting $T_E = \frac{3}{2}T_H$ the uncertainty in the final result is 0.33, while in the result of the thermal approximation (25) the uncertainty is 2. Hence, even if one considers M_E as constant, the result in the Appendix of [2] is more precise than the thermal approximation of previous literature. Thus, the derivation of eq. (7) is surely correct.

Eq. (7) has the following elegant interpretation [3, 4]. QNMs determine the position of poles of a Green's function on the given background and the Euclidean BH solution converges to a *non-strictly* thermal circle at infinity with the inverse temperature $\beta_E(\omega_n) = \frac{1}{T_E(|\omega_n|)}$ [3, 4]. Then, the spacing of the poles in eq. (7) coincides with the spacing $2\pi i T_E(|\omega_n|) = 2\pi i T_H(\frac{2M}{2M-|\omega_n|})$, expected for a *non-strictly* thermal Green's function [3, 4].
Now, we improve the analyses in [2–4]. In those works we found the physical solution for the absolute values of the frequencies (7) only for scalar and gravitational perturbations and, strictly speaking, the results of [2–4] hold true only for $j = 0, 2$. It is instead of fundamental importance to show that the analysis works for arbitrary j as in that case the quantum of area obtained from the asymptotics of $|\omega_n|$ is an intrinsic property of Schwarzschild BHs and it does not depend on the spin content of the perturbation. The key point is that as $a, b \ll |2\pi i n \times T_E(|\omega_n|)|$, for large n the leading asymptotic behavior of $|\omega_n|$ is given by the leading term in the imaginary part of the complex frequencies. Considering the leading asymptotic behavior of (7) one gets the solution in terms of of $|\omega_n|$ as

$$|\omega_n| = M \pm \sqrt{M^2 - \frac{n}{2}}. \tag{26}$$

Again, BHs cannot emit more energy than their total mass. Thus, the physical solution is the one obeying $|\omega_n| < M$, i.e.

$$E_n \equiv |\omega_n| = M - \sqrt{M^2 - \frac{n}{2}}. \tag{27}$$

E_n is interpreted like the total energy emitted by the BH at that time, i.e. when the BH is excited at a level n [2–4].

4 The Bohr-like model

Considering an emission from the ground state (i.e. a BH which is not excited) to a state with large $n = n_1$ and using eq. (27), the BH mass changes from M to

$$M_{n_1} \equiv M - E_{n_1} = \sqrt{M^2 - \frac{n_1}{2}}. \tag{28}$$

In the transition from the state with $n = n_1$ to a state with $n = n_2$ where $n_2 > n_1$ the BH mass changes again from M_{n_1} to

$$\begin{aligned} M_{n_2} \equiv M_{n_1} - \Delta E_{n_1 \to n_2} = M - E_{n_2} \\ = \sqrt{M^2 - \tfrac{n_2}{2}}, \end{aligned} \tag{29}$$

where

$$\Delta E_{n_1 \to n_2} \equiv E_{n_2} - E_{n_1} = M_{n_1} - M_{n_2} = \sqrt{M^2 - \frac{n_1}{2}} - \sqrt{M^2 - \frac{n_2}{2}}, \tag{30}$$

is the jump between the two levels due to the emission of a particle having frequency $\Delta E_{n_1 \to n_2}$. Thus, in our BH model, during a quantum jump a discrete amount of energy is radiated and, for large values of the principal quantum number n, the analysis becomes independent from the other quantum numbers. In a certain sense, QNMs represent the "electron" which jumps from a level to another one and the absolute values of the QNMs frequencies represent the energy "shells" [2]. In Bohr model [5, 6] electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation (in standard units) $E = hf$, where h is the Planck constant and f the transition frequency. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to eq. (30). The similarity is completed if one notes that the interpretation of eq. (27) is of a particle, the “electron”, quantized on a circle of length [3]

$$L = \frac{1}{T_E(E_n)} = 4\pi \left(M + \sqrt{M^2 - \frac{n}{2}} \right), \tag{31}$$

which is the analogous of the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr model [5, 6]. On the other hand, Bohr model is an approximated model of the hydrogen atom with respect to the valence shell atom model of full quantum mechanics. In the same way, our BH model should be an approximated model with respect to the definitive, but at the present time unknown, BH model arising from a full quantum gravity theory.

As E_n is interpreted like the total energy emitted at level n, considering the expressions (28) and (29) for the residual BH mass one needs also

$$M^2 - \frac{n}{2} \geq 0. \tag{32}$$

In fact, BHs cannot emit more energy than their total mass and the total energy emitted by the BH cannot be imaginary. The expression (32) gives a maximum value for the overtone number n

$$n \leq n_{max} = 2M^2, \tag{33}$$

which corresponds to $E_{n_{max}} = M$. On the other hand, we recall that, by using the Generalized Uncertainty Principle, Adler, Chen and Santiago [29] have shown that the total BH evaporation is prevented in exactly the same way that the Uncertainty Principle prevents the hydrogen atom from total collapse. In fact, the collapse is prevented, not by symmetry, but by dynamics, as the *Planck distance* and the *Planck mass* are approached [29]. That important result implies that eq. (32) has to be slightly modified, becoming (the *Planck mass* is equal to 1 in Planck units)

$$M^2 - \frac{n}{2} \geq 1. \tag{34}$$

Thus, one gets a slightly different value of the maximum value of the overtone number n

$$n \leq n_{max} = 2(M^2 - 1). \tag{35}$$

Then, the countable sequence of QNMs for emitted energies cannot be infinity although n can be extremely large [2]. In fact, restoring ordinary units and considering a BH mass of the order of 10 solar masses, one easily gets $n_{max} \sim 10^{76}$. On the other hand, we expect further corrections to our semi-classical analysis when the *Planck scale* is approached, as we need a full theory of quantum gravity to obtain a correct description of the Planck scale's physics. Here the value of n_{max} has been correct with respect to the value that we found in [2]

5 Implications on the area quantization

Setting $n_1 = n - 1$, $n_2 = n$ in eq. (30) on gets the emitted energy for a jump among two neighboring levels

$$\Delta E_{n-1 \rightarrow n} = \sqrt{M^2 - \frac{n+1}{2}} - \sqrt{M^2 - \frac{n}{2}}. \tag{36}$$

Bekenstein [30] has shown that the Schwarzschild BH area quantum is $\triangle A = 8\pi$ (the *Planck length* $l_p = 1.616 \times 10^{-33}$ cm is equal to one in Planck units). As for the Schwarzschild BH the *horizon area* A is related to the mass through the relation $A = 16\pi M^2$, a variation $\triangle M$ in the mass generates a variation

$$\triangle A = 32\pi M \triangle M \tag{37}$$

in the area. Using eqs. (28) and (36) and putting $\triangle M = -\Delta E_{n-1 \rightarrow n}$ (emission) one gets

$$\triangle A_{n-1} \equiv -32\pi M_{n-1} \Delta E_{n-1 \rightarrow n}. \tag{38}$$

Eq. (38) should give the area quantum of an excited BH for an emission from the level $n-1$ to the level n in function of the quantum number n and of the initial BH mass. Actually, we find a problem using eq. (38). In fact, an absorption from the level n to the level $n-1$ is now possible, with an absorbed energy

$$\Delta E_{n \rightarrow n-1} = -\Delta E_{n-1 \rightarrow n} = \sqrt{M^2 - \frac{n}{2}} - \sqrt{M^2 - \frac{n+1}{2}}. \tag{39}$$

In that case, one sets $\triangle M = -\Delta E_{n \rightarrow n-1} = \Delta E_{n-1 \rightarrow n}$ and the quantum of area should be

$$\triangle A_n \equiv -32\pi M_n \Delta E_{n \rightarrow n-1} = 32\pi M_n \Delta E_{n-1 \rightarrow n}. \tag{40}$$

Then, the absolute value of the area quantum for an absorption from the level n to the level $n-1$ is different from the absolute value of the area quantum for an emission from the level $n-1$ to the level n because $M_{n-1} \neq M_n$. The problem is solved if one considers the *effective mass* corresponding to the transitions between the two levels n and $n-1$, which is the same for emission and absorption

$$\begin{array}{c} M_{E(n,\ n-1)} \equiv \frac{1}{2}\left(M_{n-1}+M_{n}\right) \\ \frac{1}{2}\left(\sqrt{M^{2}-\frac{n-1}{2}}+\sqrt{M^{2}-\frac{n}{2}}\right). \end{array} \tag{41}$$

Replacing M_{n-1} with $M_{E(n,\ n-1)}$ in eq. (38) and M_n with $M_{E(n,\ n-1)}$ in eq. (40) we obtain

$$\begin{array}{ll} \triangle A_{n-1} \equiv -32\pi M_{E(n,\ n-1)}\Delta E_{n-1\rightarrow n} & \textit{emission} \\ \triangle A_{n} \equiv -32\pi M_{E(n,\ n-1)}\Delta E_{n\rightarrow n-1} & \textit{absorption}, \end{array} \tag{42}$$

and now one gets $|\triangle A_n| = |\triangle A_{n-1}|$. By using eqs. (39) and (41) one finds immediately

$$|\triangle A_n| = |\triangle A_{n-1}| = 8\pi. \tag{43}$$

Thus, eq. (43) retrieves the famous result of Bekenstein on the area quantization [30], and this *cannot* be a coincidence. It is a confirmation of the correctness of the present analysis instead.

Putting $A_{n-1} \equiv 16\pi M_{n-1}^2$, $A_n \equiv 16\pi M_n^2$, the formulas of the number of quanta of area can be written down as

$$N_{n-1} \equiv \frac{A_{n-1}}{|\triangle A_{n-1}|} = \frac{16\pi M_{n-1}^{2}}{32\pi M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} = \frac{M_{n-1}^{2}}{2M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} \tag{44}$$

before the emission, and

$$N_{n} \equiv \frac{A_{n}}{|\triangle A_{n}|} = \frac{16\pi M_{n}^{2}}{32\pi M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} = \frac{M_{n}^{2}}{2M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} \tag{45}$$

after the emission respectively. One can easily check that

$$N_{n}-N_{n-1} = \frac{M_{n}^{2}-M_{n-1}^{2}}{2M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} = \frac{\Delta E_{n-1\rightarrow n}\left(M_{n-1}+M_{n}\right)}{2M_{E(n,\ n-1)}\cdot\Delta E_{n-1\rightarrow n}} = 1, \tag{46}$$

as one expects. Thus, the famous formula of Bekenstein-Hawking entropy [1, 32, 33] reads

$$\left(S_{BH}\right)_{n-1} \equiv \frac{A_{n-1}}{4} = 8\pi N_{n-1}M_{n-1}\cdot\Delta E_{n-1\rightarrow n} = 4\pi\left(M^{2}-\frac{n+1}{2}\right) \tag{47}$$

before the emission and

$$\left(S_{BH}\right)_{n} \equiv \frac{A_{n}}{4} = 8\pi N_{n}M_{n}\cdot\Delta E_{n-1\rightarrow n} = 4\pi\left(M^{2}-\frac{n}{2}\right) \tag{48}$$

after the emission respectively. Then, we find the intriguing result that Bekenstein-Hawking entropy is a function of the QNMs principal quantum number, i.e. of

the BH quantum level. Again, we correct and improve our previous results in [2] which strictly hold only for scalar and gravitational perturbations.
On the other hand, it is a general belief that there is no reason to expect that Bekenstein-Hawking entropy will be the whole answer for a correct quantum gravity theory [8]. For a better understanding of BH entropy we need to go beyond Bekenstein-Hawking entropy and identify the sub-leading corrections [8]. Using the quantum tunnelling approach one obtains the sub-leading corrections to the second order approximation [9]. In this approach BH entropy contains three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and the inverse area term [9]

$$S_{total} = S_{BH} - \ln S_{BH} + \frac{3}{2A}. \tag{49}$$

Apart from a coefficient, this correction to BH entropy is consistent with loop quantum gravity [9], where the coefficient of the logarithmic term has been rigorously fixed at $\frac{1}{2}$ [9, 31]. In this way, the formulas of the total entropy that takes into account the sub-leading corrections to Bekenstein-Hawking entropy become

$$\begin{array}{c}(S_{total})_{n-1} = 4\pi\left(M^2 - \frac{n-1}{2}\right)\\ -\ln\left[4\pi\left(M^2 - \frac{n-1}{2}\right)\right] + \frac{3}{32\pi\left(M^2 - \frac{n-1}{2}\right)}\end{array} \tag{50}$$

before the emission, and

$$\begin{array}{c}(S_{total})_{n} = 4\pi\left(M^2 - \frac{n}{2}\right)\\ -\ln\left[4\pi\left(M^2 - \frac{n}{2}\right)\right] + \frac{3}{32\pi\left(M^2 - \frac{n}{2}\right)}\end{array} \tag{51}$$

after the emission, respectively. Thus, also the total BH entropy results a function of the BH excited state n. Eqs. (50) and (50) permit to write immediately the number of micro-states

$$\begin{array}{c}g(N_{n-1}) \propto \exp\{4\pi\left(M^2 - \frac{n-1}{2}\right)\\ -\ln\left[4\pi\left(M^2 - \frac{n-1}{2}\right)\right] + \frac{3}{32\pi\left(M^2 - \frac{n-1}{2}\right)}\end{array} \tag{52}$$

before the emission, and

$$\begin{array}{c}g(N_{n}) \propto \exp\{4\pi\left(M^2 - \frac{n}{2}\right)\\ -\ln\left[4\pi\left(M^2 - \frac{n}{2}\right)\right] + \frac{3}{32\pi\left(M^2 - \frac{n}{2}\right)},\end{array} \tag{53}$$

after the emission, respectively.

We stress that our results are in perfect agreement with existing results in the literature. In fact, as we consider large n, it is $\Delta E_{n-1\to n} \approx \frac{1}{4M}$ [2, 3]. Thus, if one neglects the difference between the original BH mass and the residual mass

M_n, i.e. $M_n \simeq M$ (which was the approximation that we used in [3, 4]), the Bekenstein-Hawking entropy reads ($n \approx n-1$ and $N_n \approx N_{n-1} \equiv N$)

$$S_{BH} = \frac{A}{4} = 8\pi NM \cdot \Delta E_{n-1\to n}, \tag{54}$$

which is consistent with the standard result [7, 34–36]

$$S_{BH} \to 2\pi N. \tag{55}$$

Again, the consistence with well known and accepted results *cannot* be a coincidence, but it is a confirmation of the correctness of the current analysis instead. Then, the total entropy reads

$$S_{total} = 8\pi NM \cdot \Delta E_{n-1\to n} - \ln\left[8\pi NM \cdot \Delta E_{n-1\to n}\right] + \frac{3}{64\pi NM \cdot \Delta E_{n-1\to n}}, \tag{56}$$

which is well approximated by

$$S_{total} \simeq 2\pi N - \ln 2\pi N + \frac{3}{16\pi N}. \tag{57}$$

Now, let us explain the way in which our Bohr-like model for BHs works. Let us consider a BH original mass M. After an emission from the ground state to a state with large $n-1$, or, alternatively, after a certain number of emissions (and potential absorptions as the BH can capture neighboring particles), the BH is at an excited level $n-1$ and its mass is $M_{n-1} \equiv M - E_{n-1}$ where E_{n-1} is the absolute value of the frequency of the QNM associated to the excited level $n-1$. We recall again that E_{n-1} is interpreted as the total energy emitted at that time [2]. The BH can further emit an energy to jump to the subsequent level: $\Delta E_{n-1\to n} = E_n - E_{n-1} = M_{n-1} - M_n$. Now, the BH is at an excited level n and the BH mass is

$$\begin{aligned} M_n &\equiv M - E_{n-1} - \Delta E_{n-1\to n} = \\ &= M - E_{n-1} + E_{n-1} - E_n = M - E_n. \end{aligned} \tag{58}$$

The BH can, in principle, return to the level $n-1$ by absorbing an energy $\Delta E_{n\to n-1} = -\Delta E_{n-1\to n}$. We have also shown that the quantum of area is *the same* for both absorption and emission, given by eq. (43), as one expects.

There are three different physical situations for excited BHs ($n \gg 1$):

i) n is large, but not enough large. It is also $E_n \ll M_n \simeq M$ and one can use eqs. (54), (56) which result a better approximation than eqs. (55), (57) which were used in previous literature in strictly thermal approximation [7, 34–36].

ii) n is very much larger than in point 1, but before arriving at the Planck scale. In that case, it can be $E_n \lesssim M$, while $M_n \simeq M$ does not hold and one must use the eqs. (47), (48), (50) and (51).

iii) At the Planck scale n is larger also than in point ii), we need a full theory of quantum gravity .

6 Some important consequences

Our Bohr-like model for BHs has important implications for the BH information paradox [37]. In fact, this paper completes our previous results [2–4], confirming that BH QNMs are really the BH quantum levels in our Bohr-like semi-classical approximation. This point implies that BHs are well defined quantum mechanical systems, having ordered, discrete quantum spectra, in perfect agreement with the unitarity of the underlying quantum gravity theory and with the idea that information should come out in BH evaporation. Consistence between our Bohr-like model and a recent approach to solve the BH information paradox [25–28] has been recently highlighted in [27]. Thus, the general conviction that BHs result in highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in an unitary theory of quantum gravity is in perfect agreement with our Bohr-like model.

Another key point, which is again connected with the information puzzle, is the following. In Hawking's original computation [1] if emission can occur for a quantum of energy E, then it can also occur for any other quantum of energy bE, where b is a continuous real parameter between 0 and $\frac{M}{E}$, where M is the BH mass. After emission of a quantum of energy bE, the BH radial coordinate is determined continuously by the continuous parameter b. In other words, emissions of Hawking quanta looks completely random. The situation looks to be similar within the semi-classical context in which Parikh-Wilczek perform their calculation [10, 11]. But here there is an important difference. It is indeed important to recall that in the approach in [10, 11] the tunnelling is a *discrete* instead of *continuous* process [3, 16]. In fact, two different *countable* BH physical states must be considered, the physical state before the emission of the particle and the physical state after the emission of the particle [3, 16]. Thus, the emission of the particle can be interpreted like a *quantum transition* of frequency ω between the two discrete states [3, 16]. In the language of the tunnelling mechanism, a trajectory in imaginary or complex time joins two separated classical turning points [10, 11]. The fundamental consequence is that the radiation spectrum is now *discrete in time* [3, 16]. Let us clarify this important issue in a better way. At a well fixed Hawking temperature and the statistical probability distribution (2) are continuous functions. On the other hand, the Hawking temperature in (2) varies in time with a character which is *discrete*. In fact, the forbidden region traversed by the emitting particle has a *finite* size [11]. Considering a strictly thermal approximation, the turning points have zero separation. Therefore, it is not clear what joining trajectory has to be considered because there is not barrier [11]. The problem is solved if we argue that the forbidden finite region from $r_{initial} = 2M$ to $r_{final} = 2(M-\omega)$ that the tunnelling particle traverses works like barrier [11]. Thus, the intriguing explanation is that it is the particle itself which generates a tunnel through the horizon [11]. The discrete behavior in time of the radiation spectrum implies the countable character of the subsequent emitted Hawking quanta and, in turn, the correspondence between the countable perturbations generated by the absorbed negative energies and the BH QNMs. The fundamental consequence is that, differently on Hawking's original computation [1], now emissions of Hawking quanta are *not* completely random. They are indeed governed by eq. (30). In fact, let us consider an emission from the BH ground state to a state with large n. After that, using eq. (33) (although we recall that the last area quantum corresponds to the final Planck mass which is prevented to evaporate by the Generalized Uncertainty Principle, see Section 4 and ref. [29]), one see that the BH will have a finite and discrete number of potential emissions given by

$$n_{max} - n = 2M^2 - n. \tag{59}$$

It is enlightening to observe that such a number of potential residual emissions, which is equal to the residual number of QNMs, is also equal to the residual number of area quanta. In fact, by using eq. (28) and recalling that $r_H = 2M$ one easily compute the area of the BH excited at level n as

$$A_n = 16\pi M_n^2 = 16\pi \left(M^2 - \frac{n}{2}\right), \tag{60}$$

which, dividing for the Bekenstein's area quantum $|\triangle A_n| = 8\pi$ [30], that we retreived in eq. (43), gives the number of area quanta for the BH excited at level n

$$N_n = 2M^2 - n. \tag{61}$$

Thus, we understand that the key point is exactly Bekenstein's idea on area quantization [30], i.e. as for large n the BH area is quantized, the BH can emit *only* energies which are consistent with such a quantization. In other words, emissions of Hawking quanta are not completely random because the BH can emit only energies which corresponds to reductions of its area which are multiples of the Bekenstein's are quantum $|\triangle A_n| = 8\pi$ given by eq. (43). Hence, our results are completely consistent with the idea that the Schwarzschild spacetime is quantized around the BH core.

7 Conclusion remarks

In this paper we have shown that the intuitive but general conviction that BHs result in highly excited states representing both the "hydrogen atom" and the "quasi-thermal emission" in quantum gravity is more than a picture, discussing a model of quantum BH somewhat similar to the historical semi-classical model of the structure of a hydrogen atom introduced by Bohr in 1913. In the model the absolute values of the QNMs frequencies represent the energy "shells", or, in other words, the "electrons" of quantum gravity which jump from a level to another. In Bohr model [5, 6] electrons can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation with an energy difference of the levels according to the Planck relation $E = hf$. In our BH model, QNMs can only gain and lose energy by jumping from one allowed energy shell to another, absorbing or emitting radiation (emitted radiation is given by Hawking quanta) with an energy difference of the levels according to eq. (30). The similarity between the two models is completed if one notes that the interpretation of eq. (27) is of a particle, the "electron", quantized on a circle of length given by eq. (31).

The model is in perfect agreement with existing results in the literature [7, 34–36], starting from the famous result of Bekenstein on the area quantization $|\triangle A_n| = 8\pi$ [30].

This paper improves, clarifies and finalizes some recent results that, also together with collaborators, we published in various peer reviewed journals [2-4], [16-19] and has important consequences on the BH information puzzle and on the non-strictly random character of Hawking radiation.

A fundamental feature of the Bohr-like model that we analysed in this paper is the discreteness of the BH horizon area as the function of the QNMs principal quantum number, which is consistent with various models of quantum gravity where the spacetime is considered fundamentally discrete [43, 44].

Preliminary results on the Bohr-like BH model in this paper have been recently discussed in an Invited Lecture at the 12th International Conference of Numerical Analysis and Applied Mathematics [42].

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