Proof of 2^e is irrational

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Abstract

2^e is rational or irrational number is not known. It is unsolved problem in analysis [1] .We proved that 2^e as irrational number.

Introduction

Irrationality measure is not defined. In this research we proved that 2^e is irrational number. We attack the proof by method of contradiction. We assume that 2^e be rational number. Then we use some logarithms properties and simplification to get a relation between '*e*' and assumed rational number, since we known that '*e*' is irrational number, we use some further simplification and method to prove that 2^e is irrational number .

Mathematical Proof

Let 2^e be rational number then,

$2^e = \frac{p}{q}$, where gcd(p,q) = 1 and $p,q \in Z^+$, $q \neq 1,0$

The reason that $q \neq 1$, because if q =1 then there exist no 'p' integer for which $p^{1/e}$ is equal to 2 . [2]

$2^e = \frac{p}{q}$

Taking $\log_j$ on both sides, where $j > 0$

$\log_j 2^e = \log_j \frac{p}{q}$

$e = \log_2 \frac{p}{q}$ [3] using $\log_h h^w = w$ [4]

So, $\log_2 \frac{p}{q}$ should be irrational number since '*e*' is irrational number .

Let us check whether $\log_2 \frac{p}{q}$ is rational, irrational or have conditional properties.

Let $\log_2 \frac{p}{q}$ be rational number then,

$\log_2 \frac{p}{q} = \frac{m}{n}$, gcd (m,n) = 1 and $n \neq 0$ where m, n$\in Z^+$

$$\frac{p}{q} = 2^{\frac{m}{n}}$$

Raising both sides by power of 'n'

$$\left(\frac{p}{q}\right)^n = 2^m$$

$\therefore$ $\left(\frac{p}{q}\right)^n$ is always a even number i.e. $\left(\frac{p}{q}\right)^n$ = 2k, where k ∈ Z^+.

$\therefore$ $\frac{p}{q}$ is also even number i.e. $\frac{p}{q} = 2d$, where d ∈ Z^+. [5]

<u>Classifying even number</u>

1. Pure even number – The even number that can written as 2^a form where $a \in Z^+$.

   Example - 2^3 , 2^5 , 2^{100}

2. Pseudo even number – The even number that can be written as product of even and odd number i.e. in form of 2t · (2u-1), where t and u ∈ Z^+ and u > 1 .

   Example- 12 = 4 · 3 , 82 = 2 · 41 , 20 = 4·5

<u>Proof strategy</u>

So in order to know 2^e is irrational or rational number, we first let that 2^e be rational number .Then we get a relation that e = $\log_2 \frac{p}{q}$. This help us to analyse that the further steps because we know that '*e*' as irrational number. After this we try to analyse whether $\log_2 \frac{p}{q}$ is rational or irrational, this is done by again letting $\log_2 \frac{p}{q}$ as rational. By simplification we get that $\frac{p}{q}$ is even number (Using [5]) and we classified $\frac{p}{q}$ as pure even number and pseudo even number, because further we get that both gives different result of rationality measure (whether rational or irrational). Then we take the $\log_2$ of both case, we see that $\log_2 2^a$ is always rational number (using theorem 1.0) and $\log_2 2t\cdot(2u-1)$ is always irrational number (using theorem 1.1). But the both cases give that 2^e is irrational number. Since, the $\log_2 2^a$ is rational number but the '*e*' is irrational number so this disproves the assumed fact (using [6]) , the next case is that $\log_2 2t\cdot(2u-1)$ is irrational number but we see that by expansion of 2^e that 2^e can never be in form of 2t· (2u-1) (using [8]) .But there is also one case when $\frac{p}{q}$ is equal to 2^{e} but we also disprove this (using [9]) .So the all cases disprove our assumption of 2^{e} as rational number , therefore 2^{e} is irrational number .

Cases arise for values of $\frac{p}{q}$:

1. $\frac{p}{q} = 2^{a}$, where $a \in Z^{+}$.

Then $\log_2 2^{a}$ is a rational number (using theorem 1.0) , but 'e' is equal to $\log_2 \frac{p}{q}$ i.e. $\log_2 2^{a}$, but 'e' is irrational number so it cannot be equal to rational number so $\frac{p}{q} \neq 2^{a}$. [6]

2. $\frac{p}{q} = 2t \cdot (2u-1)$,where t and u $\in Z^{+}$ and u > 1 .

In this case $\log_2 2t \cdot (2u-1)$ is irrational (using theorem 1.1) , but $\frac{p}{q} = 2t \cdot (2u-1)$ and $\frac{p}{q} = 2^{e}$

So, $2^{e} = 2t \cdot (2u-1)$

$2^1 \cdot 2^1 \cdot 2^{1/2!} \cdot 2^{1/3!} \cdot 2^{1/4!} \;.... = 2t \cdot (2u-1)$

Using $2^{e} = 2^1 \cdot 2^1 \cdot 2^{1/2!} \cdot 2^{1/3!} \cdot 2^{1/4!} \;....$, by Taylor expansion of 'e' [7]

This is an absurd result because in the LHS there are only '2'as factors having different powers but in RHS there is a odd factor (2u-1) , so by comparing factors of both sides we can claim that LHS $\neq$ RHS are not equal. [8]

But there is one way also which make the $\frac{p}{q}$ as irrational , when $\frac{p}{q}$ is equal to 2^{e} , so in this case $\log_2 2^{e}$ is equal to '*e*' But there are three cases arises for certain values of p and q for which $\frac{p}{q} = 2^{e}$: [9]

1. $p = 2^{e}$, q=1

But both values are impossible because $q \neq 1$ (Using [2]).

2. $p=2^{c}$ and $q= 2^{d}$, where $c - d = e$

But in this case $c - d \neq e$, because '*e*' is irrational number .

3. p and q be the distinct pseudo even number. So p = 2m(2f-1) and q = 2r(2v-1) ,where m,f,r,v $\in Z^{+}$ and f >1 , v >1 .

But in this case $\gcd(p,q) \neq 1$, since p and q are even number .

So, all the possible cases does not satisfy the condition of rationality of 2^{e} .Therefore it contradicts our assumption of 2^{e} as rational number and so 2^{e} is irrational number .

Theorems used

Theorem 1.0 : $\log_2 2^a$ is rational or irrational number according to the 'a' as rational or irrational number respectively .

Proof : Using base changing properties of logarithm (using [4]) we get $\log_2 2^a = a$. Therefore as 'a' is rational or irrational number, it decides whether $\log_2 2^a$ as rational or irrational number respectively .

Theorem 1.1: $\log_2 2t\cdot(2u-1)$ is always a irrational number , where t and u $\in Z^+$ and $u > 1$.

Proof : Let $\log_2 2t\cdot(2u-1)$ be a rational number

$$\therefore \log_2 2t\cdot(2u-1) = \frac{x}{y} \quad \text{, where } \gcd(x,y) = 1 \quad \text{, x and y} \in Z^+ \text{ and } y \neq 0$$

$$2t\cdot(2u-1) = 2^{\frac{x}{y}}$$

Raising both side by power of y

$$(2t)^y \cdot (2u-1)^y = 2^x$$

So, this is an absurd result as the RHS is always a even number, but in LHS there is odd factor i.e. $(2u-1)^y$ is present .By comparing factors of both sides we can claim that LHS is not equal to RHS . Therefore our assumption of $\log_2 2t\cdot(2u-1)$ as rational number is wrong, so $\log_2 2t\cdot(2u-1)$ is irrational number.

Reference

[1] http://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics

[4] H.S. Hall and S.R. Knight , Higher Algebra , Fourth Edition, London : Macmillan ; New York : Martin's Press ,page no. 177

[7] H.S. Hall and S.R. Knight , Higher Algebra , Fourth Edition , London : Macmillan ; New York : Martin's Press ,page no.188