The Chromatic-Covering of a Graph: Ratios, Domination, Areas and Farey Sequences.

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Abstract

The study of the chromatic number and vertex coverings of graphs has opened many avenues of research. In this paper we combine these two concepts in a ratio, to investigate the domination effect of the chromatic number, of the subgaph induced by a vertex covering of a graph G (called the cover graph of G), on the original chromatic number of G, where large number of vertices are involved. This is referred to as the *chromatic-cover domination*. If this chromatic-cover ratio is a function of n, the order of graphs belonging to a class of graph, then we discuss its horizontal *asymptotic* behavior and attach the graphs average degree to the Riemann integral of this ratio, thus associating *chromatic-cover area* with classes of graphs. We found that the chromatic-cover domination had a strongest effect on complete graph, while this chromatic-cover domination had zero effect on star graphs. We show that the chromatic-cover asymptote of all classes of graphs belong to the interval [0,1], and conjecture that complete graphs are the only class of graphs having chromatic-cover asymptote of 1 and that they also have the largest area . We construct a class of graphs, using known classes of graph where vertices are replaced with cliques on q vertices, thus generating sequences which converges to the chromatic-cover asymptote of known classes of graphs. We use a particular sequence to construct a Farey chromatic-cover sequence which is a subsequence of the famous Farey sequence.

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Key words: chromatic number, vertex cover, domination, ratios, asymptotes, areas, Farey sequences.

1. Introduction

All graphs in this paper are simple and loopless and on n vertices. We shall use the graph-theoretical notation of [6].

Chromatic number, vertex covers and ratios

Much research has been done involving the chromatic number of a graph (see[7]) and (minimum) vertex coverings of graph (see [1]). Ratios have been an important aspect of graph theoretical definitions. Examples of ratios are: expanders, (see [2]), the central ratio of a graph (see [3]), eigen-pair ratio of classes of graphs (see [9]), Independence and Hall ratios (see [4]) and tree-cover ratio of graphs (see [8].

In this paper we combine the two concepts of chromatic number and vertex covering to form a ratio, associated with a connected graph G, involving the chromatic number of the sugraph H(S) of G induced by a vertex cover S of G, called the *cover graph of G*, and the chromatic number of G. This chromatic-cover ratio allows for the investigation of the domination effect of the chromatic number of the cover graph on the original chromatic number of G, where a large number of vertices are involved – referred to as the *chromatic-cover domination*. If the chromatic cover ratio is a function of n for a particular class of graphs, then we investigated its asymptotic behavior (see [7] and [8]). The chromatic-cover domination was determined for known classes of graph. We found that, for the complete graph, the chromatic-cover domination was the strongest, and for star graphs with rays of length one, no effect at all, while for the sun graph the effect was average. By introducing the average degree of a graph together with the Riemann integral of the chromatic-cover ratio we associated chromatic –cover area with classes of graphs (see [7] and [8]). Using known classes of graph we constructed a new graph by replacing end vertices with cliques of order q creating sequences

$$\frac{2^2}{3^2},\frac{3^2}{4^2},\frac{4^2}{5^2},\ldots,\frac{q^2}{(q+1)^2}\ ;\frac{1}{3}.\frac{2}{4}.\frac{3}{5},\ldots,\frac{q-1}{q+1}\ ;\ \frac{1}{2.2^2},\frac{2^2}{2.3^2},\frac{3^2}{24^2},\ldots,\frac{(q-1)^2}{2q^2}$$

The first two converges to the chromatic-cover asymptote of complete graphs, while the third to the chromatic-cover asymptote of the sun graph. We use the root sequence associated with the first sequence to construct a Farey chromatic-cover sequence which is a sub-sequence of the famous Farey sequence.

2. Chromatic-cover ratio, asymptotes, domination and area

We combine the idea of chromatic number and vertex cover in the following definitions to allow for the measure of the domination of the chromatic number of a cover graph over the chromatic number of original graph for large values of n.

Definition 2.1

Let G be a connected graph with minimum covering S of vertices. Let H(S) be the subgraph of G induced by S, the *cover graph* of G.

The *chromatic-cover ratio* of a graph G of order n, with respect to S, is defined as:

$$\operatorname{cov}\{\chi^{S}(G)\} = \frac{|S|\chi(H(S))}{n\chi(G)}$$

where $\chi(G)$ is the chromatic number of G.

Definition 2.2

If $\operatorname{cov}\{\chi^{S}(G)\} = f(n)$ for every $G \in \Im$, where $\Im$ is a class of graphs, then the asymptotic behavior of f(n) is called the *chromatic-cover asymptote of* $\Im$ and denoted by (see [8] and [9]):

$$as\operatorname{cov}\{\chi^{S}(\Im)\}.$$

Chromatic –cover domination

This asymptote give a measure of the *domination effect* of the chromatic number of the cover graph on the chromatic number of the original graph, for large values of n, referred to as the *chromatic-cover domination.*

Definition 2.3

If $\operatorname{cov}\{\chi^S(G)\} = f(n)$ for every $G \in \mathfrak{I}$, where $\mathfrak{I}$ is a class of graphs ,then the *chromatic-cover area* is defined as(see [8] and [9]) :

$A_{\mathfrak{I}(n)}^{\chi^S} = \frac{2m}{n}\int f(n)dn$ with $A_{\mathfrak{I}(k)}^{\chi^S} = 0$ where k is the smallest number of vertices for which $\operatorname{cov}\{\chi^S(G)\} = f(n)$ is defined, and $\frac{2m}{n}$ is the average degree of $G \in \mathfrak{I}$.

Examples:

2.1 The complete graph K_n we have, with its cover graph, $H(S) = K_{n-1}$:

$$\operatorname{cov}\{\chi^S(K_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(n-1)(n-1)}{nn} = \frac{(n-1)^2}{n^2} \text{ and}$$

$$as\operatorname{cov}\{\chi^S(K_n)\} = 1$$

$$A_{K_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = (n-1)\int \frac{n^2-2n+1}{n^2}dn$$

$$= (n-1)\int (1-\frac{2}{n}+\frac{1}{n^2})dn = (n-1)(n-2\ln n - n^{-1} + c)$$

$$A_{K_2}^{\chi^S} = 0 \Rightarrow c = 2\ln 2 - \frac{3}{2}$$

2.2 The complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ we have, with S consisting of one of the partite sets on $\frac{n}{2}$ vertices, and its cove graph the set of $\frac{n}{2}$ isolated vertices:

$$\operatorname{cov}\{\chi^S(K_{\frac{n}{2},\frac{n}{2}})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n}{2})(1)}{n(2)} = \frac{1}{4} \text{ and}$$

$$as \operatorname{cov}\{\chi^S(K_{\frac{n}{2},\frac{n}{2}})\} = \frac{1}{4}$$

$$A^{\chi^S}_{K_{\frac{n}{2},\frac{n}{2}}} = \frac{2m}{n}\int f(n)dn = \frac{n}{2}\int \frac{1}{4}dn = \frac{n}{2}(\frac{n}{4}+c)$$

$$A^{\chi^S}_{K_{1,1}} = 0 \Rightarrow c = -\frac{1}{2}$$

2.3 The cycle graph C_n on an even number of vertices , we have with S having size $\frac{n}{2}$ by considering every second vertex of the cycle so that the cover graph consists of $\frac{n}{2}$ isolated vertices :

$$\operatorname{cov}\{\chi^S(C_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n}{2})(1)}{n(2)} = \frac{1}{4}$$

$$as \operatorname{cov}\{\chi^S(C_n)\} = \frac{1}{4}$$

$$A^{\chi^S}_{C_n} = \frac{2m}{n}\int f(n)dn = 2\int \frac{1}{4}dn = 2(\frac{n}{4}+c)$$

$$A^{\chi^S}_{C_4} = 0 \Rightarrow c = -1$$

2.4 The path graph P_n on an even number of vertices, we have S having size $\frac{n}{2}$ by considering the first vertex of the path and then every second vertex so that the cover graph consists of $\frac{n}{2}$ isolated vertices:

$$\operatorname{cov}\{\chi^S(P_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n}{2})(1)}{n(2)} = \frac{1}{4}$$

$$as\operatorname{cov}\{\chi^S(P_n)\} = \frac{1}{4}$$

$$A_{P_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{n+2}{2}\int \frac{1}{4}dn = \frac{n+2}{2}(\frac{n}{4}+c)$$

$$A_{P_2}^{\chi^S} = 0 \Rightarrow c = -\frac{1}{2}$$

2.5 The wheel W_n with n-1 spokes and where n is odd, we have the central vertex and every second vertex of the cycle as S so that the cover graph is the star graph with rays of length 1:

$$\operatorname{cov}\{\chi^S(W_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n-1}{2})(2)}{n(3)} = \frac{n-1}{3n}$$

$$as\operatorname{cov}\{\chi^S(W_n)\} = \frac{1}{3}$$

$$A_{W_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{4n-4}{n}\int \frac{1}{3}dn = \frac{4n-4}{n}(\frac{n}{3}+c)$$

$$A_{W_4}^{\chi^S} = 0 \Rightarrow c = -\frac{4}{3}$$

2.6 Star graphs $S_{r,1}$ on n vertices with r rays of length 1, we have S the central vertex:

$$\operatorname{cov}\{\chi^S(S_{r,1})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(1)(1)}{n(2)} = \frac{1}{2n}$$

$$as\operatorname{cov}\{\chi^S(S_{r,1})\} = 0$$

$$A_{S_{r,1}}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{2(n-1)}{n}\int \frac{1}{2n}dn = \frac{2(n-1)}{n}(\frac{\ln n}{2}+c)$$

$$A_{S_{1,1}}^{\chi^S} = 0 \Rightarrow c = -\frac{\ln 2}{2}$$

2.7 Star graphs $S_{r,2}$ with m rays of length 2, and $n = 2r+1$, we have S consisting of the middle vertex of each ray so that the cover consists of r isolated vertices.

$$\text{cov}\{\chi^S(S_{r,2})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(m)(1)}{n(2)} = \frac{\frac{n-1}{2}}{2n} = \frac{n-1}{4n}$$

$$as\,\text{cov}\{\chi^S(S_{r,2})\} = \frac{1}{4}$$

$$A_{S_{r,2}}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{4(n-1)}{2n}\int \frac{n-1}{4n}dn = \frac{2(n-1)}{n}(\frac{n}{4}-\frac{\ln n}{4}+c)$$

$$A_{S_{r,2}}^{\chi^S} = 0 \Rightarrow c = -\frac{3}{4}+\frac{\ln 3}{4}$$

2.8 The sun graph Su_n on n vertices

For the sun graph on an even number of vertices- i.e. we have an even cycle on $\frac{n}{2}$ vertices with end vertices added to each vertex of the cycle, we take S to be the vertices of the cycle so that the cover graph is the cycle and we have:

$$\text{cov}\{\chi^S(Su_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n}{2})(2)}{n(2)} = \frac{1}{2}$$

$$as\,\text{cov}\{\chi^S(Su_n)\} = \frac{1}{2}$$

$$A_{Su_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{2n}{n}\int \frac{1}{4}dn = 2(\frac{n}{4}+c)$$
$$A_{Su_6}^{\chi^S} = 0 \Rightarrow c = -\frac{3}{2}$$

2.9 The fan graph F_n on n vertices

Construct the fan graph F_n on and odd number $n \geq 3$ of vertices by taking a path on n-1 vertices and joining each vertex of the path to a single vertex, the center of the fan graph.

The chromatic number of the fan graph is 3 and we take S as the center vertex with every alternate vertex of path starting with the first vertex so that the cover graph is a star graph on $\frac{n-1}{2}+1$vertices and has chromatic number 2. Thus

$$\text{cov}\{\chi^S(F_n)\} = \frac{|S|\chi(H(S))}{n\chi(F_n)} = \frac{(\frac{n-1}{2}+1)(2)}{n(3)} = \frac{n+1}{3n}:$$

$$as\,\text{cov}\{\chi^S(F_n)\} = \frac{1}{3}$$

$$A_{F_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{4n-6}{n}\int \frac{n+1}{3n}dn = \frac{4n-6}{3n}(n+\ln n+c)$$
$$A_{F_3}^{\chi^S} = 0 \Rightarrow c = -3-\ln 3$$

2.10 The Ladder graph L_n on n vertices

Let the ladder on $n \geq 4$ vertices be formed by joining corresponding vertices of paths on $\frac{n}{2}$ vertices each. We take $\frac{n}{2}$ to be even so that the covering graph will be found by taking alternating vertices of the first path and different alternating vertices of the second so that its chromatic number is 1 and it will have $\frac{n}{4}+\frac{n}{4}=\frac{n}{2}$ vertices. The chromatic number of the ladder graph is 2.

$$\operatorname{cov}\{\chi^S(L_n)\} = \frac{|S|\chi(H(S))}{n\chi(L_n)} = \frac{(\frac{n}{2})(1)}{n(2)} = \frac{1}{4}$$

$$as\operatorname{cov}\{\chi^S(L_n)\} = \frac{1}{4}$$

$$A_{L_n}^{\chi^S} = \frac{2m}{n}\int f(n)dn = \frac{3n-4}{n}\int \frac{1}{4}dn = \frac{3n-4}{n}(\frac{n}{4}+c)$$

$$A_{L_4}^{\chi^S} = 0 \Rightarrow c = -1.$$

Theorem 2.1

The chromatic cover ratio, asymptote and area respectively for the following classes $\mathfrak{I}$ of graphs are:

1. $K_n : \frac{(n-1)^2}{n^2}; 1;\ (n-1)(n-2\ln n - n^{-1} - 2\ln 2 - \frac{3}{2})$
2. $K_{\frac{n}{2},\frac{n}{2}} : \frac{1}{4}; \frac{1}{4};\ \frac{n}{2}\int \frac{1}{4}dn = \frac{n}{2}(\frac{n}{4} - \frac{1}{2})$
3. $C_n : \frac{1}{4}; \frac{1}{4}; 2(\frac{n}{4} - 1)$
4. $P_n : \frac{1}{4};\ \frac{1}{4};\ \frac{n+2}{2}(\frac{n}{4} - \frac{1}{2})$
5. $W_n : \frac{n-1}{3n}; \frac{1}{3}; \frac{4n-4}{n}(\frac{n}{3} - \frac{4}{3})$
6. $S_{r,1} : \frac{1}{2n}; 0; \frac{2(n-1)}{n}(\frac{\ln n}{2} - \frac{\ln 2}{2})$
7. $S_{r,2} : \frac{n-1}{4n}; \frac{1}{4}\ \frac{2(n-1)}{n}(\frac{n}{4} - \frac{\ln n}{4} - \frac{3}{4} + \frac{\ln 3}{4})$

8. $Su_n : \frac{1}{2}\ ;\ \frac{1}{2} ; 2(\frac{n}{4} - \frac{3}{2})$

9. $F_n : \frac{n+1}{3n} ;\ \frac{1}{3} ;\ \frac{4n-6}{3n}(n + \ln n - 3 - \ln 3)$

10. $L_n : \frac{1}{4} ; \frac{1}{4} ; \frac{3n-4}{n}(\frac{n}{4} - 1)$

Theorem 2.2

If $\operatorname{cov}\{\chi^S(G)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = f(n)$ for each $G \in \Im$, then

$as\operatorname{cov}\{\chi^S(\Im)\} \in [0,1]$ for all such classes of graphs.

Proof

There are 5 possibilities for $\chi(G);|S|;\chi(H(S))$, where $k,t,s,k',p,q,k'',t',q',w,u,v,w'.t'',v'$ are non-negative constants :

$1. n-k; n-t; n-s$
$2. n-k'; p; q$
$3. n-k''; n-t'; q'$
$4. w; u; v$
$5. w'; n-t'', v'; \quad w' > v'$

In case 1 $as\operatorname{cov}\{\chi^S(\Im)\} = 1$.

In cases 2,3 and 4 $as\operatorname{cov}\{\chi^S(\Im)\} = 0$.

In case 5 $as\operatorname{cov}\{\chi^S(\Im)\} = \frac{v'}{w'}$; since $w' > v'$ we have $0 < \frac{v'}{w'} < 1$

2.1 Corollary

The chromatic-cover domination is the greatest for complete graphs, and is 0 for star graphs with rays of length 1, and average for sun graphs.

Conjecture 1

The complete graph possesses the strongest chromatic-cover domination of all classes of regular graphs and star graphs.

Conjecture 2

The complete graph possesses the largest chromatic-cover area of all classes of graphs.

3. The q-clique chromic-cover partners and sequences

3.1 The sun graph and its q-clique chromatic-cover partner

For the sun graph on an even number of vertices- i.e. we have an even cycle on $\frac{n}{2}$ vertices with end vertices added to each vertex of the cycle, we take S to be the vertices of the cycle so that H(S) is the cycle and showed that:

$$\mathrm{cov}\{\chi^S(Su_n)\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{n}{2})(2)}{n(2)} = \frac{1}{2}$$

$$as\,\mathrm{cov}\{\chi^S(Su_n)\} = \frac{1}{2}$$

This graph is regarded as chromatic-cover domination balanced when a large number of vertices are involved. We use this ratio to construct the K_q-*chromatic-cover partner* of $G = Su_n$ with respect to S as follows:

For q=2 we take each vertex u not in S (the cycle) adjacent to v on the cycle and replace It with K_2 and join every vertex of K_2 to v. Thus we have a triangle incident with each vertex of the cycle. Thus the new graph $H_n^{K2} = K_2 - Par_\chi^S(Su_n)$ on n vertices has a cycle on $\frac{n}{3}$ vertices and $\frac{2n}{3}$ remaining vertices not on the cycle. This graph has chromatic number 3. The vertices of S will be taken as the subgraph of H which is a sun graph on $\frac{2n}{3}$ vertices. Then

$$\operatorname{cov}\{\chi^S(H_n^{K2})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{2n}{3})(2)}{n(3)} = \frac{4}{9} = \frac{2^2}{3^2}$$

The general construction of $H_n^{Kq} = K_q - Par_{\chi}^{S}(Su_n)$ involves replacing each vertex u not on the cycle, u adjacent to v on the cycle, with a q-clique Kq and join every vertex of this clique to v. We will then have a graph on n vertices with the cycle on $\frac{n}{q+1}$ vertices and $\frac{qn}{q+1}$ vertices not on the cycle. This new graph will have chromatic number q+1 and S will be the graph with a cycle on $\frac{n}{q+1}$ vertices, each vertex of the cycle joined to a clique of size q-1 so that its chromatic number is q and is on $\frac{qn}{q+1}$ vertices. Hence:

$$\operatorname{cov}\{\chi^S(H_n^{Kq})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{(\frac{qn}{q+1})(q)}{n(q+1)} = \frac{q^2}{(q+1)^2}$$

We therefore have associated a sequence:

$$\frac{2^2}{3^2}, \frac{3^2}{4^2}, \frac{4^2}{5^2}, \ldots, \frac{q^2}{(q+1)^2}$$

With the K_q-chromatic-cover partner $H_n^{Kq} = K_2 - Par_{\chi}^{S}(Su_n)$ of the sun graph which converges to 1, the chromatic-cover asymptote of complete graphs.

3.2 The q-clique chromatic-cover partner of the star graph with rays of length 1

Consider the star graph on n vertices with r rays of length 1-i.e. $K_{1.n-1}$.To form the q-clique chromatic-cover partner of $K_{1.n-1}$ we replace each vertex u, except the center vertex v, with a q-clique where $q \geq 2$ and join every vertex of each clique connected to the center vertex v. There will be $\frac{(n-1)}{q}$ q-cliques connected to v vertices so that the total number of vertices will be n. The chromatic number of the new partner graph will be q+1 and the covering S will be on $\frac{(n-1)}{q}(q-1)+1$vertices, q-1 vertices from each q-clique and the center vertex, and will have chromatic number q. Thus the chromatic-cover ratio of this partner graph $H_n^{K_q} = K_q - Par_\chi^S(K_{1,t-1})$ will be:

$$\operatorname{cov}\{\chi^S(H_n^{K_q})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{[\frac{(n-1)(q-1)}{q}+1]q}{n(q+1)} = \frac{(n-1)(q-1)+q}{n(q+1)}$$

$$= \frac{nq-n-q+1+q}{nq+n} = \frac{nq-n+1}{nq+n}$$

Fixing q and dividing top and bottom by n gives us the ratio (for large values of n):

$\frac{q-1}{q+1}$ this yields sequence:

$$\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \ldots, \frac{q-1}{q+1}$$

This sequence also converges to 1.

3.3 The q-clique chromatic-cover partner of star graphs with r rays of length 2.

For each end vertex u , connected to the middle vertex v, of the star graph with r rays of length 2, replace it with a q-clique and join each vertex of each clique to the vertex v of this star graph.

There will be $n = 1 + r + rq$ vertices all together with $r = \frac{n-1}{q+1}$ q-cliques and $r = \frac{n-1}{q+1}$ vertices connected directly to the center vertex so that the chromatic-cover partner graph will be on n vertices We actually have r (q+1)-cliques connected directly by an edge to the center vertex. The chromatic number of the partner will be q+1. Take S to be (minimum vertex cover) q-1 vertices from each clique and a middle vertex plus the center vertex so that S has size:

$\frac{(n-1)(q)}{(q+1)} + 1$ and chromatic number q.

The chromatic-cover ratio of this graph will be:

$$\text{cov}\{\chi^S(H_n^{K_q})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{[\frac{(n-1)(q)}{(q+1)} + 1]q}{n(q+1)} = \frac{[(n-1)(q) + q + 1]q}{n(q+1)(q+1)}$$
$$= \frac{nq^2 + q}{n(q^2 + 2q + 1)}$$

Fixing q and dividing all terms by n .e get the ratio (for n large):

$\frac{q^2}{(q+1)^2}$ yielding the sequence identical to the sequence of the chromatic-cover partner of the sun graph:

$$\frac{2^2}{3^2}, \frac{3^2}{4^2}, \frac{4^2}{5^2}, \ldots, \frac{q^2}{(q+1)^2}.$$

3.4 The q-clique chromatic-cover partner of the complete end-extend graph

Take the complete end-extend graph (take the complete graph on at least two vertices and attach an end vertex to each of its vertices) and replace each end vertex u (joined to v) with a clique of order q and join each vertex of the clique to v, so that we have a q-clique chromatic partner graph on n vertices with and $\frac{n}{q+1}$ (q+1)-cliques and a clique T on $\frac{n}{q+1}$ vertices made up of a single vertex from each of the (q+1)-cliques. We take S as the collection of q vertices from each clique where one is from T. The chromatic number of the new partner graph will be q+1 and that of the cover graph q.

The chromatic-cover ratio of this q-clique partner graph G will be, with $(\frac{n}{q+1)} \leq (q-1))$:

$$\operatorname{cov}\{\chi^S(H_n^{K_q})\} = \frac{|S|\chi(H(S))}{n\chi(G)} = \frac{\frac{n}{(q+1)}qq}{n(q+1)} = \frac{q^2}{(q+1)^2}$$

This gives rise to the sequence:

$$\frac{2^2}{3^2}, \frac{3^2}{4^2}, \frac{4^2}{5^2}, \ldots, \frac{q^2}{(q+1)^2}$$

Which converges to 1.

3.5 The q-clique chromatic-cover partner of the fan end-extend graph

Take the fan end-extend graph (the fan graph on at least three vertices and attach an end vertex to each of its vertices of its path- not its center vertex) and replace each end vertex u (joined to v) with a clique of order q and join each vertex of the clique to v, so that we have a q-clique chromatic partner graph Q on n vertices with $\frac{n}{2}-1$ vertices on the path and $\frac{(\frac{n}{2}-1)}{q}$ vertices from the cliques.

The chromatic number of Q is q and we take S to be the center vertex together with q-1 vertices from each clique so that the chromatic number is q-1. Thus:

$$\operatorname{cov}\{\chi^S(H_n^{K_q})\} = \frac{|S|\chi(H(S))}{n\chi(Q)} = \frac{[(\frac{(\frac{n}{2}-1)}{q})(q-1)+1](q-1)}{nq} = \frac{[(n-2)(q-1)+2q](q-1)}{2qnq}$$

$\dfrac{(nq-n-2q+2+2q)(q-1)}{2nq^2} = \dfrac{n(q-1)^2+2(q-1)}{2nq^2}$ which has asymptote:

$\dfrac{(q-1)^2}{2q^2}$ which gives rise to sequence:

$\dfrac{1}{2.2^2}, \dfrac{2^2}{2.3^2}, \dfrac{3^2}{24^2}, \ldots, \dfrac{(q-1)^2}{2q^2}$.

3.6 The q-clique chromatic-cover partner of the end-extend ladder graph

Form the end-extend ladder graph by joining and end vertex u to each vertex of the ladder graph. Then form the q-clique chromatic cover partner by replacing each end vertex u (joined to v) with a q-clique and join each vertex of the clique to v. Thus each vertex of the original ladder graph will now belong to a clique of order q+1. The chromatic number of the partner will be q+1 and we take S to be q vertices from each (q+1)-clique where we include the vertex of the ladder subgraph. Each of the 2 paths of the original ladder will give rise to $\dfrac{n}{2(q+1)}$ vertices in the partner graph. We take S to be q vertices from each clique where one vertex comes from the original ladder graph. The chromatic number of the cover graph will be q so that the chromatic-cover ratio of the q-clique partner graph will be:

$$\operatorname{cov}\{\chi^S(L_n^{K_q})\} = \frac{|S|\chi(H(S))}{n\chi(L_n)} = \frac{[(\frac{(n)}{(q+1)})(q)](q)}{n(q+1)}$$

$= \dfrac{(q-1)^2}{q^2}$ which converges to 1.

Theorem 3.1

The following sequences arise from the q-clique partner of the classes of graphs:

$\frac{2^2}{3^2}, \frac{3^2}{4^2}, \frac{4^2}{5^2}, ..., \frac{q^2}{(q+1)^2}$ for the, ladder, complete and stars with rays of length 2 associated q-cliqued partner.

$\frac{1}{3}. \frac{2}{4}. \frac{3}{5}, ..., \frac{q-1}{q+1}$ for the star graph with rays of length 1 associated q-cliqued partner

$\frac{1}{2.2^2}, \frac{2^2}{2.3^2}, \frac{3^2}{24^2}, ..., \frac{(q-1)^2}{2q^2}$ for the fan associated with q-cliqued partner.

4. Farey q-chromatic-cover sequences and diagrams

The Farey sequence of order *n* is the sequence FY_n of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to *n*, arranged in order of increasing size. (see [5]). Farey sequences are named after the British geologist John Farey, Sr., whose letter about these sequences was published in the *Philosophical Magazine* in 1816.

For example, the sequence FY_5 is as follows:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

Interestingly, the *root-sequence* (term by term square-root) associated with the first sequence in theorem 3.1 is:

$$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$$

Forms a subsequence of the Farey sequence- generally a n-2 subsequence of FY_n.

This sequence is a *root chromatic-cover (*n-2) sub-sequence of FY_5.

The *unit-mirror* sequence of this sub-sequence is:

$\frac{1}{3},\frac{1}{4},\frac{1}{5}$ i.e. the sum of corresponding terms of the root chromatic-cover sequence and the unit-mirror sequence is 1- these pairs are called the *unit-mirror pairs.*

Finally we form the *Farey chromatic-cover sequence* by taking the union of these 2 sequences and arranging terms in ascending order to form a 2n-4 subsequence of the Farey sequence:

$\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{3},\frac{3}{4},\frac{4}{5}$

Note that the pairs $\frac{1}{5},\frac{1}{3};\frac{2}{3},\frac{4}{5}$ each have difference $\frac{2}{15}$ and they are called *duo-pairs,* i.e. pairs whose difference has 2 in the numerator.

The *Farey q-chromatic-cover diagram* for q=5 is shown in figure 1 below:

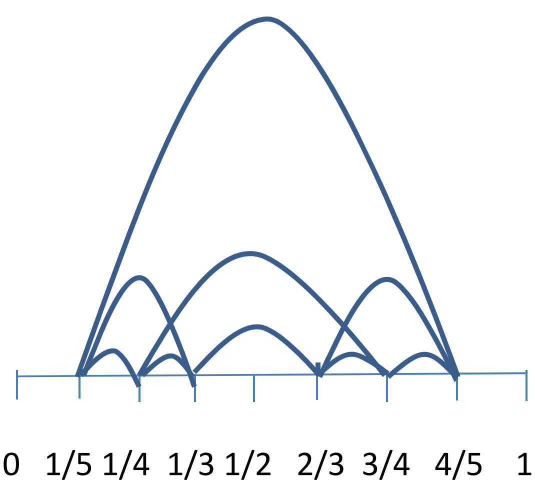
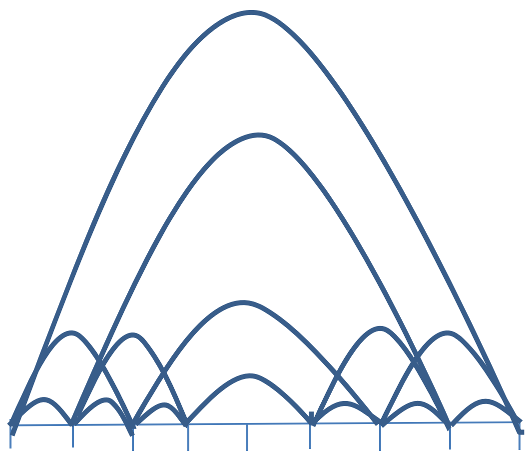


Figure 1: The Farey 5-chromatic-cover diagram

In the diagram neighbors are joined, the unit-mirror pairs are joined and the duo-pairs are joined

The Farey 6-chromatic-cover diagram is shown in figure 2 below with 6 intersections.



1/6 1/5 1/4 1/3 1/2 2/3 3/4 4/5 5/6

Figure 2: The Farey 6-chromatic-cover diagram

Total number of intersections generally will be 2+(q-5)4.

Theorem 4.1

If the neighbors of the Farey q-chromaitc cover sequence:

$A:\frac{a}{a+1},\frac{a+1}{a+2}$ have unit-mirror associate neighbors: $B':\frac{1}{a+1},\frac{1}{a+2}$.

Swop entries of B' to keep ascending order: $B:\frac{1}{a+2},\frac{1}{a+1}$

Then the midpoints of A and B are unit-mirror pairs.

Proof

The midpoint of A is: $\frac{a}{a+1}+\frac{1}{2}[\frac{a+1}{a+2}-\frac{a}{a+1}]=\frac{a}{a+1}+\frac{1}{2}[\frac{(a+1)^2-a(a+2)}{(a+2)(a+1)}]$

$$=\frac{a}{a+1}+\frac{1}{2}[\frac{1}{(a+2)(a+1)}]=\frac{2a(a+2)+1}{2(a+2)(a+1)}=\frac{2a^2+4a+1}{2(a+2)(a+1)}.$$

The midpoint of B is:

$$\frac{1}{a+2}+\frac{1}{2}[\frac{1}{a+1}-\frac{1}{a+2}]=\frac{1}{a+2}+\frac{1}{2}[\frac{(a+2)-(a+1)}{(a+2)(a+1)}]$$

$$= \frac{1}{a+2} + \frac{1}{2}[\frac{1}{(a+2)(a+1)}]$$

$$= \frac{2(a+1)+1}{2((a+2)(a+1)} = \frac{2a+3}{2((a+2)(a+1)}.$$

Midpoint of A plus midpoint of B is:

$$\frac{2a^2+4a+1}{2(a+2)(a+1)} + \frac{2a+3}{2((a+2)(a+1)} = \frac{a^2+3a+2}{(a+2)(a+1)} = 1.$$

This completes the proof

Thus through the chromatic-covser ratio of the q-cliqued partner of the complete graph we have connected the complete graph to a variation of the Farey sequence.

5. Conclusion

In this paper we combined the two concepts of chromatic number and vertex covering to form a ratio, associated with a connected graph G, involving the chromatic number of the cover graph of *G* and the chromatic number of G. This chromatic-cover ratio allowed for the investigation of the domination effect of the chromatic number of cover graph on the original chromatic number of G, where a large number of vertices are involved – referred to as the *chromatic-cover domination*. If the chromatic cover ratio is a function of n for a particular class of graphs, then we investigated its asymptotic behavior. The chromatic-cover domination was determined for known classes of graph. We found that, for the complete graph, the chromatic-cover domination was the strongest, and for star graphs with rays of length one, no effect at all, while for the sun graph the effect was average. By introducing the average degree of a graph together with the Riemann integral of the chromatic-cover ratio we associated chromatic –cover area with classes of graphs. Using known classes of graph we constructed new classes of graphs using q-cliques and created sequences. We used one of these sequences to create a Farey q-chromatic-cover sequence which is a 2n-4 subset of the famous Farey sequences and prove that the midpoints of unit-mirror neighbor pairs from this Farey q-chromatic-cover sequence are also unit-mirror pairs.

We conjectured that the chromatic-cover domination is the strongest for complete graphs over all classes of regular graphs. We also believe that complete graphs possess the greater chromatic-cover area of all classes of graphs

5. References

[1] Adiga, C. Bayad, A. Gutman, I. and Srinivas, S. A. The Minimum Covering Energy of a Graph. *Kragujevac J. Sci.* 34, 39-56. (2012).

[2] Alon, N. and Spencer, J. H. 2011. Eigenvalues and Expanders. *The Probabilistic Method* (3rd ed.). John Wiley & Sons.

[3] Buckley, F. 1982. The central ratio of a graph. *Discrete Mathematics*. 38(1): 17–21.

[4] Gábor, S. 2006. Asymptotic values of the Hall-ratio for graph powers . *Discrete Mathematics.*306(19–20): 2593–2601.

[5] Hardy, G.H., Wright, E.M. 1979. *An Introduction to the Theory of Numbers* (Fifth Edition). Oxford University Press. I

[6] Harris, J. M., Hirst, J. L. and Mossinghoff, M. 2008. *Combinatorics and Graph theory*. Springer, New York.

[7] Sopena, E. 2014. The oriented chromatic number of graphs: A short survey Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence. CNRS, LaBRI, UMR5800, F-33400 Talence. eric.sopena@labri.fr

[8] Winter, P. A. and Adewusi, F.J. 2014. Tree-cover ratio of graphs with asymptotic convergence identical to the secretary problem. *Advances in Mathematics: Scientific Journal*; Volume 3, issue 2, 47-61.

[9] Winter, P. A. and Jessop, C.L. 2014.Integral eigen-pair balanced classes of graphs: ratios, asymptotes, areas and involution complementary. To appear in: *International Journal of Graph Theory.*