Limit theorems for $k$-subadditive lattice group-valued capacities in the filter convergence setting

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Abstract

We investigate some properties of lattice group-valued positive $k$-subadditive set functions, and in particular we give some comparisons between regularity and continuity from above. Moreover we prove different kinds of limit theorems in the non-additive case with respect to filter convergence, in which it is supposed that the involved filter is diagonal.

Definitions 0.1 (a) Given a free filter $\mathcal{F}$ of $\mathbb{N}$, we say that a subset of $\mathbb{N}$ is $\mathcal{F}$-stationary iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of $\mathbb{N}$.

(b) A free filter $\mathcal{F}$ of $\mathbb{N}$ is said to be diagonal iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that $J \setminus A_n$ is finite for all $n \in \mathbb{N}$

Let $R$ be a Dedekind complete lattice group, $G$ be any infinite set, $\Sigma$ be a $\sigma$-algebra of subsets of $G$, and $k$ be a fixed positive integer.

Definitions 0.2 (a) A capacity $m : \Sigma \to R$ is a set function, increasing with respect to the inclusion and such that $m(\emptyset) = 0$.

(b) A capacity $m$ is said to be $k$-subadditive on $\Sigma$ iff

$$m(A \cup B) \leq m(A) + km(B) \quad \text{whenever } A, B \in \Sigma, A \cap B = \emptyset.$$  

\[ (1) \]

(c) We say that a capacity $m$ is continuous from above at $\emptyset$ iff

\[ (O) \lim_n m(H_n) = \bigwedge_n m(H_n) = 0 \]

whenever $(H_n)_n$ is a decreasing sequence in $\Sigma$ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$.

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(d) A capacity \( m \) is \( k\)-\( \sigma \)-subadditive on \( \Sigma \) iff

\[
m\left( \bigcup_{n=1}^{\infty} E_n \right) \leq m(E_1) + k \sum_{n=2}^{\infty} m(E_n)
\]

for any sequence \((E_n)_n\) from \( \Sigma \).

**Proposition 0.3** Let \( m : \Sigma \to R \) be a \( k \)-subadditive capacity, continuous from above at \( \emptyset \). Then \( m \) is \( k\)-\( \sigma \)-subadditive.

**Definitions 0.4**

(a) A capacity \( m : \Sigma \to R \) is said to be **continuous from above** (resp. **below**) iff

\[
m(E) = (O) \lim_{n} m(E_n) = \bigwedge_{n} m(E_n)
\]

(resp. \( m(E) = (O) \lim_{n} m(E_n) = \bigvee_{n} m(E_n) \))

whenever \((E_n)_n\) is a decreasing (resp. increasing) sequence in \( \Sigma \), with \( E = \bigcap_{n=1}^{\infty} E_n \) (resp. \( E = \bigcup_{n=1}^{\infty} E_n \)).

(b) A capacity \( m : \Sigma \to R \) is **(s)-bounded** on \( \Sigma \) iff there exists an \((O)\)-sequence \((\sigma_p)_p\) such that for each \( p \in \mathbb{N} \) and for every disjoint sequence \((C_h)_h\) in \( \Sigma \) there is a positive integer \( h_0 \) with \( m(C_h) \leq \sigma_p \) whenever \( h \geq h_0 \).

(c) Let \( \tau \) be a Fréchet-Nikodým topology on \( \Sigma \). A capacity \( m : \Sigma \to R \) is said to be \( \tau \)-**continuous** on \( \Sigma \) iff for each decreasing sequence \((H_n)_n\) in \( \Sigma \), with \( \tau\)-\( \lim H_n = \emptyset \), we get

\[
(O) \lim_{n} m(H_n) = (O) \bigwedge_{n} m(H_n) = 0.
\]

(d) Let \( \mathcal{G}, \mathcal{H} \subset \Sigma \) two lattices, such that \( \mathcal{H} \) is closed under countable unions, and the complement of every element of \( \mathcal{H} \) belongs to \( \mathcal{G} \). We say that a capacity \( m : \Sigma \to R \) is **regular** iff for every \( E \in \Sigma \) there are two sequences \((F_n)_n\) in \( \mathcal{H} \) and \((G_n)_n\) in \( \mathcal{G} \), with

\[
F_n \subset F_{n+1} \subset E \subset G_{n+1} \subset G_n \quad \text{for any } n,
\]

and \((O) \lim_{n} m(G_n \setminus F_n) = (O) \bigwedge_{n} m(G_n \setminus F_n) = 0 \).

The next result links continuous from above at \( \emptyset \) and regularity of capacities.

**Theorem 0.5** Let \( R \) be a Dedekind complete weakly \( \sigma \)-distributive lattice group, \((G,d)\) be a compact metric space, \( \Sigma \) be the \( \sigma \)-algebra of all Borel sets of \( G \), \( \mathcal{G} \) and \( \mathcal{H} \) be the lattices of all open and all compact subsets of \( G \) respectively. Then every \( k \)-subadditive regular capacity \( m : \Sigma \to R \) is continuous from above at \( \emptyset \).

Conversely, if \( R \) is also super Dedekind complete, then every \( k \)-subadditive capacity \( m : \Sigma \to R \), continuous from above at \( \emptyset \), is regular.
We now give the following limit theorems for non-additive lattice group-valued capacities with respect to filter convergence.

**Theorem 0.6** Let \( \mathcal{F} \) be a diagonal filter of \( \mathbb{N} \), \( m_j : \Sigma \to R \), \( j \in \mathbb{N} \), be an equibounded sequence of \( k \)-subadditive capacities, such that \( m_0(E) := (O\mathcal{F}) \lim m_j(E) \) exists in \( R \) for every \( E \in \Sigma \), \( m_0 \) is continuous from above at \( \emptyset \) and \( m_j \) is \( (s) \)-bounded on \( \Sigma \) for every \( j \geq 0 \).

If \( R \subset C_{\infty}(\Omega) \) is as in the Maeda-Ogasawara-Vulikh representation theorem, then for every \( I \in \mathcal{F}^* \) and for each disjoint sequence \( (C_h)_h \in \Sigma \) there exist a set \( J \subset I \), \( J \in \mathcal{F}^* \), and a meager set \( N \subset \Omega \) with

\[
(O) \lim_h \left( \bigvee_{j \in J} m_j(C_h) \right) = 0
\]
and

\[
\lim_h (\sup_{j \in J} m_j(C_h)(\omega)) = 0 \quad \text{for each } \omega \in \Omega \setminus N.
\]

**Theorem 0.7** Let \( R, \Omega, \mathcal{F} \) be as in Theorem 0.6, \( m_j : \Sigma \to R \), \( j \in \mathbb{N} \), be an equibounded sequence of \( k \)-subadditive capacities. Assume that \( m_0(E) := (O\mathcal{F}) \lim m_j(E) \) exists in \( R \) for every \( E \in \Sigma \).

Then for every \( I \in \mathcal{F}^* \) and for each decreasing sequence \( (H_n)_n \) in \( \Sigma \) with

\[
(O) \lim_n m_j(H_n) = \bigwedge_n m_j(H_n) = 0 \quad \text{for every } j \geq 0
\]
there are a set \( J \subset I \), \( J \in \mathcal{F}^* \), and a meager set \( N^* \subset \Omega \) with

\[
\lim_n (\sup_{j \in J} m_j(H_n)(\omega)) = \inf_n (\sup_{j \in J} m_j(H_n)(\omega)) = 0
\]
and

\[
(O) \lim_n \left( \bigvee_{j \in J} m_j(H_n) \right) = \bigwedge_n \left( \bigvee_{j \in J} m_j(H_n) \right) = 0.
\]

**Theorem 0.8** Let \( \mathcal{F}, R, \Omega, k, G, \Sigma \) be as in Theorem 0.7, \( \tau \) be a Fréchet-Nikodým topology on \( \Sigma \), \( m_j : \Sigma \to R \), \( j \in \mathbb{N} \), be an equibounded sequence of \( k \)-subadditive capacities, \( \tau \)-continuous (resp. continuous from above at \( \emptyset \)) on \( \Sigma \). Let \( m_0(E) := (O\mathcal{F}) \lim m_j(E) \) exist in \( R \) for every \( E \in \Sigma \), and suppose that \( m_0 \) is \( \tau \)-continuous (resp. continuous from above at \( \emptyset \)) on \( \Sigma \).

Then for every \( I \in \mathcal{F}^* \) and for each decreasing sequence \( (H_n)_n \) in \( \Sigma \), with \( \tau \lim_n H_n = \emptyset \) (resp. \( \bigcap_{n=1}^{\infty} H_n = \emptyset \)), there exist a set \( J \subset I \), \( J \in \mathcal{F}^* \), and a meager set \( N \subset \Omega \), satisfying (7) and (8).

**Theorem 0.9** Let \( \mathcal{F}, R, \Omega, G, \Sigma \) be as in Theorem 0.7, \( G, \mathcal{H} \subset \Sigma \) be two lattices, such that the complement of every subset of \( \mathcal{H} \) belongs to \( \mathcal{G} \), and \( \mathcal{H} \) is closed under countable unions. Let \( m_j : \Sigma \to R \), \( j \in \mathbb{N} \), be a sequence of \( k \)-subadditive regular capacities, such that \( m_0(E) = (O\mathcal{F}) \lim m_j(E) \) for
any $E \in \Sigma$ and $m_0$ is regular. Then we get:

(R3) for every $E \in \Sigma$ and $I \in \mathcal{F}^*$ there are $J \in \mathcal{F}^*$, $J \subset I$, and two sequences $(F_n)_n$ in $\mathcal{H}$, $(G_n)_n$ in $\mathcal{G}$, satisfying (3) and with

$$(O) \lim_n \left( \bigvee_{j \in J} m_j(G_n \setminus F_n) \right) = \bigwedge_n \left( \bigvee_{j \in J} m_j(G_n \setminus F_n) \right) = 0,$$

and furthermore there exists a meager set $N \subset \Omega$ with

$$(O) \lim_n \left( \sup_{j \in J} m_j(G_n \setminus F_n)(\omega) \right) = \inf_n \left( \sup_{j \in J} m_j(G_n \setminus F_n)(\omega) \right) = 0$$

for each $\omega \in \Omega \setminus N$. 
