Derivation of the Equation $E = m c^2$ from the Heisenberg Uncertainty Principles
(Draft version)

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Abstract

The present paper is concerned with the derivation of the Einstein's equation of equivalence of mass and energy from the Heisenberg uncertainty relations.

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1. Introduction

In 1927 Werner Heisenberg proposed two fundamental relations that would revolutionize quantum mechanics. They are known as the Heisenberg uncertainty relations or principles. These relations are:

1. The momentum-position uncertainty principle (or spatial uncertainty principle)

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{1.1a}$$

$$\Delta p_y \Delta y \geq \frac{\hbar}{2} \tag{1.1b}$$

$$\Delta p_z \Delta z \geq \frac{\hbar}{2} \tag{1.1c}$$

and

2. the Heisenberg energy-time uncertainty principle (or temporal uncertainty principle):

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{1.2}$$

The uncertainty relations are a consequence of assuming that particles can be described by wave packets. Based on these relations we shall derive the equation of equivalence of mass and energy

$$E = m c^2 \tag{1.3}$$

2. From Heisenberg Relations to Einstein's Equation

Let us consider the spatial uncertainty principle given by relation (1.1a)

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{2.1a}$$

and the temporal uncertainty principle given by relation (1.2)

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{2.1b}$$

We can make reasonably good estimates by writing the above relations as approximations

$$\Delta p_x \Delta x \approx \frac{\hbar}{2} \tag{2.2a}$$

$$\Delta E \Delta t \approx \frac{\hbar}{2} \tag{2.2b}$$

Later we shall turn these approximations into equalities. But first, let us outline the strategy we shall follow. Because we apply these two uncertainty principles to the same particle, at the same time, the uncertainty in the position of the particle, Δx, must be related to the uncertainty in time Δt. In other words we shall assume that the uncertainty in the speed of the particle is given by the ratio between the uncertainties in position and time. Then, according to experimental observations we shall consider that the maximum uncertainty in this velocity is the speed of light in vacuum, *c*. This in turns means that the maximum uncertainty in the energy of the particle will be the maximum energy of the particle itself.

To start the derivation let us divide the expression (2.2a) by the expression (2.2b)

$$\frac{\Delta p_x \Delta x}{\Delta E \Delta t} \approx 1 \tag{2.3}$$

$$\Delta p_x \frac{\Delta x}{\Delta t} \approx \Delta E \tag{2.4}$$

Let us assume that the uncertainty Δx is the distance a particle could have travelled in a a given time, Δt, thus the uncertainty in the velocity of this particle, $v_\Delta \equiv \Delta v_x$, will be given by

$$\Delta v_x = v_\Delta = \frac{\Delta x}{\Delta t} \tag{2.5}$$

From relations (2.4) and (2.5) we can write

$$\Delta p_x \Delta v_x \approx \Delta E \tag{2.6}$$

but

$$\Delta p_x = m \Delta v_x \tag{2.7}$$

Substituting Δp_x in expression (2.6) with the second side of expression (2.7) we get

$$m \Delta v_x \Delta v_x \approx \Delta E \tag{2.8}$$

$$m (\Delta v_x)^2 \approx \Delta E \tag{2.9}$$

Now we shall assume that the maximum physically possible uncertainty in the velocity of the particle is the speed of light, *c*. Thus, we take the limit on both sides of expression (2.9) when Δv_x tends to the speed of light. Mathematically we express these limits as follows

$$\lim_{\Delta v_x \to c} m (\Delta v_x)^2 = \lim_{\Delta v_x \to c} \Delta E \tag{2.10}$$

Note that, in the limit, the approximate equal sign will turn into an equality. Assuming that the mass of the particle, *m,* does not depends on the uncertainty in the speed, we can write

$$m \lim_{\Delta v_x \to c} (\Delta v_x)^2 = \lim_{\Delta v_x \to c} \Delta E \tag{2.11}$$

$$m c^2 = \lim_{\Delta v_x \to c} \Delta E \tag{2.12}$$

Now we have to find the meaning of the limit of the second side of equation (2.12). To do this we observe that, on the first side of this expression, we have assumed that the speed of light, *c*, is the maximum uncertainty in the velocity of the particle. Therefore to be consistent with this assumption (which is equivalent to being consistent with the first side of the relation), the uncertainty ΔE must be the maximum uncertainty in the energy of the particle (the maximum uncertainty in the speed of the particle will produce the maximum uncertainty in its energy and viceversa. See Appendix 1, case 3 for a more detailed explanation). From particle/anti-particle annihilation we know that a particle has a maximum energy, *E.* (this is so because we know that during annihilation the total mass of each particle of the pair disappears during the process. We are based on an experimental fact since we assume that we don't know anything about Einstein's equivalence of mass and energy). Thus ΔE must be equal to the energy, *E*, we obtain from a particle when the particle is annihilated. Mathematically

$$\Delta E = E \tag{2.13}$$

Combining equations (2.12) and (2.13) and swapping sides we obtain the famous Einstein's equation of equivalence of mass and energy

$$E = m c^2 \tag{2.14}$$

3. Conclusions

In summary, this development has shown that the Einstein's equation of equivalence between mass and energy is a special case of the Heisenberg uncertainty relations ((1.1) and (1.2)) when the uncertainty in the velocity of the particle equals the speed of light.

Appendix 1

The uncertainty principle allows the following cases

Case 1

If the momentum and the energy of a particle are known with total accuracy:

$$\Delta p = \Delta E = 0$$

then the position and duration are completely undefined:

$$\Delta x = \Delta t = \infty$$

Case 2

If the momentum and the energy of a particle are completely undefined:

$$\Delta p = \Delta E = \infty$$

Then the position and duration are known with total accuracy:

$$\Delta x = \Delta t = 0$$

Case 3

Between cases 1 and 2 there are an infinite number of possible cases in which the uncertainties are greater than zero and finite. For example:

$$\Delta p = \Delta p_1$$

$$\Delta E = \Delta E_1$$

$$\Delta x = \Delta x_1$$

$$\Delta t = \Delta t_1$$

where

$0 < \Delta p_1 < \infty$, $0 < \Delta E_1 < \infty$, $0 < \Delta x_1 < \infty$ and $0 < \Delta t_1 < \infty$

Due to physical considerations (as in this research paper) we could have maximum values for the uncertainties of the momentum and the corresponding uncertainties of time. For example

$$\Delta p = \Delta p_{max}$$

$$\Delta E = \Delta E_{max}$$

$$\Delta x = \Delta x_{min}$$

$$\Delta t = \Delta t_{min}$$

where

Δp_{max} = maximum possible value of the uncertainty in the momentum (See Note).
ΔE_{max} = maximum possible value of the uncertainty in the energy (See Note).
Δx_{min} = minimum possible value of the uncertainty in the position.
Δt_{min} = minimum possible value of the uncertainty in the duration.

Note. The maximum possible values of the above uncertainties (Δp_{max} and ΔE_{max}) are imposed by the laws of physics, such as the maximum speed of material particles ($v < c$).