Knot formation in an entropic 2+1 dimensional space

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Abstract

The formation of a trefoil knot in a measurement space with entropy is described. It is shown that for a given prime knot, invariants with the characteristics of Laurent polynomials can be developed in 2+1 dimensional measurement space. These polynomials distinguish chiral property and uniquely address charge, parity and time symmetries.

Introduction

The role of topology in explaining the structure formation in the high-energy physics is becoming more and more prominent. It is required that we first establish a fundamental topological structure before developing the description in *(t, x, y, z)* physical space. The knot theory is one of the instruments which allows us to visualize the possible topological states a system may take over a period of time. What started as a suggestion by Lord Kelvin and consequently an extensive work done by Tait on characterizing the various types of knots, has evolved into a very powerful mathematical field which has the potential to unlock the fundamental principles guiding the formation of the elementary structures [1-4].

Mathematically the knot structures are algebraically represented by knot-invariants, which are polynomials which remain unchanged for the equivalent knots. The earliest example of such polynomials is Alexander polynomial. The other well-known examples are Alexander-Conway polynomial, Jones polynomial, and HOMFLY polynomial. We must remember that these knot polynomials represent a three-dimensional topology. Conventionally we develop a mathematical structure for a physical knot based on certain mathematical rules, determine the knot invariants and correlate them to the probability amplitudes. A desirable quality of these knot-invariants is to make distinction between the knots based on properties such as chirality as was shown by Jones polynomial. Clearly more are the properties distinguished by a knot-invariant more robust is the polynomial.

In the following work we take a slightly different approach. Rather than assuming the existence of a knot, we form a knot step-by-step in a space characterized by entropy. The polynomial is written in 2+1 dimensional measurement space instead of conventional $\mathbb{R}^3$ space. Please note that 2+1 dimensional space in the context of knots, is not the 2+1 dimensional Minkowski space. Since the entropy is being considered, the order in which the moves are being made while forming the knot becomes important. We will consider the most elementary of the prime knots, the trefoil knot and develop its corresponding polynomials.

Knot formation

We consider a source represented by an infinite information space where a string is formed of the measurements made by an observer of finite capacity, who is trying to determine the nature of the source. In such case the entropy of the system will be continuously increasing and process as being measured, will be irreversible. However since the observer has a finite capacity, the process as being measured is reversible but only within a quantum '*h'*, defined by the observer's resources or its environment. (Please note that *'h'* is defined as a quantum for illustrative purposes only and it does not represent Planck's constant.)

Let us discuss how prime knots will be formed in such space and how they will be represented algebraically. The progress of measured information ***I*** in a discrete measurement space along the time-axis can be described as in Eqn. 1,

$$I(t = 0^{+}) = I(t = 0) \times e^{1-q}. \tag{1}$$

In Equation 1, *q* is a natural number (*q = 1, 2, 3 ...*), *I(t=0)* represents the information available for the initial state (t = 0), and $I(t = 0^{+})$ represents the information available at a later instant, ($t = 0^{+}$). A knot is formed when an observer cannot measure a state in a single measurement (single measurement represents zero entropy as $\log_e 1 = 0$). In that case more than one measurement is needed and the entropy comes into the picture. We consider large q values where a statistical phenomenon is established. In that case Eqn. 1, is modified to,

$$I(t = 0^{+}) = I(t = 0) \times e^{-q}. \quad (2)$$

We postulate that a knot is formed and a polynomial is created only when a measurement is made by an observer. A finite capacity observer is always in a discrete measurement space. If a state is below the observer's capacity threshold we define it as ε, where ε represents an indeterminate state for a finite capacity observer. The knot formation in a discrete measurement space is shown in Fig. 1.

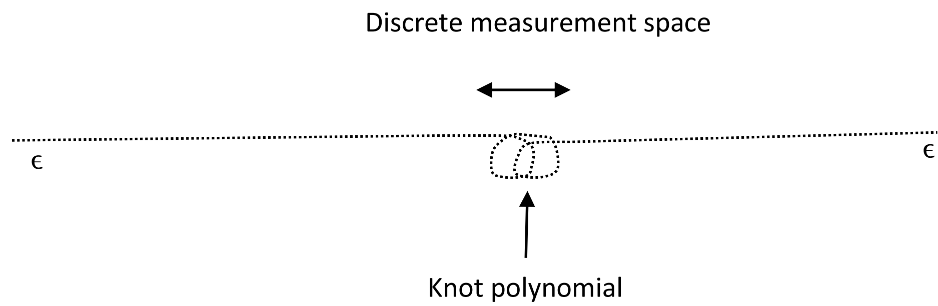

Figure-1: The knot formation in a discrete measurement space.

Trefoil knot polynomials

A trefoil knot as formed in a discrete measurement space for a given q-value, is shown in Fig. 2. It is a 2-dimensional plane with third dimension representing the motion within the quantum *'h'* along the time-axis. We have the following properties characterizing the knot,

i. The motion along time-axis is in positive (over-crossing) and negative (under-crossing) directions both. This motion is along the perpendicular to the plane of the page.
ii. The anticlockwise (ACW) or clockwise (CW) motion in the plane of the page. It represents the orientation of the knot at each crossing.

iii. The order in which the crossing are approached is important. For example in Fig. 2, U2 and O3 cannot occur before O1. Similarly O3 cannot occur before U2.

iv. The nature of the first crossing defines the knot as positive or negative. For example if the first crossing of the knot is over-crossing (O1), the knot is classified as positive knot. Similarly if the first crossing of the knot is under-crossing (U1), the knot is classified as negative knot.

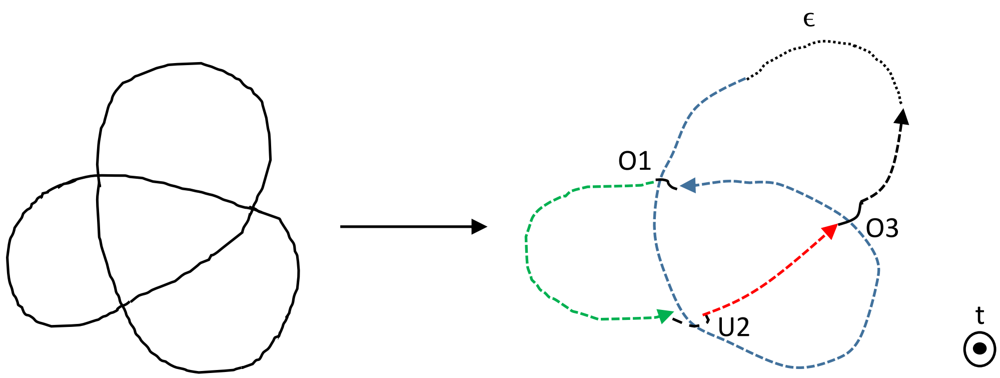

Figure-2: The representation of a positive ACW trefoil knot in a discrete measurement space.

The polynomial for ACW positive trefoil knot is calculated as shown in Table-1.

Crossing description	Sectional polynomial	Complete polynomial
Crossing 0	0_j	0_j
Crossing O1	$0_j \times e^{-h/2}$	$0_j+0_je^{-h/2}$
Crossing U2	$0_j(1+e^{-h/2})\times e^{h/2} = 0_j(e^{h/2}+1)$	$(0_j+0_je^{-h/2})+0_j(e^{h/2}+1)$ $= 20_j+0_je^{-h/2}+0_je^{h/2}$
Crossing O3	$(20_j+0_je^{-h/2}+0_je^{h/2})\times e^{-h/2}$ $= 0_j(2e^{-h/2}+e^{-h}+1)$	$(20_j+0_je^{-h/2}+0_je^{h/2})+0_j(2e^{-h/2}+e^{-h}+1)$ $= 0_j(3+3e^{-h/2}+e^{h/2}+e^{-h})$

Table-1: The polynomial calculation for positive ACW trefoil knot.

In calculating the polynomial the ACW orientation is considered positive per trigonometric conventions. The sectional polynomial represents the contribution of the specific crossing which is then added to the contributions from the previous sections. The polynomial is completed when a knot is formed. We define the polynomial variable as $t = e^{-h}$. The polynomial for a positive ACW trefoil knot can be written from the Table-1 as, $3 + 3t^{\frac{1}{2}} + t^{-\frac{1}{2}} + t$. The normalized polynomial for a positive trefoil knot is, $1 + \frac{1}{3}t^{-\frac{1}{2}} + t^{\frac{1}{2}} + \frac{1}{3}t$. The coefficients of the polynomial take integer or fractional values. The polynomial variable t represents the variation of the information within the quantum '*h*'.

We can next consider the reflection of positive ACW trefoil knot, which will be positive CW trefoil knot (Fig. 3). In this case the orientation direction of the knot changes from ACW to CW while the nature of over- and under-crossing and their respective order do not change. The plane of reflection can be either vertical or horizontal. In either case only change is in the orientation of the knot.

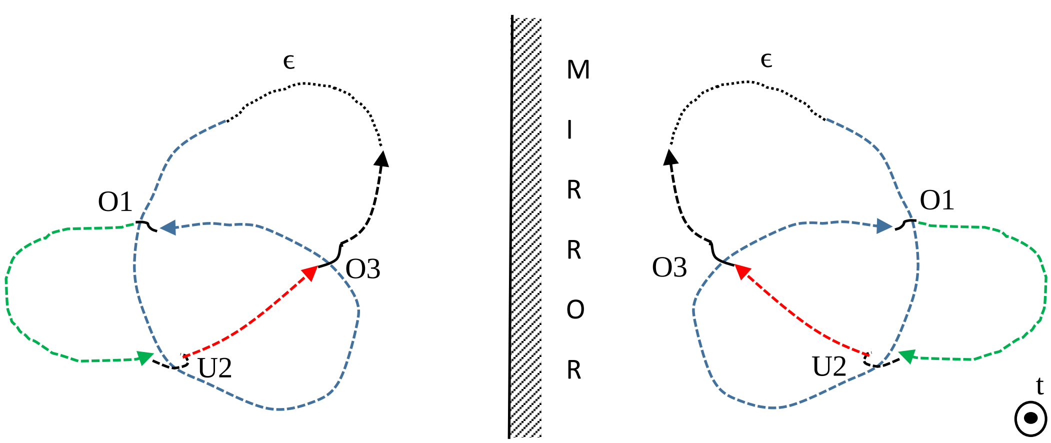

Figure-3: Positive ACW and CW trefoil knots.

The calculation of polynomial in this case is shown in Table-2. The CW orientation is negative per trigonometric convention.

Crossing description	Sectional polynomial	Complete polynomial
Crossing 0	0_j	0_j
Crossing O1	$0_j\times(-e^{-h/2})$	$0_j-0_je^{-h/2}$
Crossing U2	$0_j(1-e^{-h/2})\times(-e^{h/2}) = 0_j(-e^{h/2}+1)$	$0_j-0_je^{-h/2}+0_j(-e^{h/2}+1)$ $= 20_j-0_je^{-h/2}-0_je^{h/2}$
Crossing O3	$(20_j-0_je^{-h/2}-0_je^{h/2})(-e^{-h/2})$ $= 0_j(-2e^{-h/2}+e^{-h}+1)$	$20_j-0_je^{-h/2}-0_je^{h/2}+0_j(-2e^{-h/2}+e^{-h}+1)$ $= 0_j(3-3e^{-h/2}-e^{h/2}+e^{-h})$

Table-2: The polynomial calculation for positive CW trefoil knot.

The polynomial for positive CW trefoil knot can be written as $3 - 3t^{1/2} - t^{-1/2} + t$ where $t = e^{-h}$. We can similarly write the corresponding polynomials for negative ACW and CW knots. The normalized polynomials for all four cases i.e. positive ACW, positive CW, negative ACW, and negative CW trefoil knots are summarized in the Table-3.

Trefoil knot type	Corresponding Polynomial
Positive ACW	$1+\frac{1}{3}t^{-\frac{1}{2}}+t^{\frac{1}{2}}+\frac{1}{3}t$
Positive CW	$1-\frac{1}{3}t^{-\frac{1}{2}}-t^{\frac{1}{2}}+\frac{1}{3}t$
Negative ACW	$1+t^{-\frac{1}{2}}+\frac{1}{3}t^{\frac{1}{2}}+\frac{1}{3}t^{-1}$
Negative CW	$1-t^{-\frac{1}{2}}-\frac{1}{3}t^{\frac{1}{2}}+\frac{1}{3}t^{-1}$

Table-3: Knot polynomials for a trefoil knot in a discrete measurement space.

Discussion

For a trefoil knot four algebraic combinations are available, with the “alternating crossing” and the “order of crossing” constraints as shown in Table-3. The structures are clearly distinguishable from each other as each polynomial representation is unique. These polynomials have the characteristics of Laurent polynomials. They can be thought of a state (positive trefoil knot), its anti-state (negative trefoil knot) and their respective reflections (CW and ACW). We note that in order for a knot to form, over- and under-crossings must be alternated. A minimum of three crossings (over- and under- combined) are required.

The description of the structure is in 2+1 dimensional plane. The effect of entropy in the observer's environment is accounted for by ensuring that the crossings are ordered and '*h*' is defined in 1-dimensional plane. The quantum '*h*' signifies the interaction or the measurement made by an observer and its value represents the observer's capacity. The physical description of a knot for the observer making the measurements will be developed in the orientation plane which is 2-dimensional. The 1-dimensinal and 2-dimensional planes for a knot are orthogonal to each other.

Charge (state and anti-state, '*h'*), Parity (reflection along an axis or equivalently ACW or CW knot orientation) and Time (order of the crossings, '*h'*) symmetries are addressed within the same polynomial. Chiral property can be distinguished as the polynomials representing ACW and CW orientations for positive and negative knots are unique.

We can write a generalized algebraic expression for the normalized polynomials corresponding to the trefoil knot as in equation-3.

$$1 + s_1.K_1 t^{-\frac{1}{2}} + s_2.K_2 t^{\frac{1}{2}} + K_3 t^{-1} + K_4 t \quad . \qquad -(3)$$

In Eqn. 3 the coefficients *(K_1, K_2, K_3, K_4)* take the values $(\frac{1}{3}, 1, 0, \frac{1}{3})$ for positive trefoil knot, and values $(1, \frac{1}{3}, \frac{1}{3}, 0)$ for negative trefoil knot. The coefficients s_1 and s_2 take the values (1, 1) and (-1, -1) for ACW and CW orientations of the trefoil knot.

We can write an additional set of the normalized polynomials for positive and negative trefoil knots for the values equal to *(1, -1)* and *(-1, 1)* for the coefficients *(s_1, s_2)* as shown in eqn. 4. The values of the coefficients *(K_1, K_2, K_3, K_4)* are left unchanged.

$$1 + \frac{1}{3} t^{-\frac{1}{2}} - t^{\frac{1}{2}} + \frac{1}{3} t$$

$$1 - \frac{1}{3} t^{-\frac{1}{2}} + t^{\frac{1}{2}} + \frac{1}{3} t \qquad - \quad (4)$$

$$1 + t^{-\frac{1}{2}} - \frac{1}{3} t^{\frac{1}{2}} + \frac{1}{3} t^{-1}$$

$$1 - t^{-\frac{1}{2}} + \frac{1}{3} t^{\frac{1}{2}} + \frac{1}{3} t^{-1}$$

We therefore have a set of eight normalized polynomials for a trefoil knot in a discrete measurement space. The physical interpretation of the polynomials represented by Eqn. 4 is yet to be ascertained.

It should be noted that the 2-dimensional knot-orientation plane is essentially 1-dimensional if the knot orientation values are taken to be +1 for ACW and -1 for CW, as we have done so far. The description in this case is 1+1 dimensional. However we will require a 2-dimensional plane because in a discrete measurement space while forming a knot a finite capacity observer cannot retrace its path exactly and therefore a loop is formed. For example an observer with lower capacity, is likely to form a much larger loop in the knot orientation plane compared to the loop formed by a higher capacity observer. Therefore a 2-dimensional orientation plane is required for the comparison of the observers.

Summary

The development of the knot polynomials in an entropic 1+2 dimensional discrete measurement space for a trefoil knot is described. The polynomial variable represents the capacity of the observer making the measurements. The polynomials which are of Laurent nature, are able to uniquely describe charge, parity and time symmetries. The methodology described can be used to develop polynomials for higher order knot structures.

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The aim of this work is to develop a fundamental mathematical structure which can be used for physical description in (t,x,y,z) space. The knot invariants provide such structures. The knot polynomials are calculated in a measurement space which accounts for the entropy inherent in a physical process. We are able to develop polynomials for the most basic of the prime knots, known as trefoil knot. These polynomials describe the basic symmetries such as charge, parity, and time. We expect to calculate polynomials for higher order knots.