In/Equivalence of Klein-Gordon and Dirac Equation

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Abstract. It will be proven that Klein-Gordon and Dirac equation, when defined on an F-space of distributions, have the same set of solutions, which makes the two equations equivalent on that vector space of distributions.
Some consequences of this for quantum field theory are shortly discussed.

1. Klein-Gordon And Dirac Equation

I assume $\hbar \equiv 1$ and $c \equiv 1$ throughout, denote with $x = (x_0, \ldots, x_3) \in \mathbb{R}^4$ a point in space-time, where x_0 is the time coordinate, and base the Minkowsi metric tensor g on the signature $(+, -, -, -)$. With this convention, the Klein-Gordon is:

$$\Box\Psi := (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\Psi = m^2\Psi, \tag{1.1}$$

and the Dirac equation is given as a matrix equation by

$$(i\gamma_0\partial_0 - \cdots - i\gamma_3\partial_3)\Psi = m\Psi, \tag{1.2}$$

where $\gamma_0, \ldots \gamma_3$ are anticommuting 4×4-matrices satisfying $\gamma_0^2 = -\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = \mathbb{1}_4$, with $\mathbb{1}_4$ denoting the 4×4-unit matrix. The Dirac matrices are a possible representation of these matrices. (As a reference I refer to [4], [9], [2], [3], or any good book on quantum field theory.)

2. Definition of the F-space

The operator $\gamma_0\partial_0 - \cdots - \gamma_3\partial_3$ is called *Dirac-operator*. Rather than to discuss it in terms of Hilbert spaces, where it is unbounded and non-selfadjoint, I prefer a super space X, say, on which this operator is a linear, continuous mapping.

Definition 2.1. Let $Cl_{1,3}(\mathbb{C})$ denote the complex algebra generated by the $\gamma_0, \dots, \gamma_3$, which by itself is a finite-dimensional, in fact 16-dimensional, vector space, hence a Banach space (when equipped with its natural norm). An F-space (or Frechet space) is defined as a metrizable, complete toplogical vector space (see: [6]). The space $\mathcal{S}(\mathbb{R}^4)$ of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^4$ into $\mathbb{C}$ is such an F-space, and so its strong dual $\mathcal{S}'(\mathbb{R}^4)$, called space of tempered distributions on $\mathbb{R}^4$, is (see also: [6]). Then the tensor product $X := Cl_{1,3}(\mathbb{C}) \otimes \mathcal{S}'(\mathbb{R}^4)$ of the finite-dimensional Clifford algebra with the space of tempered distribution is an F-space.
Let $D := i(\gamma_0\partial_0 - \dots - \gamma_3\partial_3) : X \to X$. Then D is a continuous, linear operator on X. Let $kern(D) \subset X$ be the kernel of D and $ran(D)$ its range.

Because the origin $\{0\} \subset X$ is a closed subspace of X, its preimage $kern(D) \subset X$ is closed in X, and the quotient space $X/kern(D)$ is a vector space.
Now, given any metric $d : X \times X \ni (x, y) \mapsto d(x, y) \in [0, \infty)$ defining the topology of X,

$$\hat{d} : (X/kern(D)) \times (X/kern(D)) \ni (x, y) \mapsto \min_{s,t \in kern(D)} d(x+s, y+t) \in [0, \infty)$$

is a metrics on $X/kern(D)$, for which the canonical projection $\pi : X \ni x \mapsto x + kern(D) \in X/kern(D)$ is a continuous surjection.
As an F-space, X is a Baire space (see: [6]); consequently, the open mapping theorem holds (see: [6] or [7]), and the canonical projection π is open. Therefore, $X = ran(D) \oplus kern(D)$ is the topological direct sum of $ran(D)$ and $kern(D)$, and the restriction $D|_{ran(D)}$ of D to $ran(D)$ is a continuous and open isomorphism. It follows that $kern(D^2) = kern(D)$. But on X: $D^2 = -\Box = -(\partial_0^2 - \dots - \partial_3^2)$.
Finally, for $\Psi \in X$ and $m \in \mathbb{R}$: $(D-m)\Psi = 0 \Rightarrow \Box\Psi = (D-m)(D+m)\Psi = 0$, so the set $\{\Psi \in X : D\Psi = m\Psi\}$ is contained in $\{\Psi \in X : \Box\Psi = m^2\Psi\}$.
It's the converse that poses the problem: Applying the open-mapping theorem to $D+m$ and $D-m$, we conclude that $(\Box - m^2)\Psi = 0$ and $(D-m)\Psi = 0$ have the same solutions if and only if $kern(D-m) = kern(D+m)$, which is apparently not the case.
However, look again at the definition of the matrices γ_μ: They are defined as anti-commuting 4×4-matrices for which $\gamma_0^2 = -\gamma_1^2 = \dots - \gamma_3^2 = \mathbb{1}_4$ holds, so the γ_μ, and therefore D, are defined up to unitary equivalence; that is: for each unitary transformation $U : \mathbb{C}^4 \to \mathbb{C}^4$, $\gamma'_\mu := U^{-1}\gamma_\mu U$, $(0 \le \mu \le 3)$, is equivalent to γ_μ, $(0 \le \mu \le 3)$. In other words, D is to be defined modulo $U(4)$, where $U(4)$ stands for the group of unitary transformations on $\mathbb{C}^4$.
Now, with $U = \gamma_5 := i\gamma_0 \cdots \gamma_3$, we have $U^{-1}\gamma_\mu U = -\gamma_\mu$, $(0 \le \mu \le 3)$, which maps $kern(D-m)$ unitarily onto $kern(D+m)$ and therefore proves that the set of solutions of $D^2\Psi = m^2\mathbb{1}_4\Psi$ and $D\Psi = m\mathbb{1}_4\Psi$ are identical. And, two equations, which share the same set of solutions are (mathematically) equivalent! So, we proved the equivalence of the Klein-Gordon equation with the Dirac equation.

3. Demystifying the Dirac operator

The Fourier transformation $\mathcal{F} : \mathcal{S}(\mathbb{R}^4) \ni f \mapsto \hat{f} := (2\pi)^{-2} \int_{\mathbb{R}^4} f(x) e^{ix \cdot k} d^4x \in \mathcal{S}(\mathbb{R}^4)$ is known to be a topological isomorphism (see: [6]). Its dual therefore is a topological isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^4) \ni T \mapsto \hat{T} \in \mathcal{S}'(\mathbb{R}^4)$, and it maps $T \mapsto DT$ into $\hat{T}_k \mapsto (2\pi)^{-2}(\gamma_0 k_0 - \cdots - \gamma_3 k_3)\hat{T}_k$. That reduces the whole eigenvalue problem to an algebraic problem within the finite dimensional algebra $Cl_{1,3}(\mathbb{C})$:
By writing the scalar $p_0^2 - \cdots - p_3^2 = m^2$ as a matrix equation $(p_0^2 - \cdots p_3^2)\mathbb{1}_4 = m^2 \mathbb{1}_4$, the unitary group $U(4)$ is added as a global symmetry on top of the scalar equation, all solutions of that matrix equation are unique up to $U(4)$-equivalence, and every choice of an orthonormal basis $\Psi_1, \ldots, \Psi_4 \in \mathbb{C}^4$ is arbitrary up to unitary equivalence. There is no absolute meaning of Ψ_1 to signify a spin-up and positive energetic state of a particle, wheras Ψ_4 has to be spin-down state of negative energy: $\Psi_1, \ldots, \Psi_4$ just form an orthonormal basis of $\mathbb{C}^4$! Plus, we saw that these γ-matrices are not just a sophisticated quantum theoretical artefact: as purely algebraic mathematical objects, they are part of any relativistic dynamical system!
They even introduce the notion of "state" into classical physics, because with each normal element $\Psi_1 \in \mathbb{C}$ and each real-valued λ, $e^{i\lambda}\Psi_1$ would equivalently do the job! In other words: the notion of phase invariance is introduced to all of relativistic physics along with the γ_μ! (Not just by chance, classical electromagnetic waves do exhibit phase invariance.)

4. The Consequences

Relativistic field theory always starts from a Lagrangian, which is a functional $\mathcal{L} : \phi(x) \mapsto \mathcal{L}(t, x, \phi(x), \partial_0\phi(x), \ldots, \partial_3\phi(x), \ldots) \in \mathbb{C}$, where $x = (x_0, \ldots, x_3) \in \mathbb{R}^4$ is supposed to represent the space-time coordinates (see: [9]), $\phi : \mathbb{R}^4 \to X$, and X is some topological vector space, which can be $\mathbb{C}^n$ for some $n \in \mathbb{N}$ as in classical field theory or a space of unbounded linear operators on a Hilbert space, as is the case for quantum field theory, and the dots in the argument of $\mathcal{L}$ stand for additional, optional higher derivatives of ϕ (and/or time derivatives of the space coordinates) as arguments.
Given $\mathcal{L}$, its corresponding Euler-Lagrange equations yield the equations of motions, and the symmetries of $\mathcal{L}$ yield the invariant quantities by means of Noether's theorem (see: [9] or [4]).
That makes the Lagrangian the final point of truth for the appropriate physical model under consideration.

Now, as long as different Lagrangians deliver inequivalent equations of motion, it might not be considered a flaw to guess at the Lagrangian from the equations of motion, which it defines. However, should it turn out that two distinct Lagrangians deliver equivalent equations, the whole process of the Lagrangian-centric approach will become questionable.

The above proven equivalence of Klein-Gordon and Dirac equation just gives us an example of two distinct Lagrangians that have equivalent equations of motions:
Citing [9, p.162 and p.218ff.], $\mathcal{L} = \bar{\phi}(\gamma^\mu i\partial_\mu - m)\phi$ yields the Euler-Lagrange equation $i\gamma^\mu\partial_\mu\phi = m\phi$, which is the Dirac equation; on the other hand, the Euler-Lagrange equation for the Lagrangian $\mathcal{L} = (1/2)(\partial_\mu\phi)(\partial^\mu\phi) - (1/2)m^2\phi^2$ is $\partial_\mu\partial^\mu\phi^* = m^2\phi^*$, which is the Klein-Gordon equation (see: [9, p. 218]).

Relativistic (quantum) field theory now is in need to explain, why one or the other Lagrangian is to be chosen as the "correct one". Moreover, there might be even simpler Lagrangians that deliver equivalent equations, and it is to expect that the simplest candidate will also yield precious intuitive physical insight!
Just to rely on any Lagrangian with suitable Euler-Lagrange equations and to make it more and more complex by adding more and more intricated symmetrical terms, does not do: This proves no better than appending more and more epicycles to the Ptolemaic model: It adds confidence through a better approximation, but nothing more!

If at all physics is to survive with future generations, then undoubtly the current relativistic quantum field theory will be assembled anew on firmer grounds than of today. The question is only when.

References

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