Rates of convergence of lognormal extremes under power normalization

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Abstract. Let $\{X_n, n \geq 1\}$ be an independent and identically distributed random sequence with common distribution F obeying the lognormal distribution. In this paper, we obtain the exact uniform convergence rate of the distribution of maxima to its extreme value limit under power normalization.

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1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution function (df) $F(x)$. Suppose that there exist constants $a_n > 0, b_n \in \mathbb{R}$ and a non-degenerate distribution $G(x)$ such that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{1.1}$$

for all $x \in C(G)$, the set of all continuity points of G, where $M_n = \max_{1\leq i\leq n} X_i$ denotes the largest of the first n. Then $G(x)$ must belong to one of the following three classes:

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\}, & \text{if } x \geq 0, \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$\Lambda(x) = \exp\{-e^{-x}\}, \ x \in \mathbb{R},$$

where α is one positive parameter. We say that F is in the max domain of attraction of G if (1.1) holds, denoted by $F \in D_l(G)$. Criteria for $F \in D_l(G)$ and the choice of normalizing constants a_n and b_n can be found in Galambos[1], Leadbetter et al.[2], Resnick[3], and De Haan and Ferreira[4].

The limit distributions of maxima under power normalization was first derived by Pancheva[5]. A df F is said to belong to the max domain of attraction of a non-degenerate df H under power normalization, written as $F \in D_p(H)$, if there exist constants $\alpha_n > 0$ and $\beta_n > 0$ such that

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \leq x\right) = \lim_{n\to\infty} F^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) = H(x), \tag{1.2}$$

where $\mathrm{sign}(x) = -1, 0$ or 1 according as $x < 0, x = 0$ or $x > 0$. Pancheva[5] showed that H can be one of only power types of *df*'s, that is:

$$H_{1,\alpha}(x) = \begin{cases} 0, & \text{if } x \le 1, \\ \exp\{-(\log x)^{-\alpha}\}, & \text{if } x > 1, \end{cases}$$

$$H_{2,\alpha}(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{-(-\log x)^{\alpha}\}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \ge 1, \end{cases}$$

$$H_{3,\alpha}(x) = \begin{cases} 0, & \text{if } x \le -1, \\ \exp\{-(-\log(-x))^{-\alpha}\}, & \text{if } -1 < x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

$$H_{4,\alpha}(x) = \begin{cases} \exp\{-(-\log(-x))^{\alpha}\}, & \text{if } x < -1, \\ 1, & \text{if } x \ge -1, \end{cases}$$

$$H_{5,\alpha}(x) = \Phi_1(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{-x^{-1}\}, & \text{if } x > 0, \end{cases}$$

$$H_{6,\alpha}(x) = \Psi_1(x) = \begin{cases} \exp\{x\}, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

where α is a positive parameter. Necessary and sufficient conditions for F to satisfy (1.2) have been given by Christoph and Fark[6], Mohan and Ravi[7], Mohan and Subramanya[8] and Subramanya[9].

The logarithmic normal distribution (*lognormal distribution* for short) is one of the most widely applied distributions in statistics, biology and some other disciplines. In this paper, we are interested in considering the uniform rate of convergence of (1.2) with X_n following the lognormal distribution. The probability density function of the lognormal distribution is given by

$$F'(x) = \frac{x^{-1}}{\sqrt{2\pi}} \exp\left\{-\frac{(\log x)^2}{2}\right\}, \; x > 0.$$

One interesting problem in extreme value analysis is to estimate the rate of uniform convergence of $F^n(\cdot)$ to its extreme value distribution. For power normalization, Chen et al.[10] derived convergence rates of the distribution of maxima for random variables obeying the general error distribution. Convergence rates of distributions of extremes under linear normalization, see De Haan and Resnick[11] under second-order regular variation and special cases see Hall[12] and Nair[13] for normal distribution, which also is extended to those such as general error distribution, logarithmic general error distribution, see recent work of Peng et al.[14] and Liao and Peng[15]. For other related work on the convergence rates of some given distributions, see Castro[16] for the gamma distribution, Lin et al.[17] for the short-tailed symmetric distribution due to Tiku and Vaughan[18], and Liao et al.[19] for skew normal distribution which extended the results of Nair[13]. The aim of this paper is to study the uniform and point-wise convergence rates of the distribution of power normalized maxima under power normalization to its limits, respectively.

The contents of this article is organized as follows: some auxiliary results are given in Section 2. In Section 3, we provide our main results with related proofs are deferred to Section 4.

2 Preliminaries

To prove our results, we first cite some results from Liao and Peng[15], Mohan and Ravi[7].

In the sequel, let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with common *df* F which follows the lognormal distribution. As before, let $M_n = \max_{1\leq i\leq n} X_i$ represent the partial maximum of $\{X_n, n \geq 1\}$. Liao and Peng[15] defined

$$a_n = \frac{\exp\left((2\log n)^{1/2}\right)}{(2\log n)^{1/2}},\ b_n = (\exp((2\log n)^{1/2}))\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}}\right), \tag{2.1}$$

and obtained

$$\lim_{n\to\infty} P((M_n - b_n)/a_n \leq x) = \exp(-e^{-x}) =: \Lambda(x). \tag{2.2}$$

From (2.2) we immediately derive $F \in D_l(\Lambda)$. The following Mills ratio of the lognormal distribution is due to Liao and Peng[15]:

$$\frac{1 - F(x)}{F'(x)} \sim \frac{x}{\log x}, \tag{2.3}$$

as $x \to \infty$, where $F'(x)$ is the density function of the lognormal distribution $F(x)$. According to Liao and Peng[15], we have

$$1 - F(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right),$$

for sufficiently large x, where $c(x) \to (2\pi e)^{-1/2}$ as $x \to \infty$, $g(x) = 1 + (\log x)^{-2}$ and

$$f(x) = \frac{x}{\log x}. \tag{2.4}$$

Noting that $f'(x) \to 0$ and $g(x) \to 1$ as $x \to \infty$.

Lemma 2.1. *[15] Let F denote the lognormal distribution function. Then*

$$1 - F(x) = \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \gamma(x) \tag{2.5}$$

$$= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)\left(1 - (\log x)^{-2}\right) + \mathcal{S}(x), \tag{2.6}$$

for $x > 1$, where

$$0 < \gamma(x) < \frac{1}{\sqrt{2\pi}}(\log x)^{-3}\exp\left(-\frac{(\log x)^2}{2}\right) \tag{2.7}$$

and

$$0 < \mathcal{S}(x) < \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right). \tag{2.8}$$

In order to obtain the main results, we need the following two lemmas.

Lemma 2.2. *[7] Let F denote a df and $r(F) = \sup\{x : F(x) < 1\}$. Suppose that $F \in D_l(\Lambda)$ and $r(F) = \infty$, then $F \in D_p(\Phi_1)$, where normalizing constants $\alpha_n = b_n$, $\beta_n = a_n/b_n$.*

Lemma 2.3. *[7] Let F denote a df, if $F \in D_p(\Phi_1)$ if and only if*

(i)$r(F) > 0$, and

(ii) $\lim_{t\uparrow r(F)} \frac{1-F(t\exp(y\bar{f}(t)))}{1-F(t)} = e^{-y}$, for some positive valued function f.

If (ii) holds for some $\bar{f}$, then $\int_a^{r(F)}((1-F(x))/x)\,\mathrm{d}x < \infty$ for $0 < a < r(F)$ and (ii) holds with the choice $\bar{f}(t) = \int_t^{r(F)}((1-F(x))/x)\,\mathrm{d}x/(1-F(t))$. The normalizing constants may be chosen as $\alpha_n = F^{\leftarrow}(1-1/n)$ and $\beta_n = \bar{f}(\alpha_n)$, where $F^{\leftarrow}(x) = \inf\{y : 1-F(y) \geq x\}$.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed lognormal random variables. Then $F \in D_p(\Phi_1)$ and the normalizing constants can be chosen as $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$, where a_n and b_n are given by (2.1).*

Proof. Note that F follows the lognormal distribution, which implies $F \in D_p(\Phi_1)$ and $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$ by Lemma 2.2, where a_n and b_n are defined by (2.1). □

By Lemma 2.3 and (2.3) and combining with Proposition 1.1(a)[3], a natural way to choose constants α_n and β_n is to solve the following equations:

$$2\pi(\log\alpha_n)^2\exp((\log\alpha_n)^2) = n^2 \tag{2.9}$$

and

$$\beta_n = \frac{f(\alpha_n)}{\alpha_n} = \frac{1}{\log\alpha_n}, \tag{2.10}$$

where f is given by (2.4). The solution of (2.9) may be expression as

$$\alpha_n = (\exp((2\log n)^{1/2}))\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}} + o(\frac{1}{(\log n)^{1/2}})\right) \tag{2.11}$$

and easily check that $\beta_n \sim (2\log n)^{-1/2}$.

3 Main results

In this section, we give two main results. Theorem 3.1 proves the result that the rate of uniform convergence of $F^n(\alpha_n x^{\beta_n})$ to its extreme value limit is proportional to $1/\log n$. Theorem 3.2 establishes the result that the pointwise rate of convergence of $|M_n/\alpha_n|^{1/\beta_n}\,\mathrm{sign}(M_n)$ to the extreme value *df* $\exp(-x^{-1})$ is of the order of $O(x^{-1}e^{-1/x}(\log n)^{-1})$.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ denote an independent identically distributed random variables sequence with common df F following the lognormal distribution. Then there exist absolute constants $0 < \mathcal{C}_1 < \mathcal{C}_2$ such that*

$$\frac{\mathcal{C}_1}{\log n} < \sup_{x>0}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < \frac{\mathcal{C}_2}{\log n}$$

for large $n > n_0$, where α_n and β_n are determined by (2.9) and (2.10), respectively.

Theorem 3.2. *Let α_n and β_n be given by (2.9) and (2.10). Then, for large n*

$$|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \sim x^{-1}e^{-1/x}\frac{1}{\log n},$$

as $x > 0$.

4 Proofs

Firstly, we provide the proof of Theorem 3.2 for it is relatively easy.

Proof of Theorem 3.2. By Lemma 2.1, we have

$$\begin{aligned}1-F(\alpha_n x^{\beta_n}) &= \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2})\\ &\times(1-(\log(\alpha_n x^{\beta_n}))^{-2})+\mathcal{S}(\alpha_n x^{\beta_n})\\ &=: T_1(x)T_2(x)+T_3(x)\end{aligned}$$

for $x>0$, where $T_1(x)=\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}), T_2(x)=1-(\log(\alpha_n x^{\beta_n}))^{-2}$ and $T_3(x)=\mathcal{S}(\alpha_n x^{\beta_n})$.

Firstly, we calculate the $T_1(x)$. By (2.9) and (2.10), we have

$$\begin{aligned}T_1(x) &= \frac{1}{\sqrt{2\pi}}(\log\alpha_n)^{-1}\exp\left(-\frac{(\log\alpha_n)^2}{2}\right)(1+(\log\alpha_n)^{-1}\beta_n\log x)^{-1}\\ &\times\exp\left(-(\log\alpha_n)\beta_n\log x-\frac{\beta_n^2\log^2 x}{2}\right)\\ &=\frac{1}{nx}(1+\beta_n^2\log x)^{-1}\exp\left(-\frac{\beta_n^2\log^2 x}{2}\right)\\ &=\frac{1}{nx}(1-\beta_n^2\log x+O(\beta_n^4))\left(1-\frac{\beta_n^2\log^2 x}{2}+O(\beta_n^4)\right)\\ &=\frac{1}{nx}\left(1-\beta_n^2(1+\frac{1}{2}\log x)\log x+O(\beta_n^4)\right).\end{aligned}\tag{4.1}$$

Secondly, we estimate the $T_2(x)$ and the $T_3(x)$ for $x>0$. By (2.10), we derive

$$\begin{aligned}T_2(x) &= 1-\beta_n^2(1+\beta_n^2\log x)^{-2}\\ &=1-\beta_n^2(1-2\beta_n^2\log x+O(\beta_n^4))\\ &=1-\beta_n^2+O(\beta_n^4),\end{aligned}\tag{4.2}$$

and by Lemma 2.1, we have

$$\begin{aligned}T_3(x) &\le \frac{3}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-5}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\\ &=3\beta_n^4(1+\beta_n^2\log x)^{-4}T_1(x)\\ &=O(n^{-1}\beta_n^4).\end{aligned}\tag{4.3}$$

By (4.1)-(4.3), we have

$$1-F^n(\alpha_n x^{\beta_n})=\frac{1}{nx}\left(1-\beta_n^2(1+(1+\frac{1}{2}\log x)\log x)+O(\beta_n^4)\right).$$

Thus, we obtain

$$F^n(\alpha_n x^{\beta_n})-\Phi_1(x)=\left(1-\frac{1}{nx}(1-\beta_n^2(1+(1+\frac{1}{2}\log x)\log x)+O(\beta_n^4))\right)^n-\exp(-\frac{1}{x})$$

$$
\begin{aligned}
&= \exp(-\frac{1}{x})\left(\exp(\frac{1}{x}(\beta_n^2(1+(1+\frac{1}{2}\log x)\log x)+O(\beta_n^4)))-1\right)\\
&= \exp(-\frac{1}{x})\left(\beta_n^2\frac{1}{x}(1+(1+\frac{1}{2}\log x)\log x)+O(\beta_n^4)\right) \qquad (4.4)
\end{aligned}
$$

for large n and $x > 0$. We immediately get the result of Theorem 3.2 by (4.4). □

Proof of Theorem 3.1. By Theorem 3.2 we can prove that there exists an absolute constant $\mathcal{C}_1$ such that

$$\sup_{x>0}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| > \frac{\mathcal{C}_1}{\log n}.$$

In order to obtain the upper bound for $x > 0$, we need to prove:

$$(a).\quad \sup_{1\le x<\infty}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < d_1\beta_n^2, \qquad (4.5)$$

$$(b).\quad \sup_{c_n\le x<1}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < d_2\beta_n^2, \qquad (4.6)$$

$$(c).\quad \sup_{0<x<c_n}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < d_3\beta_n^2, \qquad (4.7)$$

for $n > n_0$, where $d_i > 0,\ i = 1, 2, 3$ are absolute constants and

$$c_n = \frac{1}{2\log\log\alpha_n}$$

is positive for $n > n_0$. By (2.9), we have

$$0.4(2\log n)^{1/2} < \log\alpha_n < (2\log n)^{1/2}$$

for $n > n_0$.

Firstly, consider the case of $x \ge c_n$. Set

$$R_n(x) = -[n\log F(\alpha_n x^{\beta_n}) + n\Psi_n(x)],\ B_n(x) = \exp(-R_n),\ A_n(x) = \exp(-n\Psi_n(x) + \frac{1}{x}),$$

where $\Psi_n(x) = 1 - F(\alpha_n x^{\beta_n})$ and $A_n(x) \to 1$, as $x \to \infty$. Since

$$
\begin{aligned}
\Psi_n(x) \le \Psi_n(c_n) &< \frac{1}{\sqrt{2\pi}}(\log(\alpha_n c_n^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n c_n^{\beta_n}))^2}{2}\right)\\
&= \frac{1}{n}(1+\beta_n^2\log c_n)^{-1}\exp\left(-\log c_n - \frac{\beta_n^2\log^2 c_n}{2}\right)\\
&< \frac{1}{n}(1+\beta_n^2\log c_n)^{-1}c_n^{-1}\\
&= \left(1-\frac{\log(2\log\log\alpha_n)}{(\log\alpha_n)^2}\right)^{-1}\frac{2\log\log\alpha_n}{n}\\
&< \tilde{c}_4 < 1
\end{aligned}
$$

for $n > n_0$. So,

$$\inf_{x>c_n}(1-\Psi_n(x)) > 1-\tilde{c}_4 > 0.$$

Since

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x,$$

for $0 < x < 1$, we obtain

$$
\begin{aligned}
0 < R_n(x) &\leq \frac{n\Psi_n^2(x)}{2(1-\Psi_n(x))} < \frac{n\Psi_n^2(c_n)}{2(1-\Psi_n(x))} \\
&< \frac{n^{-1}(1+\beta_n^2\log c_n)^{-2}c_n^{-2}}{2(1-\Psi_n(x))} \\
&< \frac{n^{-1}(1+\beta_n^2\log c_n)^{-2}c_n^{-2}(\log\alpha_n)^2}{2(1-\tilde{c}_4)\beta_n^{-2}} \\
&= \frac{2}{\sqrt{2\pi}(1-\tilde{c}_4)}\left(1-\frac{\log(2\log\log\alpha_n)}{(\log\alpha_n)^2}\right)^{-2}\frac{(\log\log\alpha_n)^2\log\alpha_n}{\exp(\frac{(\log\alpha_n)^2}{2})}\beta_n^2 \\
&< \tilde{c}_5\beta_n^2
\end{aligned}
$$

for $n > n_0$.
Hence, we have

$$
n^{-1}\beta_n^{-2}(1+\beta_n^2\log c_n)^{-2}c_n^{-2} < \tilde{c}_6
$$

for $n > n_0$. Thus,

$$
|B_n(x)-1| < R_n < \tilde{c}_5\beta_n^2, \tag{4.8}
$$

for $n > n_0$. By (4.8), we have

$$
\begin{aligned}
&|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \\
&\leq \Phi_1(x)B_n(x)|A_n(x)-1| + |B_n(x)-1| \\
&< \Phi_1(x)|A_n(x)-1| + \tilde{c}_5\beta_n^2
\end{aligned} \tag{4.9}
$$

for $x \geq c_n$.

We now prove (4.5). By (2.9), (2.10) and the definition of $A_n(x)$, we have

$$
\begin{aligned}
A_n'(x) &= A_n(x)\left(\frac{1}{x^{1+\beta_n}}\frac{\log\alpha_n}{\alpha_n}\exp(-\frac{1}{2}\beta_n^2\log^2 x) - \frac{1}{x^2}\right) \\
&< 0
\end{aligned}
$$

for $x > 1$. Since

$$
0 < n\gamma(\alpha_n) < \beta_n^2, \quad \text{and} \quad e^x - 1 \leq xe^x, \text{ for } 0 \leq x \leq 1,
$$

$$
\text{and} \quad \exp(n\gamma(\alpha_n)) < \exp(\beta_n^2) < \exp(\frac{25}{8\log n}) < \exp(\frac{25}{8\log n_0}), \text{ for } n > n_0,
$$

and by (2.5), (2.9), we have

$$
\begin{aligned}
\sup_{x\geq 1}|A_n(x)-1| &= |A_n(1)-1| \\
&= |\exp(n\gamma(\alpha_n))-1| \\
&\leq n\gamma(\alpha_n)\exp(n\gamma(\alpha_n)) \\
&\leq \tilde{c}_7\beta_n^2
\end{aligned} \tag{4.10}
$$

for $n > n_0$.
Combine (4.9) with (4.10), we have

$$\sup_{x\ge 1}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < (\tilde{c}_5 + \tilde{c}_7)\beta_n^2.$$

Secondly, consider the situation of $c_n \le x < 1$. By Lemma 2.1, we obtain

$$\begin{aligned}
-n\Psi_n(x) + \frac{1}{x} &= -n\left(\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}) - \gamma(\alpha_n x^{\beta_n})\right) + \frac{1}{x} \\
&= -n\Bigg(\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}) \\
&- \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-3}q_n(\alpha_n x^{\beta_n})\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2})\Bigg) + \frac{1}{x} \\
&= \frac{1}{x}\Bigg(1 + \beta_n^2\log x)^{-1}(-(1 - (\log\alpha_n)^{-2}q_n(\alpha_n x^{\beta_n})(1 + \beta_n^2\log x)^{-2}) \\
&\times \exp(-\frac{1}{2}\beta_n^2\log^2 x) + 1 + \beta_n^2\log x\Bigg) \\
&= \frac{1}{x}(1 + \beta_n^2\log x)^{-1}Q_n(x),
\end{aligned}$$

where $0 < q_n(x) < 1$ and

$$Q_n(x) = -\left(1 - \beta_n^2 q_n(\alpha_n x^{\beta_n})(1 + \beta_n^2\log x)^{-2}\right)\exp(-\frac{1}{2}\beta_n^2\log^2 x) + 1 + \beta_n^2\log x.$$

Since $e^{-x} > 1 - x$, as $x > 0$, we have

$$\begin{aligned}
Q_n(x) &< -(1 - \beta_n^2 q_n(\alpha_n x^{\beta_n})(1 + \beta_n^2\log x)^{-2})(1 - \frac{1}{2}\beta_n^2\log^2 x) + 1 + \beta_n^2\log x \\
&< \beta_n^2((1 + \beta_n^2\log x)^{-2} + \frac{1}{2}\log^2 x).
\end{aligned}$$

But

$$\begin{aligned}
Q_n(x) &> \beta_n^2 q_n(\alpha_n x^{\beta_n})(1 + \beta_n^2\log x)^{-2} + \beta_n^2\log x \\
&> \beta_n^2\log x.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
|Q_n(x)| &< \beta_n^2((1 + \beta_n^2\log x)^{-2} + \frac{1}{2}\log^2 x + |\log x|) \\
&< \beta_n^2\left((1 - \frac{\log(2\log\log\alpha_n)}{\log^2\alpha_n})^{-2} + \frac{1}{2}\log^2 x + |\log x|\right) \\
&< \beta_n^2(\tilde{c}_8 + \frac{1}{2}\log^2 x + |\log x|)
\end{aligned}$$

for $n > n_0$, where $c_n \le x < 1$. Therefore,

$$|-n\Psi_n(x) + \frac{1}{x}| < \beta_n^2(\tilde{c}_8 + \frac{1}{2}\log^2 x + |\log x|)x^{-1}(1 + \beta_n^2\log x)^{-1}$$

$$
\begin{aligned}
&< \beta_n^2(\tilde{c}_8 + \frac{1}{2}\log^2 c_n + |\log c_n|)c_n^{-1}(1+\beta_n^2\log c_n)^{-1} \\
&< \tilde{c}_9
\end{aligned}
$$

for $n \geq n_0$. Thus, there exists a positive number θ satisfying $0 < \theta < 1$ such that

$$
\begin{aligned}
\Phi_1(x)|A_n(x)-1| &< \Phi_1(x)\exp(\theta(-n\Psi_n(x)+\frac{1}{x}))|-n\Psi_n(x)+\frac{1}{x}| \\
&< \exp(\tilde{c}_9)\beta_n^2 \sup_{c_n \leq x < 1} |(\tilde{c}_8 + \frac{1}{2}\log^2 x + |\log x|)x^{-1}|(1+\beta_n^2\log c_n)^{-1} \\
&< \tilde{c}_{10}\beta_n^2. \qquad (4.11)
\end{aligned}
$$

By (4.9) and (4.11), the proof of (4.6) is complete.

Thirdly, consider the circumstance of $0 < x < c_n$. In this case

$$\Phi_1(x) < \Phi_1(c_n) = \beta_n^2,$$

we have

$$
\begin{aligned}
\sup_{0<x<c_n} |F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| &< F^n(\alpha_n c_n^{\beta_n}) + \Phi_1(c_n) \\
&< \sup_{c_n<x<1} |F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| + 2\Phi_1(c_n) \\
&< (\tilde{c}_5 + \tilde{c}_{10})\beta_n^2 + \beta_n^2 \\
&< \tilde{c}_{11}\beta_n^2.
\end{aligned}
$$

The proof of Theorem 3.1 is finished. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JH obtained the theorem and completed the proof. SC corrected and improved the final version. Both authors read and approved the final manuscript.

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