

# New Meanings of the Fine Structure Constant

Rodolfo A. Frino – October 2014  
Electronics Engineer  
Degree from the National University of Mar del Plata

## Abstract

*In this paper I shall develop the simplest relativistic model of the hydrogen atom. On one hand, this model is incomplete since it neglects the spin of the electron. On the other hand, it incorporates relativistic effects which are not taken into account by the atomic models of Bohr and Schrodinger. In the light of this simple theory it would seem that the Rydberg constant, as far as the spectral lines is concerned, is just the result of the non-relativistic approach used by Bohr and not a true indication of the dependence of the spectral lines on this constant. This formulation emphasizes the significance of the fine structure constant in the atomic structure of hydrogen and reveals two new meanings of this constant and its connection with life. Additionally this investigation introduces three new constants of Nature a) the minimum speed of light for the existence of hydrogen,  $c_H$ , b) the energy-distance constant,  $u_d$ , and c) the quantum energy-distance constant,  $u_q$ .*

**Keywords:** Hydrogen atom, spectral lines, Planck's constant, Balmer formula, Rydberg formula, Rydberg constant, fine structure constant.

## 1. Introduction

Spectroscopic experiments carried out in the late 19th century showed that atoms emitted radiation at specific frequencies. Moreover, it was found that these frequencies were quantized.

In 1885 Johann Balmer discovered an empirical formula describing the spectral lines emitted by hydrogen. This emission happens when an electron makes a transition between energy levels  $n$  and energy level 2. Balmer formula is

$$\frac{1}{\lambda} = R_{\infty} \left( \frac{1}{2^2} - \frac{1}{n^2} \right)$$

Where  $R_{\infty}$  is the Rydberg constant. This constant is known with greatest accuracy (also referred as precision) and, according to CODATA 2010, its value is

$$R_{\infty} = 10\,973\,731.568\,539(55) m^{-1}.$$

In 1889, Johannes Rydberg generalized Balmer formula by proposing another relationship to describe the hydrogen emission lines. This emission takes place when an electron makes a transition between energy levels  $n_i$  and  $n_f$ . His formula is

$$\frac{1}{\lambda} = R_{\infty} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

Rydberg formula becomes identical to Balmer's when  $n_f$  equals 2.

The explanation of the frequencies of the radiation emitted by the atom remained a mystery until 1913 when Niels Bohr introduced the first quantum theory of the hydrogen atom. This theory successfully explained not only the spectral lines predicted by Rydberg formula but also the spectral lines of ionized Helium (and other hydrogenoid) and the size of the hydrogen atom.

Bohr's idea was that when an electron makes a transition from a higher energy state ( $n_i$ ) to a lower energy state ( $n_f$ ) the excess energy must be given out as a photon of energy  $hf$ . Then and according to Einstein he was able to write the following relationship

$$hf = E_i - E_f$$

Bohr also postulated the electron angular momentum was an integer multiple of  $\hbar$ . Mathematically this means that

$$mvr = n\hbar$$

Furthermore, he demanded the electron did not lose energy in its orbit around the proton. In the present formulation we use the de Broglie condition

$$2\pi r = n\lambda$$

which states that the circumference of the electron's orbit must be an integer multiple of the de Broglie wavelengths of the electron. Both postulates are equivalent since they are the same equation written in a slightly different way.

The present formulation is relativistic in nature but unlike the Sommerfeld theory it only considers circular orbits and therefore uses only one quantum number. Thus this is the simplest relativistic atomic theory. This theory has almost all the limitations found in the Bohr theory, except that the fine structure constant,  $\alpha$ , emerges as a natural feature of the formulation. In Bohr's theory the fine structure constant is only "visible" in the expression for the wavelengths of the spectral lines (see table 2) and, as far as I know, nobody realized about the existence of this constant until 1916 when Sommerfeld developed his atomic theory. The relativistic treatment of this theory is what allows us to unveil the alpha-dependent formulas for the allowed atomic radii, the allowed energy levels and the allowed frequencies/wavelengths of the spectral emission lines of hydrogen.

The significance of this theory is, therefore, based on the fact that this formulation is the simplest relativistic model of the hydrogen atom that is capable of predicting the fundamental relationship between the fine structure constant and the atomic structure of hydrogen.

On the contrary to Bohr's theory, the radius of the atom according to this formulation, as we shall see in section 2.3, depends on the fine structure constant. This is a profound conceptual difference between the two theories.

## 2. The Relativistic Model of the Hydrogen Atom

In order to improve the structure, reading and understanding of this paper, the development of this theory is divided into the following subsections:

- 2.1. Nomenclature and Equations
- 2.2. Electron Orbital Velocity (Allowed velocities)
- 2.3. Radii of the Orbits (Allowed radii)
- 2.4. Atomic Energy Levels (Allowed energy levels)
- 2.5. Frequency of the Spectral Lines (Allowed spectral frequencies)
- 2.6. Wavelength of the Spectral Lines (Allowed spectral wavelengths)

## 2.1. Nomenclature and Equations

We shall use the following nomenclature for constants and variables used in this paper

### Nomenclature

- $n$  = principal quantum number
- $i$  = another symbol for the principal quantum number  $n$   
(used in frequency and wavelength formulas)
- $j$  = another symbol for the principal quantum number  $n$   
(used in frequency and wavelength formulas)
- $h$  = Planck's constant
- $\hbar$  = Planck's constant over  $2\pi$  (reduced Planck's constant)
- $k$  = proportionality constant of Coulomb law
- $\epsilon_0$  = permittivity constant
- $\alpha$  = fine structure constant (electromagnetic coupling constant)
- $q_1$  = charge 1 (e.g. proton charge)
- $q_2$  = charge 2 (e.g. electron charge)
- $e$  = elementary charge
- $m$  = relativistic mass of the electron
- $m_0$  = electron rest mass
- $p$  = relativistic momentum of the electron
- $\beta$  = velocity ratio (electron orbital velocity over the speed of light in vacuum)
- $\lambda$  = de Broglie wavelength of the electron (also the wavelength of spectral lines)
- $\lambda_{ce}$  = Compton wavelength for the electron
- $\lambda_{ce}$  = Compton wavelength for the electron over  $2\pi$  (reduced Compton wavelength)
- $\lambda_{ij}$  = wavelength of the spectral line corresponding to the electron transition between the energy levels  $i$  and  $j$
- $\lambda_{ij}(A)$  = wavelength of the spectral line corresponding to the transition between the energy levels  $i$  and  $j$  in Angstroms
- $f$  = frequency of the spectral lines
- $f_{ij}$  = frequency of the spectral line corresponding to the electron transition between the energy levels  $i$  and  $j$
- $r$  = hydrogen atomic radius
- $v$  = orbital velocity of the electron
- $v_n$  = orbital velocity of the electron for level  $n$
- $c$  = speed of light in vacuum
- $F$  = attractive electrostatic force between the proton and the electron
- $F_C$  = centripetal force
- $F_{Jev}$  = conversion factor from Joules to electron-volts

$E_{rel}$  = total relativistic energy of the hydrogen atom  
 $E$  = energy levels of the hydrogen atom  
 $E_n$  = energy levels of the hydrogen atom  
 $K$  = kinetic relativistic energy of the electron of a hydrogen atom  
 $K_p$  = kinetic relativistic energy of the proton of a hydrogen atom (neglected)  
 $U$  = potential energy of the electron in a hydrogen atom

## Equations

The equations given below is the complete set of equations we shall use in the development of this theory (the are other equations, such as the Compton wavelength of the electron, that are not included here because they are not essential to the development of this theory). I have not included the equation for the fine structure constant in this section as I assume that we do not know of its existence yet. This constant will emerge explicitly in the next sections.

Thus, we shall assume that

I) the circumference of the electron's orbit equals a whole number of wavelengths (pilot wavelength) of the electron. Mathematically this means that

$$2\pi r = n\lambda \quad \text{de Broglie standing wave condition} \quad (2.1-1)$$

where

$$n = 1, 2, 3, 4, \dots$$

II) the wavelength and the relativistic momentum of the electron are related through the de Broglie's relationship

$$\lambda = \frac{h}{p} \quad \text{de Broglie equation (pilot wavelength)} \quad (2.1-2)$$

III) the momentum of the electron is

$$p = mv \quad \text{Momentum} \quad (2.1-3)$$

IV) the mass of the electron is given by the Einstein's relativistic mass

$$m = \frac{m_0}{\sqrt{1-\beta^2}} \quad \text{Relativistic mass} \quad (2.1-4a)$$

$$\beta = \frac{v}{c} \quad \text{Velocity ratio} \quad (2.1-4b)$$

V) the electrostatic force between the proton and the electron is described by the Coulomb's law

$$F = k \frac{q_1 q_2}{r^2} \quad \text{Coulomb's law} \quad (2.1-5a)$$

$$k = \frac{1}{4\pi\epsilon_0} \quad \text{Electric constant in terms of the permittivity constant} \quad (2.1-5b)$$

VI) the centripetal force exerted on the electron is relativistic

$$F_C = \frac{mv^2}{r} \quad \text{Relativistic centripetal force} \quad (2.1-6)$$

VII) The potential energy of the electron at a distance  $r$  from the proton is given by the following law

$$U = -k \frac{q_1 q_2}{r} \quad \text{Potential energy} \quad (2.1-7)$$

we shall assume that the potential energy depends on the spatial coordinates and does not depend on time.

VIII) The kinetic energy of the electron is given by its relativistic kinetic energy

$$K = \sqrt{(pc)^2 + (m_0 c^2)^2} - m_0 c^2 \quad \text{Kinetic energy of the electron} \quad (2.1-8)$$

The kinetic energy of the proton,  $K_p$ , is neglected in this theory.

IX) The total relativistic energy of the electron is given by the Einstein's relationship

$$E_{rel} = \sqrt{(pc)^2 + (m_0 c^2)^2} \quad \text{Relativistic energy of the electron} \quad (2.1-9)$$

X) The total mechanical energy of the atom equals the kinetic energy of the electron plus the kinetic energy of the proton ( $K_p$  is neglected) plus the potential energy between the electron and the proton

$$E = K + U \quad \text{Total energy} \quad (2.1-10)$$

XI) The wavelength of a monochromatic electromagnetic wave as a function of its frequency

$$\lambda = \frac{c}{f} \quad \text{Wavelength of the spectral lines} \quad (2.1-11)$$

## 2.2. Electron Orbital Velocity

We start the analysis from equations (2.1-5a), (2.1-5b) and (2.1-6) observing that

$$q_1 = q_2 = e \quad (2.2-1)$$

Thus we can write

$$k \frac{e^2}{r^2} = \frac{m v^2}{r} \quad (2.2-2a)$$

Now we use equation (2.1-4a)

$$\frac{k e^2}{r} = \frac{m_0 v^2}{\sqrt{1-\beta^2}} \quad (2.2-2b)$$

solving for  $r$  we get

$$r = \frac{k e^2 \sqrt{1-\beta^2}}{m_0 v^2} \quad (2.2-3)$$

From equations (2.1-1), (2.1-2b) and (2.1-3) we get the radius  $r$  in terms of the momentum of the electron

$$r = n \frac{h}{2 \pi m v} \quad (2.2-4)$$

Using the relativistic mass of the electron, given by equation (2.1-4a), we get

$$r = n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0 v} \quad (2.2-5)$$

solving for the electron orbital velocity  $v$  we get the formula

$$v = n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0 r} \quad (2.2-6)$$

Substituting  $r$  in equation (2.2-6) with the second side of equation (2.2-3) produces

$$v = \left( n \frac{h \sqrt{1-\beta^2}}{2 \pi m_0} \right) \left( \frac{m_0 v^2}{k e^2 \sqrt{1-\beta^2}} \right) \quad (2.2-7)$$

After a simple mathematical work we solve this equation for the velocity  $v$ . Thus we get

$$v = \frac{2 \pi k e^2}{h} \frac{1}{n} \quad (2.2-8)$$

Substituting the electric constant  $k$  with equation (2.1-5b), the electron orbital velocity,  $v$ , takes the following shape

$$v_n = \frac{e^2}{2 \epsilon_0 h} \frac{1}{n} \quad (2.2-9)$$

$$n = 1, 2, 3, 4, \dots$$

We have also substituted  $v$  with  $v_n$  to indicate that the orbital velocity of the electron depends on the quantum number  $n$ . It is worthy to note that since the speed of light  $c$  is missing in equation (2.2-9),  $v_n$  does not depend (explicitly) on the fine structure constant.

**Note**

I would like to clarify an important point here. Some authors write this equation as follows

$$v_n = \frac{e^2 c}{2 \epsilon_0 h c} \frac{1}{n} = \alpha c \frac{1}{n} \tag{2.2-9b}$$

claiming that  $v_n$  depends on the fine structure constant. This is incorrect since if the speed of light were to change, the fine structure constant would not change. This is an indication that  $v_n$  does not depend on this constant. But let us forget about the fine structure constant for the time being as we have assumed that we don't know about its existence.

Now we define the velocity  $c_H$ , for reasons that will be explained later, as

$$c_H = \frac{e^2}{2 \epsilon_0 h} \tag{2.2-10}$$

It is worthy to note that  $c_H$  is the orbital velocity of the electron in its lowest orbit. In other words  $c_H$  is the orbital velocity of the electron for  $n=1$ . The reasons of using  $c_H$  instead of  $v_1$  will become clear later after stepping further into this theory. However it is important to keep in mind that

$$c_H = v_1 \tag{2.2-11}$$

Now we rewrite equation (2.2-9) incorporating this new definition

$$v = c_H \frac{1}{n} \tag{2.2-12}$$

$$n = 1, 2, 3, 4, \dots$$

### 2.3. Radii of the Orbits

Let us reconsider equation (2.2-5) and let us write it as follows

$$r = n \frac{h}{2 \pi v} \frac{1}{m_0} \sqrt{1 - \beta^2} \tag{2.3-1}$$

considering the definition of the variable  $\beta$  (equation (2.1-4b)) this equation can be written as

$$r = n \frac{h}{2\pi m_0} \sqrt{\frac{1}{v^2} - \frac{1}{c^2}} \quad (2.3-2)$$

Substituting  $v$  with the second side of equation (2.2-9) we can calculate the radii of the orbits of the hydrogen atom

$$r = n \frac{h}{2\pi m_0 c} \sqrt{\left(\frac{n 2\epsilon_0 h c}{e^2}\right)^2 - \frac{1}{c^2}} \quad (2.3-3)$$

After some simple mathematical work we get

$$r = n \frac{h}{2\pi m_0 c} \sqrt{n^2 \left(\frac{2\epsilon_0 h c}{e^2}\right)^2 - 1} \quad (2.3-4)$$

We observe that the variables inside the parenthesis of equation (2.3-4) form a dimensionless quantity. Thus, we call the inverse of this quantity,  $\alpha$ , and then we write it as follows

$$\alpha = \frac{e^2}{2\epsilon_0 h c} \quad (2.3-5)$$

Thus we have rediscovered the most famous and enigmatic constant: the *fine structure constant*,  $\alpha$ .

This allows us to simplify equation (2.3-4) as follows

$$r_n = \frac{h}{2\pi m_0 c} \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \quad (2.3-6)$$

$n = 1, 2, 3, 4 \dots$

Equation (2.3-6) gives the atomic radii of the hydrogen atom as a function of the principal quantum number  $n$  (We have substituted  $r$  with  $r_n$  to indicate that the atomic radii are a function  $n$ ).

It is worthy to observe that the fine structure constant,  $\alpha$ , has appeared naturally in this equation without introducing any additional postulates and without multiplying and dividing by  $c$ . Only a quantum relativistic treatment of the atom is able to produce this result.

Now if we consider the Compton wavelength for the electron

$$\lambda_{Ce} = \frac{h}{m_0 c} \quad (2.3-7)$$

we can express equation (2.3-6) in terms of this wavelength

$$r_n = \frac{\lambda_{Ce}}{2\pi} \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \quad (2.3-8)$$

$n = 1, 2, 3, 4 \dots$

Finally considering that

$$\lambda_{Ce} = \frac{\lambda_{Ce}}{2\pi} \quad (2.3-9)$$

we can write equation (2.3-8) in terms of  $\lambda_{Ce}$

$$r_n = \lambda_{Ce} \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \quad (2.3-10)$$

$n = 1, 2, 3, 4 \dots$

In **Appendix 1** we have calculated the value of the atomic radius using formula (2.3-6)

## 2.4. Atomic Energy Levels

To find the expression for the energy levels for the hydrogen atom we use equations (2.1-7), (2.1-8) and (2.1-10)

$$E = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 - \frac{ke^2}{r} \quad (2.4-1)$$

$$\left( E + m_0c^2 + \frac{ke^2}{r} \right)^2 = (pc)^2 + (m_0c^2)^2 \quad (2.4-2)$$

now we square both sides of the equation to get

$$\left( E + m_0c^2 + \frac{ke^2}{r} \right)^2 = \frac{m_0^2v^2c^2}{(\sqrt{1-\beta^2})^2} + (m_0c^2)^2 \quad (2.4-3)$$

then we work on the second side of the equation

$$\left( E + m_0c^2 + \frac{ke^2}{r} \right)^2 = \frac{m_0^2c^4}{1 - \frac{v^2}{c^2}} \quad (2.4-4)$$

Now we use the equation for the orbital velocity of the electron (2.2-9). This yields

$$\left( E + m_0c^2 + \frac{ke^2}{r} \right)^2 = \frac{m_0^2c^4}{1 - \left( \frac{1}{n} \frac{e^2}{2\epsilon_0 h} \right)^2 \frac{1}{c^2}} \quad (2.4-5)$$

which can also be written as

$$\left( E + m_0 c^2 + \frac{k e^2}{r} \right)^2 = \frac{m_0^2 c^4}{1 - \frac{1}{n^2} \left( \frac{e^2}{2 \epsilon_0 h c} \right)^2} \quad (2.4-6)$$

Now we introduce the fine structure constant given by equation (2.3-5). This simplifies the last equation as follows

$$\left( E + m_0 c^2 + \frac{k e^2}{r} \right)^2 = \frac{m_0^2 c^4}{1 - \frac{\alpha^2}{n^2}} \quad (2.4-7)$$

Because the next mathematical steps are quite long (but very simple) we shall skip them and we shall write the final formula directly

$$E_n = \left( \sqrt{1 - \frac{\alpha^2}{n^2}} - 1 \right) m_0 c^2 \quad (2.4-8)$$

n= 1, 2, 3, 4, ...

This is the formula for the energy of the allowed stationary levels (also known as allowed stationary states) of hydrogen. According to the conventional interpretation proposed by Bohr, the stationary levels are those atomic energy levels in which the electron does not radiate energy.

Equation (2.4-8) gives the atomic energy as a function of the quantum number  $n$ . Thus electrons in this model (as in Bohr's model) can orbit the nucleus in only certain discrete energy levels. This means that the energy of the electrons is quantized. The lowest energy level of the hydrogen atom is known as the ground state while the excited states are the states with higher energies. Since the energy levels are negative quantities, the less negative the value of the level, the higher the energy.

It is worthy to observe that the fine structure constant appears naturally in this formula. This is due to three laws of physics:

- a) the Coulomb law
- b) the de Broglie law, and
- c) The relativistic mass law

Each of these laws “contribute” with one or more constants to make the fine structure constant. Thus the Coulomb law contributes with two constants: the elementary charge ( $e$ ) and the permittivity ( $\epsilon_0$ ). The de Broglie law contributes with the Plank's constant ( $h$ ) and Einstein's relativistic mass law contributes with the speed of light ( $c$ ). But which formula contributes with the number (1/2)? Probably the Coulomb law.

The energy of the fundamental level (ground level) can be calculated through equation (2.4-8) taking  $n=1$ . This gives

$$E_1 = \left( \sqrt{1 - \frac{\alpha^2}{1^2}} - 1 \right) m_0 c^2 = -2.179\,901\,193 \times 10^{-18} \text{ J}$$

and considering the conversion factor,  $F_{JeV}$ , between Joules and electron-volts

$$F_{JeV} = 1.602\,176\,564 \times 10^{-19} \frac{\text{J}}{\text{eV}}$$

we can express the energy of the ground state in electron-volts

$$E_1 = \frac{-2.179\,901\,193 \times 10^{-18} \text{ J}}{F_{JeV}} \approx -13.605\,873\,66 \text{ eV}$$

The measured value according to CODATA 2010 is

$$E_{measured} \approx -13.598\,434\,005\,136 \text{ eV}$$

The absolute error,  $\Delta E$ , is

$$\begin{aligned} \Delta E &= E_1 - E_{measured} \approx -13.605\,873\,66 \text{ eV} - (-13.598\,434\,005\,136 \text{ eV}) \approx \\ \Delta E &\approx -0.0074 \text{ eV} \end{aligned}$$

As expected, the error is very small.

## 2.5. Frequency of the Spectral Lines

A hydrogen atom can absorb and emit electromagnetic radiation. Electrons “orbiting” the proton can a) make a transition to a higher energy level by absorbing electromagnetic radiation (photons). This process is responsible for the emission spectrum, or b) make a transition to a lower energy level by emitting electromagnetic radiation (photons). This process produces the emission spectrum.

Since the ground state corresponds to a state of minimum energy, electrons in this state cannot emit electromagnetic radiation. The frequency of the photon emitted by a hydrogen atom can be calculated using the Einstein's relationship

$$hf = E_i - E_j \quad (2.5-1)$$

where

$$\begin{aligned} E_i &= \text{upper energy level (also known as initial energy level)} \\ E_j &= \text{lower energy level (also known as } E_f \text{ or final energy level)} \end{aligned}$$

$$f = \frac{1}{h} (E_i - E_j) \quad (2.5-2)$$

Equation (2.4-8) allows us to write the energy of the levels  $i$  and  $j$

$$E_i = \left( \sqrt{1 - \frac{\alpha^2}{i^2}} - 1 \right) m_0 c^2 \quad (2.5-3a)$$

$$E_j = \left( \sqrt{1 - \frac{\alpha^2}{j^2}} - 1 \right) m_0 c^2 \quad (2.5-3b)$$

From equations (2.5-2), (2.5-3a) and (2.5-3b) we obtain the frequency  $f_{ij}$  of the spectral lines

$$f_{ij} = \left( \sqrt{1 - \frac{\alpha^2}{i^2}} - 1 \right) \frac{m_0 c^2}{h} - \left( \sqrt{1 - \frac{\alpha^2}{j^2}} - 1 \right) \frac{m_0 c^2}{h} \quad (2.5-4)$$

After a short mathematical work we get to the final formula

$$f_{ij} = \left( \sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}} \right) \frac{m_0 c^2}{h} \quad (2.5-5)$$

This is the formula for the frequency of the spectral lines of the hydrogen atom. It is worthy to note that the Rydberg constant does not appear in this formula.

## 2.6. Wavelength of the Spectral Lines

The formula for the wavelength,  $\lambda_{ij}$ , of the spectral lines of hydrogen can be derived substituting in equation (2.1-11) the frequency,  $f$ , with the second side of equation (2.5-5). This gives

$$\lambda_{ij} = \left( \frac{1}{\sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}}} \right) \frac{h}{m_0 c} \quad (2.6-1)$$

It is worthy to note that the Rydberg constant does not appear in this formula either.

Sometimes it is convenient to express the wavelength of the spectral lines in Angstroms. This can be multiplying the previous formula by  $10^{10}$ . This produces the following formula

$$\lambda_{ij}(A) = \left( \frac{10^{10}}{\sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}}} \right) \frac{h}{m_0 c} \quad (2.6-2)$$

where  $\lambda_{ij}(A)$  is the wavelength in Angstroms.

To illustrate this theory we shall calculate the wavelengths corresponding to the Balmer series with formula (2.6-2) by substituting the quantum number for the lower energy level  $j$  with 2. Thus, formula (2.6-2) becomes

$$\lambda_{ij}(A) = \left( \frac{10^{10}}{\sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{2^2}}} \right) \frac{h}{m_0 c} \quad \text{Equivalent to Balmer formula} \quad (2.6-3)$$

$i = 3, 4, 5, \dots$

It is worthy to remark that formula (2.6-3) yields the value of the wavelengths directly in Angstroms.

The following table shows some selected wavelengths for the Balmer series ( $j=2$  and  $i = 3, 4, 5, \dots$ )

$j$ (Quantum number corresponding to the lower energy level)	$i$ (Quantum number corresponding to the higher energy level)	$\lambda$ (Theoretical wavelength in Angstroms)	$\lambda_{\text{experimental air}}$ (Measured wavelength in Angstroms CODATA 2010. NIST Atomic Spectra Database Lines Data [2])
2	3	6561.091 217	6562.22
2	4	4860.070 714	4861.29671127 4861.29776054
2	5	4339.350 151	4340.472
2	6	4100.686 560	4101.70746200 4101.71027719 4101.734
2	7	3969.059 974	3970.075
2	8	3888.058 997	3889.02486539 3889.02515372 3889.02593309 3889.064
2	9	3834.409 000	3835.397
2	$\infty$	3645.056 068	3656.65

**Table 1:** Theoretical wavelengths for selected lines of the Balmer series.

### 3. A New Meaning for the Fine Structure Constant

Let us consider the equation of the atomic radius we derived in section 2.3

$$r = \frac{h}{2\pi m_0 c} \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \quad (3-1 = 2.3-6)$$

This equation tells us the new meaning of the fine structure constant. For the hydrogen atom to exist the square root of equation (3-1) must be positive. This means that

$$\frac{n^2}{\alpha^2} - 1 > 0 \quad (3-2)$$

and this leads to the following result

$$\alpha < n \quad (3-3)$$

The lowest value of the first term of the first side of inequation (3-3) occurs when  $n = 1$ . Thus we can rewrite the condition for the existence of the hydrogen atom as

$$\alpha < 1 \quad (3-4)$$

or

$$c > \frac{e^2}{2\epsilon_0 h} \quad (3-5)$$

Now we define the constant  $c_H$  as

$$c_H = \frac{e^2}{2\epsilon_0 h} \quad (3-6)$$

$$c_H = 2\,187\,691.266 \frac{m}{S}$$

$c_H$  is the speed of the electron in its fundamental or lowest energy level. This velocity is also known as  $v_1$ . However, since this constant is a threshold for the speed of light (in vacuum) to ensure the existence of the hydrogen atom, we shall use a new nomenclature,  $c_H$ , to indicate that this constant is a new constant of Nature.

Now inequation (3-5) can be rewritten as

$$c > c_H \quad (3-7)$$

This inequation reveals one of the most important conditions for the existence of hydrogen in the Universe

### Condition for the existence of Hydrogen

*A necessary condition for the existence of hydrogen atoms is that the speed of light in vacuum to be greater than  $c_H$  :  $c > c_H$  ; or in other words, that the fine structure constant to be less than 1:  $\alpha < 1$  .*

*Since  $\alpha = 0.007\,297\,352\,57$ , this condition is satisfied.*

We now express the fine structure constant as a function of  $c_H$  as follows

$$\alpha = \frac{c_H}{c} \quad (3-8)$$

in other words

$$c = \frac{1}{\alpha} c_H \quad (3-9)$$

$$c \approx 137.035999074 c_H$$

This result shows that the speed of light in vacuum satisfies condition (3-7).

## 4. Summary

The following table compares the formulas of the Bohr model with the corresponding formulas of this formulation.

Variable	Bohr Model ( $n = 1,2,3,4, \dots$ )	Relativistic Model ( $n = 1,2,3,4, \dots$ )
Orbital velocity of the electron	$v_n = \frac{1}{n} \frac{e^2}{2\epsilon_0 h}$	$v_n = \frac{1}{n} \frac{e^2}{2\epsilon_0 h}$
Atomic radii	$r_n = n^2 \frac{h^2 \epsilon_0}{\pi m_0 e^2}$	$r_n = \left( n \sqrt{\frac{n^2}{\alpha^2} - 1} \right) \frac{h}{2\pi m_0 c}$
Energy levels	$E_n = -\frac{1}{n^2} \frac{m_0 e^4}{8\epsilon_0^2 h^2}$	$E_n = \left( \sqrt{1 - \frac{\alpha^2}{n^2}} - 1 \right) m_0 c^2$
Frequency of the spectral lines	$f_{ji} = \left( \frac{1}{j^2} - \frac{1}{i^2} \right) \left( \frac{m_0 e^4}{8\epsilon_0^2 h^3} \right)$	$f_{ij} = \left( \sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}} \right) \frac{m_0 c^2}{h}$
Wavelength of the spectral lines	$\lambda_{ji} = \frac{1}{\left( \frac{1}{j^2} - \frac{1}{i^2} \right) \left( \frac{m_0 e^4}{8\epsilon_0^2 h^3 c} \right)}$	$\lambda_{ij} = \frac{1}{\left( \sqrt{1 - \frac{\alpha^2}{i^2}} - \sqrt{1 - \frac{\alpha^2}{j^2}} \right)} \frac{h}{m_0 c}$
Important constant of the theory	Rydberg constant $R_\infty = \frac{m_0 e^4}{8\epsilon_0^2 h^3 c}$	Fine structure constant $\alpha = \frac{e^2}{2\epsilon_0 h c}$

**Table 2:** Formula comparison

## 5. Conclusions

One of the most amazing things about the Bohr model is that it was remarkably successful in explaining the empirical Rydberg formula and other aspects of the hydrogen atom from a set of hybrid physical laws.

However Nature had a hidden surprise for us. The present relativistic formulation of the atom shows that the Rydberg constant does not appear anywhere in the formula for the spectral lines. But then we can ask: Where are  $e$  and  $\epsilon_0$  in this formulation? The answer is that Nature “incorporated” these two constants, along with  $h$  and  $c$ , into the fine structure constant while the rest mass of the electron appears separately in the formulas for the atomic radius, the energy levels and the wavelength of the spectral lines (see table 2).

The square roots in the formulas I presented in this paper (see table 2) clearly indicate the relativistic nature of this formulation. Despite the fact that the Bohr model and the model presented here produce similar numerical results there is a profound difference between these two models. The Bohr model ignores relativity and therefore it ignores one of the most important laws of nature. The result, this is the price we have to pay, is the absence of the fine structure constant in the formulas of the atomic radius, the energy levels and the frequencies of that theory (see table 2). However the Bohr formula for the wavelength of the spectral lines

*Bohr formula for the wavelengths of spectral lines  
(in terms of the Rydberg constant)*

$$\lambda_{ji} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) \left(\frac{m_0 e^4}{8 \epsilon_0^2 h^3 c}\right)} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) R_\infty}$$

can show the fine structure constant if we write it as follows

*Bohr formula for the wavelengths of spectral lines  
(in terms of the fine structure constant)*

$$\lambda_{ji} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) \left(\frac{m_0 e^2}{4 \epsilon_0 h^2}\right) \left(\frac{e^2}{2 \epsilon_0 h c}\right)} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) \left(\frac{m_0 e^2}{4 \epsilon_0 h^2}\right) \alpha} = \frac{1}{\left(\frac{1}{j^2} - \frac{1}{i^2}\right) \left(\frac{m_0}{2 h}\right) c_H \alpha}$$

As far as I know, no-one at the time Bohr published his theory (1913) realized that the above formula (*Bohr formula for the wavelengths of spectral lines*) contained the most enigmatic constant of Nature: *the fine structure constant*. The world would have to wait until 1916 for Sommerfeld to discover this constant through his relativistic theory of the hydrogen atom with elliptical orbits [2].

In the light of the relativistic theory presented in this paper it seems that the Rydberg constant, as far as the spectral lines is concerned, is just the result of the non-relativistic approach used by Bohr and not a true indication of the dependence of the spectral lines on this constant. Having said that we have to clarify that the Rydberg constant still “appears” implicitly in this formulation. To obtain the Rydberg constant we need to divide the classical kinetic energy of the electron (*we must use the electron rest mass*) in its lowest orbit by  $hc$  (*the Planck's constant times the speed of light in*

vacuum). Mathematically

$$\frac{\frac{1}{2} m_o v_1^2}{hc} = \frac{1}{2} \frac{m_o c \alpha^2}{h} = \frac{m_o e^4}{8 \epsilon_0^2 h^3 c} = R_\infty$$

Another prediction of this theory concerns with the existence of hydrogen. We have seen that one of the necessary conditions for the existence of hydrogen, and therefore for the existence of life, is that the fine structure constant to be smaller than 1. Mathematically

$$\alpha < 1$$

Nature as we all know, “has passed this test” with flying colors.

One important point to observe is that we have defined the new constant of nature  $c_H$  as

*A new constant of Nature*

$$c_H = \frac{e^2}{2 \epsilon_0 h}$$

If we observe the first row of table 2 which corresponds to the orbital velocity of the electron we find that both theories yield exactly the same equation for the velocity of the electron. This suggests that the velocity  $c_H$  should be an exceptionally important physical parameter. The evidence is perhaps the fact that  $c_H$  is part of the fine structure constant as shown by the following expression

*The fine structure constant in terms of the new constant of Nature*

$$\alpha = \frac{c_H}{c}$$

This in turn suggests that there is another even more fundamental constant of Nature that I shall define as the energy-distance constant  $u_d$ . The definition of this new constant is

*The energy-distance constant*

$$u_d = \frac{e^2}{2 \epsilon_0}$$

The relationship between these three fundamental constants is

$$\alpha = \frac{u_d}{hc}$$

Since  $hc$  has the same dimensions as  $u_d$  I shall define the quantum energy-distance constant,  $u_q$ , as

*The quantum energy-distance constant*

$$u_q = hc$$

Finally the fine structure constant can be rewritten in terms of the ratio between these two new energy-distance constants

*The fine structure constant in terms of the energy-distance constants*

$$\alpha = \frac{u_d}{u_q}$$

This relationship indicates that there is a deep connection between electromagnetism (through  $u_d$ ) and the Planck scale (through  $u_q$ ). Therefore this equation seems to represent one of the most fundamental meanings of the fine structure constant. Any “theory of everything” should be able to explain this relationship from even more fundamental principles. However this term is very unfortunate since no theory will ever explain everything. For example a true theory of everything should explain what you and I are thinking right now, why we are not intelligent enough so that we wouldn't need a theory of everything to understand the universe we live in, what cities will look like in 3000 years time (unless time travel into the future were possible), and that seems to be impossible.

Another point to observe is that even if the fine structure constant were ten times larger than the value we observe today, Nature would still satisfy the above condition ( $\alpha < 1$ ) and therefore hydrogen atoms would still have formed (we are not considering any other constant tuning in this analysis), and life would still have emerged. This implies that, as far as the hydrogen atom is concerned, Nature has a relatively wide “built-in” margin to ensure the emergence of life. This suggests that, in this context and only in this context, there is no fine tuning as far as the fine structure constant is concerned.

## Appendix 1 Atomic Radius

The atomic radius of the hydrogen atom can be calculated from equation (2.2-6) for  $n = 1$

$$r_1 = \frac{h}{2\pi m_0 c} \left( 1 \sqrt{\frac{1^2}{\alpha^2} - 1} \right)$$

The standard or usual nomenclature for the atomic radius is  $a_0$  therefore the previous equation becomes

$$a_0 = \frac{h}{2\pi m_0 c} \left( \sqrt{\frac{1}{\alpha^2} - 1} \right)$$

the value this equation yields is

$$a_0 = 5.291\,631\,206 \times 10^{-11} m$$

$$a_0 = 0.529\,163\,120\,6 \text{ \AA}$$

The diameter of the hydrogen atom is

$$d_0 = 1.058\,326\,241 \times 10^{-10} \text{ m}$$

$$d_0 = 1.058\,326\,241 \text{ \AA}$$

## REFERENCES

- [1] NIST, *Atomic Spectra Database Lines Data*, retrieved October 4, 2014, from <http://physics.nist.gov/asd>
- [2] C. Churchill, *The structure of hydrogen*, Chapter 9, Draft Version, (2010)