Derivation of the Equation $E = mc^2$ from the Heisenberg Uncertainty Principles

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Abstract

The present paper is concerned with the derivation of the Einstein's formula of equivalence of mass and energy, $E = mc^2$ *, from the Heisenberg uncertainty relations. Thus, this paper unifies two of the most important laws of physics as provides the proof of the quantum mechanical nature of the above famous formula.*

Keywords: *Heisenberg uncertainty principle, spatial uncertainty principle, temporal uncertainty principle, annihilation, Planck force, Newton's law of universal gravitation.*

1. Introduction

Since the development of quantum mechanics, it has been suggested that all laws of physics should have their origins in this formulation. However, the Einstein's equation of equivalence of mass and energy resisted all efforts to frame it under the new quantum theory.

In 1927 Werner Heisenberg proposed two fundamental relations that would revolutionize quantum mechanics. They are known as the Heisenberg uncertainty relations or principles. These relations are:

1. The Heisenberg momentum-position uncertainty relations (or spatial uncertainty principle):

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{1.1a}$$

$$\Delta p_y \Delta y \geq \frac{\hbar}{2} \tag{1.1b}$$

$$\Delta p_z \Delta z \geq \frac{\hbar}{2} \tag{1.1c}$$

and

2. the Heisenberg energy-time uncertainty relation (or temporal uncertainty principle):

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{1.2}$$

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The uncertainty relations are a consequence of assuming that a particle can be described by wave packet. Based on these relations, and considering the results of particle-antiparticle annihilation experiments, the Planck force and the Newton law of universal gravitation, we shall derive the equation of equivalence of mass and energy

$$E = m c^2 \tag{1.3}$$

The purpose of using the Planck force and the Newton's law of universal gravitation is to determine the value of the real constant *k* I shall introduce in the next section.

2. From Heisenberg Relations to Einstein's Equation

Let us consider the spatial uncertainty principle given by relation (1.1a)

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{2.1a}$$

and the temporal uncertainty principle given by relation (1.2)

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{2.1b}$$

We can make reasonably good estimates by writing the above relations as approximations

$$\Delta p_x \Delta x \approx \frac{\hbar}{2} \tag{2.2a}$$

$$\Delta E \Delta t \approx \frac{\hbar}{2} \tag{2.2b}$$

Later we shall turn these approximations into equalities. But first, let us outline the strategy we shall follow. Because we apply these two uncertainty principles to the same particle, at the same time, the uncertainty in the position of the particle, Δx , must be related to the uncertainty in time Δt. In other words we shall assume that the uncertainty in the speed of the particle is given by the ratio between the uncertainties in position and time. Then, according to experimental observations we shall consider that the maximum uncertainty in this velocity is the speed of light in vacuum, *c*. This in turns means that the maximum uncertainty in the energy of the particle will be the maximum energy of the particle itself.

To start the derivation let us divide the expression (2.2a) by the expression (2.2b)

$$\frac{\Delta p_x \Delta x}{\Delta E \Delta t} \approx 1 \tag{2.3}$$

$$\Delta p_x \frac{\Delta x}{\Delta t} \approx \Delta E \tag{2.4}$$

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Let us assume that the uncertainty Δx is the distance a particle could have travelled in a given time, Δt, thus the uncertainty in the velocity of this particle, $v_\Delta \equiv \Delta v_x$, will be given by

$$\Delta v_x = v_\Delta = \frac{\Delta x}{\Delta t} \tag{2.5}$$

From relations (2.4) and (2.5) we can write

$$\Delta p_x \Delta v_x \approx \Delta E \tag{2.6}$$

but

$$\Delta p_x = m \Delta v_x \tag{2.7}$$

Substituting Δp_x in expression (2.6) with the second side of expression (2.7) we get

$$m \Delta v_x \Delta v_x \approx \Delta E \tag{2.8}$$

$$m (\Delta v_x)^2 \approx \Delta E \tag{2.9}$$

Now we shall assume that the maximum physically possible uncertainty in the velocity of the particle is the speed of light, *c*. Thus, we take the limit on both sides of expression (2.9) when Δv_x tends to the speed of light. Mathematically we express these limits as follows

$$\lim_{\Delta v_x \to c} m (\Delta v_x)^2 \approx \lim_{\Delta v_x \to c} \Delta E \tag{2.10}$$

Because the mass of the particle, *m,* does not depends on the speed uncertainty, we can write

$$m \lim_{\Delta v_x \to c} (\Delta v_x)^2 \approx \lim_{\Delta v_x \to c} \Delta E \tag{2.11}$$

$$m c^2 \approx \lim_{\Delta v_x \to c} \Delta E \tag{2.12}$$

Now we have to find the meaning of the limit of the second side of equation (2.12). To do this we observe that, on the first side of this expression, we have assumed that the speed of light, *c*, is the maximum uncertainty in the velocity of the particle. Therefore to be consistent with this assumption (which is equivalent to being consistent with the first side of the relation), the uncertainty ΔE must be the maximum uncertainty in the energy of the particle (the maximum uncertainty in the speed of the particle will produce the maximum uncertainty in its energy and vice versa. See Appendix 1, case 3 for a more detailed explanation). From particle-antiparticle annihilation experiments we know that a particle has a maximum energy, *E.* (this is so because we know that during annihilation the total mass of each particle of the pair disappears during the process. We are based on an experimental fact since we assume that we don't know the Einstein's equation of equivalence of mass and energy). Thus ΔE must be equal to the total energy, *E*, we obtain from a particle when the particle is annihilated. Mathematically

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$$\Delta E = E \tag{2.13}$$

Combining equations (2.12) and (2.13) and swapping sides we obtain the following approximate expression

$$E \approx m c^2 \tag{2.14}$$

Now we need to transform this approximation into an equation. We know, from thousand of annihilation experiments, that if the kinetic energy of the particle before annihilation increases, the energy of the corresponding photons after annihilation increase proportionally. Therefore we can write

$$E = k m c^2 \tag{2.15}$$

Now all we need to do is to find the value of the constant *k*. In order to do that we shall derive the Newton's law of universal gravitation from equation (2.15) (including the constant *k*) and from the Planck force. The derivation, which is shown in **Appendix 3,** produces the gravitation law if and only if the value of *k* is either 1 or -1. We shall adopt *k* = 1 (See note 1). **Appendix 2** contains the derivation of the Planck force which is used in Appendix 3. Thus, in the light of the result of Appendix 3, we can write equation (2.15) as follows

(Equation 2.15 with *k* = 1)

$$E = m c^2 \tag{2.16}$$

We need to consider two cases:

Case 1) The particles are at rest before annihilation.
Case 2) The particles are not at rest before annihilation.

Case 1) Particles are at rest before annihilation

If the particles are at rest before annihilation, then equation (2.16) becomes

$$E_{min} = m_{min} c^2 \tag{2.17}$$

where m_{min} is the minimum mass of the particle. Because this mass is the mass of the particle when the particle is at rest, this mass must be the rest mass, m_0 , of the particle so we can write

$$m_0 = m_{min} \tag{2.18}$$

Consequently E_{min} must be the rest energy, E_0 , of the particle. So we can write

$$E_0 = E_{min} \tag{2.19}$$

Thus, the special case of equation (2.16) occurs when the particle is at rest ($m = m_0$ and $E = E_0$). Mathematically

$$E_0 = m_0 c^2 \tag{2.20}$$

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Case 2) Particles are not at rest before annihilation
If the particles are at not rest before annihilation (this means that the particles have kinetic energy) then the total mass of each particle will be greater that their corresponding rest masses and therefore the total energy of each particle will be greater than its rest energy. Based on particle-antiparticle annihilation experiments we know that the greater the kinetic energy of the particles before annihilation the greater the energy of the photons produced after annihilation. So equation (2.16) must hold for all possible values of the kinetic energies of the particles involved in the annihilation process. Based on these experimental facts we deduce that the energy *E* in equation (2.16) must represent the total energy of the particle of mass *m.* As a consequence, we also deduce that the mass of the particle, *m*, must depend on the velocity of the particle (group velocity). It is worthy to remark that despite the fact that we have assumed that we do not know the formula for the relativistic mass, we have found that the mass of the particle must depend on its velocity. Thus we have proved that the general formula that governs the transformation of mass into energy and vice versa is:

$$E = m c^2 \tag{2.21}$$

which is Einstein's famous formula of equivalence of mass and energy.

Note 1

The analysis suggests that the complete form of Einstein's formula should be

$$E = \pm m c^2$$

The minus sign suggests that a particle with positive mass could posses negative energy. This conclusion is supported by Einstein's formula for the total relativistic energy

$$E = \pm \sqrt{p^2 c^2 + m_0^2 c^4}$$

and also by the formula for the relativistic mass

$$m = \frac{m_0}{\pm \sqrt{1 - \frac{v^2}{c^2}}}$$

Which, in addition, suggests the existence of negative mass.
These equations support the existence of negative energy black holes [1].

3. Conclusions

This paper shows that the Einstein's equation of equivalence of mass and energy (eq. 2.21) is a special case of the Heisenberg uncertainty relations when the uncertainty in the velocity of the particle equals the speed of light. The conventional derivation of the formula of equivalence of mass and energy is based on the relativistic mass law

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$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

, the work-energy theorem and on Newton's second law of motion. This derivation can be found in most books dealing with special relativity. It is worthy to remark that we derived Einstein's equation without using the above law. So we have also proved that this law, along with the other above mentioned laws, are sufficient but not necessary to derive the famous formula of equivalence of mass and energy.

One point that could be criticized is that this research is based not only on first principles but also on annihilation experiments. However this is also the case of other theories. Einstein's special theory of relativity, for example, is based on the results of experiments (Michelson and Morley) which proved that the speed of light in vacuum is independent of the motion of the light source (a postulate known as: the invariance of *c*). Einstein adopted this experimental result as one of his postulates to formulate his remarkable and revolutionary theory.

Two conclusions emerge from this paper:

1. the special theory of relativity turned out to be based on quantum mechanical principles (the Heisenberg uncertainty relations), which means that two of the most important laws of physics have now been unified.

2. the Heisenberg uncertainty relations underline a deeper truth than that of the Einstein's equation of equivalence of mass and energy.

In summary, this paper shows the full glory of Heisenberg uncertainty relations.

REFERENCES

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Appendix 1

The Three Cases of the Uncertainty Relations

The uncertainty principle allows the following cases

Case 1

If the momentum and the energy of a particle are known with total accuracy:

$$\Delta p = \Delta E = 0$$

then the position and duration are completely undefined:

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$$\Delta x = \Delta t = \infty$$

Case 2

If the momentum and the energy of a particle are completely undefined:

$$\Delta p = \Delta E = \infty$$

Then the position and duration are known with total accuracy:

$$\Delta x = \Delta t = 0$$

Case 3

Between cases 1 and 2 there are an infinite number of possible cases in which the uncertainties are greater than zero and finite. For example:

$$\Delta p = \Delta p_1$$

$$\Delta E = \Delta E_1$$

$$\Delta x = \Delta x_1$$

$$\Delta t = \Delta t_1$$

where

$$0 < \Delta p_1 < \infty \quad , \quad 0 < \Delta E_1 < \infty \quad , \quad 0 < \Delta x_1 < \infty \quad \text{and} \quad 0 < \Delta t_1 < \infty$$

Due to physical considerations (as in this research paper) we could have maximum values for the uncertainties of the momentum and the corresponding uncertainties of time. For example

$$\Delta p = \Delta p_{max}$$

$$\Delta E = \Delta E_{max}$$

$$\Delta x = \Delta x_{min}$$

$$\Delta t = \Delta t_{min}$$

where

Δp_{max} = maximum possible value of the uncertainty in the momentum (See Note).
ΔE_{max} = maximum possible value of the uncertainty in the energy (See Note).
Δx_{min} = minimum possible value of the uncertainty in the position.
Δt_{min} = minimum possible value of the uncertainty in the duration.

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Note. The maximum possible values of the above uncertainties (Δp_{max} and ΔE_{max}) are imposed by the laws of physics, such as the maximum speed of material particles ($v < c$).

Appendix 2

Derivation of the Planck Force

In this appendix I derive the expression for the Planck force through two different methods:
a) Derivation of the Planck force from Coulomb's law, and
b) Derivation of the Planck force from the Newton's second law of motion

a) Derivation of the Planck Force from Coulomb's Law

I shall derive the Planck force from Coulomb's law

$$F_E = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \qquad \text{(Coulomb's law)} \qquad \text{(A2a.1)}$$

We shall substitute both charges q_1 *and* q_2 with the Planck charge Q_P and the distance r with the Planck length L_P, this yields

$$F_P = \frac{Q_P^2}{L_P^2} \qquad \text{(A2a.2)}$$

Where

$$Q_P \equiv \sqrt{2\varepsilon_0 h c} \qquad \text{(Planck charge)} \qquad \text{(A2a.3)}$$

$$L_P \equiv \sqrt{\frac{hG}{2\pi c^3}} \qquad \text{(Planck length)} \qquad \text{(A2a.4)}$$

$$F_P = \frac{1}{4\pi \varepsilon_0} \left(\frac{2\varepsilon_0 h c}{\frac{hG}{2\pi c^3}} \right) \qquad \text{(A2a.5)}$$

Finally the Planck force is

$$F_P = \frac{c^4}{G} \qquad \text{(Planck Force)} \qquad \text{(A2a.6)}$$

b) Derivation of the Planck Force from Newton's Second Law

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of Motion

I shall derive the Planck force from Newton's second law of motion

$$F = ma \qquad \text{(Newton's second law of motion)} \qquad \text{(A2b.1)}$$

The Planck force can also be defined as

$$F_P \equiv M_P a_P \qquad \text{(A2b.2)}$$

Where

F_P = Planck force
M_P = Planck mass
a_P = Planck acceleration

The Planck mass is defined as

$$M_P \equiv \sqrt{\frac{hc}{2\pi G}} \qquad \text{(Planck mass)} \qquad \text{(A2b.3)}$$

The Planck acceleration can be defined as

$$a_P \equiv \frac{c}{T_P} \qquad \text{(A2b.4)}$$

Where

c = speed of light in vacuum
T_P = Planck time

$$T_P \equiv \sqrt{\frac{hG}{2\pi c^5}} \qquad \text{(Planck time)} \qquad \text{(A2b.5)}$$

In equation (A2b.4) we substitute T_P with the second side of equation (A2b.5) . This yields

$$a_P = c\sqrt{\frac{2\pi c^5}{hG}} \qquad \text{(A2b.6)}$$

$$a_P = \sqrt{\frac{2\pi c^7}{hG}} \qquad \text{(Planck acceleration)} \qquad \text{(A2b.7)}$$

in equation (A2b.2) we substitute M_P and a_P with the second side of equations (A2b.3) and (A2b.7) respectively yields

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$$F_P = \sqrt{\frac{h\,c}{2\pi\,G}\,\frac{2\pi\,c^7}{hG}} \qquad \text{(A2b.8)}$$

Finally the expression for the Planck force is

$$F_P = \frac{c^4}{G} \qquad \text{(Planck Force)} \qquad \text{(A2b.9)}$$

Appendix 3
Derivation of Newton's Law of Universal Gravitation

In this section I shall derive Newton's law of universal gravitation from the scale law, the Planck force and the equation $E = k\,mc^2$. In 1687 Isaac Newton published his Principia where he introduced his universal law of gravitation through the following equation

$$F_G = G\frac{m_1 m_2}{r^2} \qquad \text{(Newton's law of universal gravitation)} \qquad \text{(A3.1)}$$

Where

F_G = Gravitational force between two any bodies of masses m_1 and m_2 (this force is also known as force of universal gravitation, gravity, gravity force, force of gravitational attraction, force of gravity, Newtonian force of gravity, force of universal mutual gravitation, etc.)
G = Gravitational constant (also known as constant of gravitation, constant of gravity, gravitational force constant, universal constant of gravity, universal gravitational constant, Newtonian gravitational constant, etc.)
m_1 = mass of body 1
m_2 = mass of body 2
r = distance between the centres of body 1 and body 2

We start the analysis by observing that the numerator of the Planck force equation (A2b.9) [or equation (A2a.6)] is c^4. This suggests that the equation we wish to derive for should originate from the product of two (relativistic) energies such as $E_1 = k\,m_1 c^2$ and $E_2 = k\,m_2 c^2$. We also notice that the c^4 factor of the product $k^2 m_1\,m_2 c^4$ should cancel out with the numerator of equation (A2b.9). Thus, taking into consideration these facts, we draw the following scale table

Work	Work	Energy	Energy
W_G	W	E_1	E_2

***Table 1**: This scale table is used to derive Newton's law of universal gravitation.*

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In summary, the quantities shown in Table 1 must be defined as follows

$W_G \equiv F_G\, r$ (Work done by F_G) (A3.2)

$W \equiv F_P\, r$ (Work done by F_P) (A3.3)

$E_1 \equiv k\, m_1 c^2$ (energy of body 1) (A3.4)

$E_2 \equiv k\, m_2 c^2$ (energy of body 2) (A3.5)

From the table we establish the following relationship

$$W_G W = S\, E_1\, E_2 \tag{A3.6}$$

where *S* is a real scale factor. It is worthy to observe that we could have postulated equation (A3.6) directly without drawing any table. This table is only a visual tool to help us to establish relationships. Replacing the variables W_G, W, E_1 and E_2 by equations (A3.2), (A3.3), (A3.4) and (A3.5) respectively, we get

$$F_G r\, F_P r = S\, k^2\, m_1 c^2\, m_2 c^2 \tag{A3.7}$$

$$F_G = S\, k^2 \frac{G}{c^4} \frac{1}{r^2} m_1\, m_2 c^4 \tag{A3.8}$$

$$F_G = S\, k^2 G \frac{m_1\, m_2}{r^2} \tag{A3.9}$$

The last equation turns into the Newton's law of universal gravitation (see equation A3.1) if and only if $S = 1$ and $k = \pm 1$. Thus, with these values of *S* and *k,* we get equation (A3.1):

$$F_G = G \frac{m_1\, m_2}{r^2} \tag{A3.10}$$

Conclusion

The Newton's law of universal gravitation can be derived from the proposed mass-energy equation: $E = k\, mc^2$ if and only if the value of the constant *k* is either 1 or -1.

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