Conjecture that states that a Fermat number is either prime either divisible by a 2-Poulet number

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Abstract. In this paper I make a conjecture which states that any Fermat number (number of the form 2^(2^n) + 1, where n is natural) is either prime either divisible by a 2-Poulet number. I also generalize this conjecture stating that any number of the form N = ((2^m)^p + 1)/3^k, where m is non-null positive integer, p is prime, greater than or equal to 7, and k is equal to 0 or is equal to the greatest positive integer such that N is integer, is either a prime either divisible by at least a 2-Poulet number (I will name this latter numbers Fermat-Coman numbers) and I finally enunciate yet another related conjecture.

Note:

For a list of 2-Poulet numbers see the sequence A214305 which I posted on OEIS. For a list of Fermat numbers see the sequence A000215 in OEIS.

Conjecture 1:

Any Fermat number F = 2^(2^n) + 1 is either prime either divisible by a 2-Poulet number.

Note:

It is known that the first 5 Fermat numbers (3, 5, 17, 257, 65537) are primes. Also, for n = 5 is obtained F = 4294967297 = 641*6700417, which is, indeed, a 2-Poulet number (for the next two (composite) Fermat numbers, 18446744073709551617 340282366920938463463374607431768211457, semiprimes, I couldn't verify if they are 2-Poulet numbers).

Conjecture 2:

Any Fermat-Coman number of the form N = ((2^m)^p + 1)/3^k, where m is non-null positive integer, p is prime, greater than or equal to 7, and k is equal to 0 or is equal to the greatest positive integer such that N is integer, is either a prime either divisible by at least a 2-Poulet number.

Verifying the conjecture:

(For m = 1 and the first eight such p)

: $(2^7 + 1)/3 = 43$, prime;
: $(2^{11} + 1)/3 = 683$, prime;
: $(2^{13} + 1)/3 = 2731$, prime;
: $(2^{17} + 1)/3 = 43691$, prime;
: $(2^{19} + 1)/3 = 174763$, prime;
: $(2^{23} + 1)/3 = 2796203$, prime;
: $(2^{29} + 1)/3 = 178956971 = 59*3033169$, a 2-Poulet number;
: $(2^{31} + 1)/3 = 715827883$, prime;

Verifying the conjecture:
(For m = 2 and the first four such p)

: $4^7 + 1 = 16385 = 5*29*113$, which is divisibe by $3277 = 29*113$, a 2-Poulet number;
: $4^{11} + 1 = 4194305 = 5*397*2113$, which is divisibe by $838861 = 397*2113$, a 2-Poulet number;
: $4^{13} + 1 = 67108865 = 5*53*157*1613$, which is divisibe by:
  : $8321 = 53*157$, a 2-Poulet number;
  : $85489 = 53*1613$, a 2-Poulet number;
  : $253241 = 157*1613$, a 2-Poulet number;
: $4^{17} + 1 = 17179869185 = 5*137*953*26317$, which is divisibe by:
  : $130561 = 137*953$, a 2-Poulet number;
  : $3605429 = 137*26317$, a 2-Poulet number;
  : $25080101 = 953*26317$, a 2-Poulet number.

Verifying the conjecture:
(For m = 3 and the first two such p)

: $(8^7 + 1)/3^2 = 233017 = 43*5419$, a 2-Poulet number;
: $(8^{11} + 1)/3^2 = 954437177 = 67*683*20857$, which is divisibe by $1397419 = 67*20857$, a 2-Poulet number.

Note:

The Fermat-Coman primes (Fermat-Coman numbers which are primes) seems to be very rare.

Conjecture 3:

For any prime p greater than or equal to 7 the number $(4^p + 1)/5$ is either prime either a product of primes $p_1*p_2*...p_n$ such that all the numbers p_i*p_j are 2-Poulet numbers for $1 \le i < j \le n$ (this Conjecture is verified for p up to 17 (see the Conjecture 2 above).